Sufficient Conditions for Self-Scheduled Bounded Amplitude Control

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Abstract

This paper considers the problem of synthesizing a parameter-dependent output feedback controller for continuous-time linear parameter varying (LPV) systems so that a prespecified bound on signal amplitude is satisfied by the controlled system. The main result of the paper is a sufficient condition for the existence of an LPV controller which guarantees an upper bound on the induced-$\mathcal{L}_\infty$ norm of the controlled system. The condition takes the form of linear matrix inequalities (LMIs) which must be solved pointwise over a parameter set. A set of LPV controllers which satisfy the performance constraints are parameterized in terms of feasible solutions to the LMI existence conditions. In the event that the parameter variations are removed, the conditions reduce to necessary and sufficient LTI synthesis conditions for bounded amplitude performance. The existence conditions are amenable to efficient numerical techniques.

1 Introduction

A measure of system performance which frequently arises in control problems is the *peak value* of an appropriately selected plant signal. Control problems in which this performance measure arise are prevalent in the literature: motor control problems with electrical (voltage or current) or mechanical (motion) restrictions and process control problems with chemical concentration restrictions are two examples. In each of these examples, violation of the amplitude restrictions can lead to performance degradation and possibly catastrophic system failure. Furthermore, energy-based design techniques, such as the $\mathcal{H}_\infty$ design paradigm, are often inadequate for these types of performance problems.

In this paper, we consider the problem of synthesizing controllers for finite-dimensional linear parameter varying (LPV) plants so that the controlled system satisfied bounded amplitude performance constraints. LPV systems are linear systems whose state-space matrices depend in a continuous fashion on a time-varying parameter vector, $\theta(t) \in \mathbb{R}^s$. While the trajectory of $\theta(t)$ is not known *a priori*, it is assumed to

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be measurable in real-time, thus providing real-time information on the behavior of the system. Intuition alone suggests that incorporating such information into the controller should improve overall performance of the controlled system since such information allows the controller to adjust, or *schedule* to the change in operating conditions $\theta(t)$. A controller which incorporates real-time parameter measurements in this fashion is called a *self-scheduled controller.* Such controllers are justified empirically by the widely successful gain-scheduling technique.

The relationship between classical gain-scheduling and parameter-dependent systems is seen in [13] and [14]. Early work on parameter-dependent controllers is seen in [10]; more recently, parameter-dependent synthesis approaches based on Lyapunov and small-gain techniques have appeared [12][5][2] which frame the synthesis problem as a convex optimization problem. These approaches generalize $\mathcal{H}_\infty$ techniques for LTI systems to LPV systems. We note that while this list is not exhaustive, current literature on self-scheduled control is limited in scope to stability and $\mathcal{L}_2$, or $\mathcal{H}_\infty$, performance problems.

In fact, bounded amplitude control is a difficult problem even for systems with no parameter dependence (i.e. LTI systems). While bounded amplitude control of discrete-time linear systems is well-understood [15][8], bounded amplitude control of continuous-time systems remains a difficult task [9]. A relatively new approach to this problem for deterministic, linear time-invariant systems was recently proposed in [11]. At the heart of this approach is the identification of an invariant ellipsoid which bounds the reachable states of the system under the assumption of an amplitude-bounded exogenous input. The ellipsoid is characterized by a positive definite solution to a Riccati matrix inequality coupled with an additional linear matrix inequality (LMI) [7]. This approach was extended in [6] to the synthesis of constant gain state-feedback controllers for LPV systems.

In this paper, we generalize the results of [6] and [1] to the synthesis of parameter-dependent controllers for the bounded amplitude performance problem. The approach presented in this paper is similar in nature to the parameter-dependent controller results presented in [5]. The primary difference between this new result is the nature of the performance problem. In [5], results are presented for an $\mathcal{L}_2$ ($\mathcal{H}_\infty$) performance problem; here, we consider a bounded amplitude, or induced-$\mathcal{L}_\infty$ performance problem. As with other self-scheduling, LTI synthesis results are recovered when parameter dependence is removed (i.e. the state-space matrices are constant). The derivation presented here follows that presented in [11] for bounded amplitude LTI performance with notable exceptions. First, and most apparent, the problem considered here deals with a more general class of systems, namely, LPV systems. Because of this, the algebraic Riccati equation used in the LTI synthesis are no longer useful. The ARE must be relaxed to an appropriate Riccati inequality, much as was done in the $\mathcal{L}_2$ self-scheduling problem.

The remainder of this paper is organized as follows. Section 2 presents the mathematical notation used throughout the paper. Section 3 presents some preliminary results which characterize bounded amplitude performance of LPV systems and states the output feedback synthesis results for LTI systems. The main results concerning synthesis of parameter dependent control for LPV systems are located in section 4. Computation issues are discussed in section 5.

2 Mathematical Background

Many of the definitions for LPV systems presented in this section are generalized from [5] and the references therein.

Definition 2.1 *The infinity norm of a signal* $f : \mathbb{R}^+ \to \mathbb{R}^n$ *is defined as*

$$\|f\|_\infty := \operatorname*{ess\,sup}_{t \geq 0} \|f(t)\|$$

where $\|\cdot\|$ *denotes the Euclidean vector norm.* $\mathcal{L}_\infty^n$ *denotes the normed* n*-dimensional signal space in the usual fashion under the above signal norm definition;* $\mathcal{BL}_\infty^n$ *denotes the subset of* $\mathcal{L}_\infty^n$ *with signal norms bounded above by unity.*

Definition 2.2 (Parameter Variation Set) *Given a compact subset* $\Theta \subset \mathbb{R}^s$*, the* parameter variation set $\mathcal{F}_\Theta$ *denotes the set of all continuous functions mapping* $\mathbb{R}^+$ *into* Θ.

The notation $\theta \in \mathcal{F}_\Theta$ denotes a function in the parameter variation set; $\theta \in \Theta$ denotes a vector in a compact subset of $\mathbb{R}^s$. Note that both $\mathcal{F}_\Theta$ and, for instance, $\mathcal{L}_\infty^n$ represent signal spaces. Technically, $\mathcal{F}_\Theta \subset \mathcal{L}_\infty^s$ since $\mathcal{F}_\Theta$ consists of supremum bounded s-dimensional vectors which vary continuously in time. In this paper, $\mathcal{F}_\Theta$ will always refer to parameter signals or parameter variations; $\mathcal{L}_\infty^n$ will refer to signals in the plant input, output or state space.

Definition 2.3 (LPV System) *Given a compact set* $\Theta \subset \mathbb{R}^s$*, and continuous functions* $A : \mathbb{R}^s \to \mathbb{R}^{n \times n}$*,* $B : \mathbb{R}^s \to \mathbb{R}^{n \times n_w}$*,* $C : \mathbb{R}^s \to \mathbb{R}^{n_z \times n}$*, and* $D : \mathbb{R}^s \to \mathbb{R}^{n_z \times n_w}$*, an* n^{th} *order* linear parameter varying (LPV) system *is a dynamical system whose dynamics evolve as*

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A(\theta(t)) & B(\theta(t)) \\ C(\theta(t)) & D(\theta(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \tag{1}$$

where $\theta \in \mathcal{F}_\Theta$.

The performance measure used in this paper is the induced-$\mathcal{L}_\infty$ norm which, as defined below, is related to the peak-to-peak gain of the system.

Definition 2.4 *Let the LPV system of definition 2.3 be denoted by* $\Sigma(\Theta, A, B, C, D)$. *For any* $\theta \in \mathcal{F}_\Theta$, *the linear time-varying system described in equation 1 is denoted* Σ_θ. *The state-transition matrix of* Σ_θ *is denoted* $\Phi_\theta(t, t_0)$. *For* $x(t_0) = 0$, *the causal linear input/output mapping,* $H_\theta : \mathcal{L}_\infty^{n_w} \to \mathcal{L}_\infty^{n_z}$, *of* Σ_θ *is defined as*

$$H_\theta w(t) \quad = \quad \int_{t_0}^{t} C(\theta(\tau))\Phi_\theta(t,\tau)B(\theta(\tau))w(\tau)d\tau + D(\theta(t))w(t)$$

The induced-$\mathcal{L}_\infty$ *norm of* H_θ *is given by*

$$\|H_\theta\|_{i\infty} := \sup_{w \in \mathcal{B}\mathcal{L}_\infty^{n_w}} \|H_\theta w\|_\infty. \tag{2}$$

The shorthand $\|H_{\mathcal{F}_\Theta}\|_{i\infty} \leq \gamma$ *for some* $\gamma > 0$ *means that for all* $\theta \in \mathcal{F}_\Theta$, $\|H_\theta\|_{i\infty} \leq \gamma$ *where* $H_{\mathcal{F}_\Theta} := \{H_\theta : \theta \in \mathcal{F}_\Theta\}$.

Finally, consider an LPV plant input/output mapping, P_θ, which maps input vectors $[w' \; u']'$ to output vectors $[z' \; y']'$ and LPV feedback control mapping, K_θ, which maps y to u. The closed loop mapping from w to z will be denoted $\mathcal{F}_\ell(P_\theta, K_\theta)$.

Remark: When the parameter set Θ is a singleton, parameter variations are fixed at a single point. In this case, all of the above definitions reduce to the corresponding notions for LTI systems. $\cdot$

3 Preliminary Results

This section contains preliminary results establishing conditions for bounded amplitude performance of LPV systems and stating some previously derived results[1] for synthesis of output feedback control which maintains a bound on the induced-$\mathcal{L}_\infty$ norm of LTI systems. These results all concern strictly proper systems. Some of the notation for this section is borrowed from [1].

3.1 Bounded Amplitude Performance

Consider an LPV system, H, with state-space realization

$$H(\theta) \quad := \quad \left[\begin{array}{c|c} A(\theta) & B(\theta) \\ \hline C(\theta) & 0 \end{array}\right]. \tag{3}$$

defined for a parameter set Θ. The following result is a minor variation of the result proven in [6]; the proof is omitted.

Theorem 3.1 *Fix $\alpha > 0$ and performance level $\gamma > 0$ and consider the LPV system of (3) with $(A(\theta), B(\theta))$ controllable for all $\theta \in \Theta$. If there exists a matrix $Q > 0$ such that*

$$A(\theta)Q + QA'(\theta) + \alpha Q + \frac{1}{\alpha}B(\theta)B'(\theta) \quad \leq \quad 0 \tag{4}$$

$$C'(\theta)C(\theta) \quad < \quad \gamma^2 Q^{-1} \tag{5}$$

for all $\theta \in \Theta$, then

$$\|H_{\mathcal{F}_\Theta}\|_{i\infty} < \gamma$$

For a fixed α, let the set V_α be the set of all γ such that the matrix inequalities (4)-(5) admit a solution, i.e.

$$V_\alpha := \left\{ \gamma \;\middle|\; \exists Q > 0 \text{ such that } A(\theta)Q + QA'(\theta) + \alpha Q + \frac{1}{\alpha}B(\theta)B'(\theta) \leq 0,\ C'(\theta)C(\theta) < \gamma^2 Q^{-1}\ \forall \theta \in \Theta \right\} \tag{6}$$

V_α is the set of all upper bounds on $\|H_{\mathcal{F}_\Theta}\|_{i\infty}$ corresponding to a fixed parameter α; the smallest of these will be denoted

$$N_\alpha(H_{\mathcal{F}_\Theta}) := \begin{cases} \inf V_\alpha, & \alpha \in (0, \kappa) \\ \infty & \alpha \geq \kappa \end{cases} \tag{7}$$

where

$$\kappa := -2 \max_{\theta \in \Theta} \max_{i=1,\ldots,n} (\Re(\lambda_i(A(\theta)))),$$

is the smallest upper bound corresponding to α when V_α is nonempty. Here, $\lambda_i(A(\theta))$ denotes the i^{th} eigenvalue of $A(\theta)$ for some $\theta \in \Theta$.

Remark: $N_\alpha(H_{\mathcal{F}_\Theta}) := \infty$ for $\alpha \geq \kappa$ for the following reason. (4) can be written as

$$(A(\theta) + \frac{1}{2}\alpha I)Q + Q(A(\theta) + \frac{1}{2}\alpha I)' + \frac{1}{\alpha}B(\theta)B'(\theta) \leq 0.$$

For $\alpha \geq \kappa$, there exists a $\theta \in \Theta$ such that $A(\theta) + \frac{1}{2}\alpha I$ is no longer Hurwitz, so no $Q > 0$ can satisfy (4) for this θ.

The following corollary to theorem 3.1 is now apparent.

Corollary 3.1 *Fix $\alpha > 0$ and performance level $\gamma > 0$ and consider the LPV system of (3) with $(A(\theta), B(\theta))$ controllable for all $\theta \in \Theta$. Then for all $\theta \in \mathcal{F}_\Theta$, $N_\alpha(H_{\mathcal{F}_\Theta}) < \gamma$ if and only if there exists a matrix $Q > 0$ such that (4)-(5) admit a solution for all $\theta \in \Theta$.*

Clearly, from the corollary,

$$\|H_{\mathcal{F}_\Theta}\|_{i\infty} \leq \inf_{\alpha>0} N_\alpha(H_{\mathcal{F}_\Theta})$$

Thus, corollary 3.1 generalizes the LTI results of [11].

3.2 LTI System Results

In the case when Θ is a singleton, the LPV system may be treated as an LTI system. Output feedback synthesis results for LTI systems were previously derived in [1]. Some of these results are restated here so that they may be applied in the next section.

Consider an LTI system, P, with state space realization

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(y) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(y) \end{bmatrix} \tag{8}$$

The system in (8) satisfies the following assumptions.

(LTI1) (A, B_1, C_2) controllable and detectable. This ensures that an appropriately defined Riccati equation has a unique positive definite solution.

(LTI2) $B_1 D_{21}' = 0$. This means that process noise $B_1 w$ is entirely decoupled from measurement noise $D_{12} w$.

(LTI3) D_{21} has full row rank. This means that measurement noise can corrupt all measurements.

The primary result of this section is stated in the following theorem.

Theorem 3.2 ([1]) *Fix any number $\alpha > 0$ and performance level γ and consider the system defined in (8) under assumptions LTI1-LTI3. The following are equivalent:*

1. *There exists a strictly proper, finite-dimensional,LTI controller K which internally stabilizes the system and renders $N_\alpha(\mathcal{F}_\ell(P, K)) < \gamma$.*

2. *The LMI (in $Z = Z'$ and V)*

$$\begin{bmatrix} AZ + ZA' + B_2 V + V' B_2' + \alpha Z & B_1 \\ B_1' & -\alpha I \end{bmatrix} \leq 0 \tag{9}$$

$$\begin{bmatrix} (Z - Y_\alpha) & (Z - Y_\alpha) C_1' + V' D_{12}' \\ C_1 (Z - Y_\alpha) + D_{12} V & \gamma^2 I - C_1 Y_\alpha C_1' \end{bmatrix} > 0 \tag{10}$$

admits a solution with $Z > 0$, where Y_α is the stabilizing solution to the algebraic Riccati equation

$$(A + \frac{1}{2}\alpha I)Y_\alpha + Y_\alpha(A + \frac{1}{2}\alpha I)' - \alpha Y_\alpha C_2'(D_{21}D_{21}')^{-1}C_2 Y_\alpha + \frac{1}{\alpha}B_1 B_1' = 0 \tag{11}$$

Moreover, if either (hence, both) of these statements hold, then one controller that renders $N_\alpha(\mathcal{F}_\ell(P,K)) < \gamma$ is given by

$$K = \left[\begin{array}{c|c} A + B_2\bar{J} - \alpha Y_\alpha C_2'(D_{21}D_{21}')^{-1}C_2 & \alpha Y_\alpha C_2'(D_{21}D_{21}')^{-1} \\ \hline \bar{J} & 0 \end{array}\right] \tag{12}$$

where $\bar{J} := V(Z - Y_\alpha)^{-1}$ and Z, V and Y_α satisfy the conditions of part 2.

Theorem 3.2 may be used to synthesis strictly proper LTI controllers which enforce an amplitude constraint on the controlled plant. The approach is to perform a line search for values of $\alpha > 0$, evaluating the conditions of the theorem at each point until specified performance, γ, is achieved.

4 Strictly Proper LPV Output Feedback

The plant description is now generalized to the LPV system, $P(\theta)$, with state space realization

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(\theta) & B_1(\theta) & B_2(\theta) \\ C_1(\theta) & 0 & D_{12}(\theta) \\ C_2(\theta) & D_{21}(\theta) & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} \tag{13}$$

defined over a parameter set Θ, which satisfies the following assumptions.

(LPV1) $(A(\theta), B_1(\theta), C_2(\theta))$ controllable and detectable for all $\theta \in \Theta$.

(LPV2) $B_1(\theta)D_{21}'(\theta) = 0$ for all $\theta \in \Theta$.

(LPV3) $D_{21}(\theta)$ has full row rank for all $\theta \in \Theta$.

To generalize the previous results to self-scheduled controllers for LPV plants, the Riccati equation must be relaxed to a matrix inequality. We make the following definition.

Definition 4.1 *Suppose that $(A(\theta), R(\theta))$ is stabilizable for all $\theta \in \Theta$. Let the mapping $\mathcal{Q} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be defined by*

$$\mathcal{Q}(Y) := A(\theta)Y + YA'(\theta) + YR(\theta)Y + M(\theta). \tag{14}$$

Suppose that $R(\theta) \leq 0$ *for all* $\theta \in \Theta$. *Then a positive definite matrix* Y_{min} *satisfying*

$$\mathcal{Q}(Y_{min}) \leq 0, \ \ \forall\theta \in \Theta$$

is called minimal stabilizing solution *if*

$$Y_{min} \leq Y \quad \textit{for all} \quad Y > 0, \ \mathcal{Q}(Y) \leq 0, \ \ \forall\theta \in \Theta.$$

Remark: When Θ is a singleton, e.g. $\Theta = \theta_0$, then the minimal stabilizing solution, in the sense defined above, is the stabilizing solution to the algebraic Riccati equation

$$A(\theta_0)Y_{min} + Y_{min}A'(\theta_0) + Y_{min}R(\theta_0)Y_{min} + M(\theta_0) \quad = \quad 0 \tag{15}$$

To see this, take any positive definite Y such that $\mathcal{Q}(Y) \leq 0$ for all $\theta \in \Theta$. Then

$$\mathcal{Q}(Y) - \mathcal{Q}(Y_{min}) \leq 0$$

Equivalently,

$$\begin{gathered}(A(\theta_0) + Y_{min}R(\theta_0))(Y - Y_{\min}) + (Y - Y_{\min})(A(\theta_0) + Y_{min}R(\theta_0))' \\ \alpha(Y - Y_{\min}) + (Y - Y_{\min})R(\theta_0)(Y - Y_{\min}) \leq 0\end{gathered}$$

for all $\theta \in \Theta$. This implies the existence of a matrix $\hat{M}(\theta_0)$ so that

$$\begin{gathered}(A(\theta_0) + Y_{min}R(\theta_0))(Y - Y_{\min}) + (Y - Y_{\min})(A(\theta_0) + Y_{min}R(\theta_0))' \\ \alpha(Y - Y_{\min}) + (Y - Y_{\min})R(\theta_0)(Y - Y_{\min}) + \hat{M}(\theta_0) = 0\end{gathered} \tag{16}$$

Since Y_{min} is a stabilizing solution to the ARE in (15), $A(\theta_0) + Y_{min}R(\theta_0)$ is stable. This implies that $Y - Y_{\min} \geq 0$ (see, e.g. [17]), or $Y \geq Y_{min}$.

We are now ready to state the main result of the paper.

Theorem 4.1 *Fix any number* $\alpha > 0$ *and performance level* γ *and consider the LPV plant in (13) under assumptions LPV1-LPV3. Then a sufficient condition for the existence of a strictly proper, finite-dimensional, LPV controller* $K(\theta)$ *which internally stabilizes the system and for all* $\theta \in \mathcal{F}_\Theta$, *renders* $N_\alpha(\mathcal{F}_\ell(P(\theta), K(\theta))) < \gamma$ *is the existence of matrices* $Z = Z' > 0$ *and* V *such that*

$$\begin{bmatrix} A(\theta)(Z - Y_\alpha) + (Z - Y_\alpha)A'(\theta) + B_2(\theta)V + V'B_2'(\theta) + \alpha(Z - Y_\alpha) & Y_\alpha C_2'(\theta) \\ C_2(\theta)Y_\alpha & -\frac{1}{\alpha}D_{21}(\theta)D_{21}'(\theta) \end{bmatrix} \leq 0 \tag{17}$$

$$\begin{bmatrix} (Z-Y_\alpha) & (Z-Y_\alpha)C_1'(\theta)+V'D_{12}'(\theta) \\ C_1(\theta)(Z-Y_\alpha)+D_{12}(\theta)V & \gamma^2 I - C_1(\theta)Y_\alpha C_1'(\theta) \end{bmatrix} > 0 \quad (18)$$

for all $\theta \in \Theta$ where Y_α is the minimal stabilizing solution (in the sense of definition 4.1) of the Riccati inequality

$$(A(\theta)+\frac{1}{2}\alpha I)Y_\alpha + Y_\alpha(A(\theta)+\frac{1}{2}\alpha I)' - \alpha Y_\alpha C_2'(\theta)(D_{21}(\theta)D_{21}'(\theta))^{-1}C_2(\theta)Y_\alpha + \frac{1}{\alpha}B_1(\theta)B_1'(\theta) \le 0 \qquad (19)$$

In addition, if the above matrix inequalities admit a solution, then one controller that renders $N_\alpha(\mathcal{F}_\ell(P(\theta),K(\theta))) < \gamma$ for all $\theta \in \mathcal{F}_\Theta$ is given by

$$K = \left[\begin{array}{c|c} A(\theta)+B_2(\theta)\bar{J} - \alpha Y_\alpha C_2'(\theta)(D_{21}(\theta)D_{21}(\theta)')^{-1}C_2(\theta) & \alpha Y_\alpha C_2'(\theta)(D_{21}(\theta)D_{21}(\theta)')^{-1} \\ \hline \bar{J} & 0 \end{array}\right] \qquad (20)$$

where $\bar{J} := V(Z-Y_\alpha)^{-1}$ and Z, V and Y_α satisfy the conditions of part 2.

Furthermore, if Θ is a singleton, then the above conditions are strengthened to necessary and sufficient.

Proof of Theorem 4.1: Suppose that the LMIs of equation 17 and 18 in variables Z and V admits a solution with $Z > 0$, and suppose the loop is closed with the realization given in equation 20. Let x and $\hat{x}$ denote the state of the plant and controller, respectively. Define $W(\theta) := (D_{21}(\theta)D_{21}'(\theta))^{-1}$; by assumption LPV3, $W(\theta)$ is well-defined for all $\theta \in \Theta$. The closed loop system with states $[x' \;\; x'-\hat{x}']'$ has realization

$$\mathcal{F}_\ell(P(\theta),K(\theta)) = \left[\begin{array}{c|c} \bar{A}(\theta) & \bar{B}(\theta) \\ \hline \bar{C}(\theta) & 0 \end{array}\right]$$

$$:= \left[\begin{array}{cc|c} A(\theta)+B_2(\theta)\bar{J} & -B_2(\theta)\bar{J} & B_1(\theta) \\ 0 & A(\theta)-\alpha Y_\alpha C_2'(\theta)W(\theta)C_2(\theta) & B_1(\theta)-\alpha Y_\alpha C_2'(\theta)W(\theta)D_{21}(\theta) \\ \hline C_1(\theta)+D_{12}(\theta)\bar{J} & -D_{12}(\theta)\bar{J} & 0 \end{array}\right]$$

By corollary 3.1, $N_\alpha(\mathcal{F}_\ell(P(\theta),K(\theta))) < \gamma$ for all $\theta \in \mathcal{F}_\Theta$ if and only if

$$\begin{aligned} \bar{A}(\theta)Q + Q\bar{A}'(\theta) + \alpha Q + \frac{1}{\alpha}\bar{B}(\theta)\bar{B}'(\theta) &\le 0 \\ \bar{C}'(\theta)\bar{C}(\theta) &< \gamma^2 Q^{-1} \end{aligned}$$

Now, suppose that a matrix Q is set to

$$Q = \begin{bmatrix} Z & Y_\alpha \\ Y_\alpha & Y_\alpha \end{bmatrix},$$

Since $Z > 0$ and $Y_\alpha > 0$, $Q > 0$ if and only if $Z > Y_\alpha$, which must be true since $Z - Y_\alpha$ is a principle minor of the matrix in (18).

Now, consider

$$E(\theta) := \begin{bmatrix} E_{11}(\theta) & E_{12}(\theta) \\ E'_{12}(\theta) & E_{22}(\theta) \end{bmatrix} = \bar{A}(\theta)Q + Q\bar{A}'(\theta) + \alpha Q + \frac{1}{\alpha}\bar{B}(\theta)\bar{B}'(\theta)$$

Performing the algebra, the terms of the matrix simplify to

$$\begin{aligned} E_{11}(\theta) &= A(\theta)Z + ZA'(\theta) + B_2(\theta)V + V'B_2'(\theta) + \alpha Z + \frac{1}{\alpha}B_1(\theta)B_1'(\theta) &(21)\\ E_{12}(\theta) &= A(\theta)Y_\alpha + Y_\alpha A'(\theta) - \alpha Y_\alpha C_2'(\theta)W(\theta)C_2(\theta)Y_\alpha + \alpha Y_\alpha + \frac{1}{\alpha}B_1(\theta)B_1'(\theta) \\ &\quad -B_1(\theta)D_{21}(\theta)W(\theta)C_2(\theta)Y_\alpha &(22)\\ E_{22}(\theta) &= A(\theta)Y_\alpha - \alpha Y_\alpha C_2'(\theta)W(\theta)C_2(\theta)Y_\alpha + Y_\alpha A'(\theta) + \alpha Y_\alpha \\ &\quad +\frac{1}{\alpha}[B_1(\theta)B_1'(\theta) - \alpha B_1(\theta)D_{21}'(\theta)W(\theta)C_2(\theta)Y_\alpha - \alpha Y_\alpha C_2'(\theta)W(\theta)D_{21}(\theta)B_1'(\theta)] &(23)\end{aligned}$$

Under the assumption $B_1(\theta)D'_{12}(\theta) = 0$, $E_{12}(\theta) = E_{22}(\theta)$, so $E(\theta) \leq 0$ for all $\theta \in \Theta$ if $E_{11}(\theta) \leq E_{22}(\theta) \leq 0$.

That $E_{22}(\theta) \leq 0$ for all $\theta \in \Theta$ is immediately evident from (19). From (17), for all $\theta \in \Theta$,

$$A(\theta)(Z - Y_\alpha) + (Z - Y_\alpha)A'(\theta) + B_2(\theta)V + V'B_2'(\theta) + \alpha(Z - Y_\alpha) + \alpha Y_\alpha C_2'(\theta)W(\theta)C_2(\theta)Y_\alpha \leq 0$$

which is equivalent to

$$A(\theta)Z + ZA'(\theta) + B_2(\theta)V + V'B_2'(\theta) + \alpha Z \leq A(\theta)Y_\alpha + Y_\alpha A'(\theta) + \alpha Y_\alpha - \alpha Y_\alpha C_2'(\theta)W(\theta)C_2(\theta)Y_\alpha$$

and

$$\begin{aligned} & A(\theta)Z + ZA'(\theta) + B_2(\theta)V + V'B_2'(\theta) + \alpha Z + \frac{1}{\alpha}B_1(\theta)B_1(\theta)' \\ \leq\ & A(\theta)Y_\alpha + Y_\alpha A'(\theta) + \alpha Y_\alpha - \alpha Y_\alpha C_2'(\theta)W(\theta)C_2(\theta)Y_\alpha + \frac{1}{\alpha}B_1(\theta)B_1(\theta)' &(24)\end{aligned}$$

This expression, in turn, simplifies to

$$E_{11}(\theta) \leq E_{22}(\theta) \ \ \forall\theta \in \Theta$$

thus implying that $E(\theta) \leq 0$, or

$$\bar{A}(\theta)Q + Q\bar{A}'(\theta) + \alpha Q + \frac{1}{\alpha}\bar{B}(\theta)\bar{B}'(\theta) \leq 0 \ \ \forall\theta \in \Theta \tag{25}$$

Now to demonstrate that $\gamma^2 Q^{-1} > \bar{C}'(\theta)\bar{C}(\theta)$. Through an elementary Schur complement argument, it is easy to show that

$$\gamma^2 Q^{-1} > \bar{C}'(\theta)\bar{C}(\theta) \Leftrightarrow \begin{bmatrix} Q & Q\bar{C}(\theta) \\ \bar{C}'(\theta)Q & \gamma^2 I \end{bmatrix} > 0$$

Substituting for Q and $\bar{C}(\theta)$, and multiplying on both sides by $blockdiag(I, Y_\alpha^{-1}, I)$, this expression is equivalent to

$$\begin{bmatrix} Z & I & ZC_1'(\theta) + V'D_{12}'(\theta) \\ I & Y_\alpha^{-1} & C_1'(\theta) \\ C_1(\theta)Z + D_{12}(\theta)V & C_1(\theta) & \gamma^2 I \end{bmatrix} > 0$$

$$\Leftrightarrow \begin{bmatrix} Z & ZC_1'(\theta) + V'D_{12}'(\theta) & I \\ C_1(\theta)Z + D_{12}(\theta)V & \gamma^2 I & C_1'(\theta) \\ I & C_1(\theta) & Y_\alpha^{-1} \end{bmatrix} > 0 \quad \forall\theta \in \Theta$$

Since $Y_\alpha > 0$, this last expression is equivalent, via Schur complement arguments, to

$$\begin{bmatrix} (Z - Y_\alpha) & (Z - Y_\alpha)C_1'(\theta) + V'D_{12}'(\theta) \\ C_1(\theta)(Z - Y_\alpha) + D_{12}(\theta)V & \gamma^2 I - C_1(\theta)Y_\alpha C_1'(\theta) \end{bmatrix} > 0$$

which is just (18).

To show that the equation is necessary and sufficient when θ is a singleton, θ_0, note that Y_α is now the solution to the ARE

$$(A(\theta_0) + \frac{1}{2}\alpha I)Y_\alpha + Y_\alpha(A(\theta_0) + \frac{1}{2}\alpha I)' - \alpha Y_\alpha C_2'(\theta)W(\theta_0)C_2(\theta_0)Y_\alpha + \frac{1}{\alpha}B_1(\theta_0)B_1'(\theta_0) = 0$$

Then, the matrix inequality in (17), which is equivalent to (24), is now equivalent to

$$A(\theta_0)Z + ZA'(\theta_0) + B_2(\theta_0)V + V'B_2'(\theta_0) + \alpha Z + \frac{1}{\alpha}B_1(\theta_0)B_1(\theta_0)' \leq 0$$

which, by Schur complements, is equivalent to

$$\begin{bmatrix} A(\theta_0)Z + ZA'(\theta_0) + B_2(\theta_0)V + V'B_2'(\theta_0) + \alpha Z & B_1(\theta_0) \\ B_1'(\theta_0) & -\alpha I \end{bmatrix} \leq 0$$

By theorem 3.2, (17) and (18) are necessary and sufficient conditions. $\square$

Remark: The restriction that Y_α in (19) be a minimal stabilizing solution is overly restrictive for the existence of a self-scheduled controller. However, if this restriction is removed, the results do not reduce to necessary and sufficient conditions in the LTI case.

5 Implementation

Theorem 4.1 represents the key analytical device for the synthesis of LPV controllers satisfying prespecified amplitude performance constraints. The approach is identical in nature to that proposed earlier for LTI systems [1] and for LTI controllers for LPV systems [6]: perform a line search for values of $\alpha > 0$, evaluating (at each α) the conditions of theorem 4.1 to determine if a controller exists satisfying performance constraint, γ. When values of α are found such that the matrix inequality conditions (17)-(18) admit a solution, then the LPV controller with realization given in (20) will achieve closed loop performance goals.

The matrix inequality conditions (17)-(18) of the theorem must be satisfied pointwise for all $\theta \in \Theta$. There are several options to tackle this problem which have been proposed in the literature for similar matrix inequality conditions. When Θ is a polytope, then it is sufficient to evaluate the matrix inequalities simultaneously at the vertices of the polytope. The resulting set of *linear* matrix inequalities can typically be solved quickly when the dimension of the parameter space is small enough. this approach is illustrated in [3]. If, in addition, the parameter dependence in the system matrix coefficients is linear fractional, then the S-procedure[16] may also be applied as in [6]; however, this approach is known to give conservative results[4].

When the parameter set is not polytopic, different approaches must be taken. Perhaps the most obvious approach is to form a grid over the parameter set Θ and simultaneously solve a set of *linear* matrix inequalities, refining the grid until it is apparent that the solution will be valid for points not on the grid. This is the approach taken in [10]. Another approach is to attempt to satisfy the conditions for a polytopic set which contains Θ. The problem here is that the results are potentially more conservative since they must account for parameter variations which will not occur; worse, controllability or observability may be lost at some points in the larger parameter set.

In summary, there are a variety of methods to handle the parameter variations of the matrix inequality conditions (17)-(18). The most attractive require simple parameter set geometry and linear fractional parameter dependence. In other cases, more *ad hoc* approaches must be adopted.

6 Conclusions

This paper has presented sufficient conditions for the synthesis of parameter-dependent output feedback controllers which guarantee an upper bound on the induced-$\mathcal{L}_\infty$ norm of the controlled system. These conditions reduce to necessary and sufficient conditions when the parameter variation is removed, i.e. when plant is LTI. The condition takes the form of linear matrix inequalities (LMIs) which must be solved pointwise over a parameter set. A set of LPV controllers which satisfy the performance constraints are parameterized in terms of feasible solutions to the LMI existence conditions.

The results extend the class of performance problems which may be treated with self-scheduled control techniques; current literature appears to contain results for the $\mathcal{L}_2$ performance problem only. Furthermore, the results presented here provide a means of bounded amplitude control of class of continuous-time systems more general than previously considered, without the problem of controller dimensionality explosion. They are intuitively satisfying because they are similar in flavor to the $\mathcal{H}_\infty$ results of [5]; parameter-dependent controller synthesis requires the feasibility of a 'controller' Riccati inequality (17), an 'observer' Riccati inequality (19) and a 'spectral radius' coupling inequality (18). Thus, the approach proposed here for bounded amplitude control is no more difficult than similar approaches, e.g. $\mathcal{L}_2$ control, though they may be more computationally intensive. However, note that the results presented here do not rely on the loop-shifting arguments used in, e.g. [12],[2], so that well-posedness issues in the controller implementation are not an issue on the results presented here.

Finally, it is apparent that the conditions presented here might be strengthened to necessary and sufficient if the Riccati matrices Y_{min}, Z and V of the theorem are allowed to be parameter dependent. In that case, specific assumptions must be imposed in the rate of parameter variation, as in the $\mathcal{L}_2$ case.

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