On Linear Fractional Representations of Multidimensional Rational Matrix Functions

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Abstract

This report provides a practical and tutorial proof for the existence of linear fractional representations of multidimensional rational matrix mappings. The existence result is used extensively in the μ-analysis/synthesis and linear parameter varying system paradigms which assume that the description of linear systems subject to parametric uncertainties and variations can be expressed in a linear fractional manner. Also briefly addressed is the question of minimal linear fractional realizations of the uncertain systems; the report provides a brief overview of current research in that direction.

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1 Introduction

In the past decade, significant advances have been made in the design and analysis of linear (control) systems which are subject to *structural* perturbations. Arguably the most notable of these techniques are the μ-analysis/synthesis (see, e.g., [9][11]) and linear parameter varying (LPV) system (e.g., [18] [1]) paradigms. These design and analysis techniques often require that parametric uncertainty and (possibly known) parametric variations be represented in linear fractional form, i.e.

$$\mathcal{F}_u(M, \Delta) = M_{22} + M_{21}\Delta(I - M_{11})^{-1}M_{12} \tag{1}$$

where

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

represents the nominal system and the structure of the perturbations is denoted by the matrix, Δ. Copies of the uncertain or time-varying parameters, $\delta_1, \delta_2, \ldots, \delta_s \in \mathbb{R}$ lie on the diagonal entries of Δ,

$$\Delta(\delta) = \text{diag}(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_s I_{r_s})$$

where I_k denotes the $k \times k$ identity matrix. The dimension $r_1, r_2, \ldots, r_s$ depend on structure of the parametric uncertainty or variations. The quantity

$$r = \sum_{i=1}^{s} r_i$$

defines the *order* of the representation given in equation 1. The problem of finding M and Δ to model the uncertainty or parametric variation is equivalent to a multidimensional realization problem [5].

In the formulation of M and Δ, the first and foremost is the question of existence: *Does a linear fractional representation of the structurally perturbed plant exist?* For a large class of practical problems, where the perturbed system matrices are multidimensional rational matrix mappings, the answer is affirmative. This fact is often stated in the literature, however, it is done so either with no reference or with reference to a work not part of the general body of literature [17][1]. This is interesting since much of the current research on, for example, μ-analysis/synthesis and LPV system techniques, rely on the existence of a linear fractional uncertainty model as described above at the beginning of the problem formulation.

The objective of this report is to provide a practical and tutorial proof for the existence of linear fractional representations of multidimensional rational matrix mappings. The results presented here are not new and

[1] This cite was found in [21]

the proofs provided would not be considered innovative since they are based on simple algebraic techniques. This work was performed in early stages of research on LPV systems when it was not known that the existence proofs had appeared in either conference proceeding or journal articles. Since that time, it has come to light that the results have, in fact, appeared in similar form in [19] and [20].

This report will sparingly address the question of linear fractional representations which naturally follows the existence question, namely: *If a linear fractional representation exists, is it unique and if not, what is the best representation?* Actually, the question of uniqueness is easily answered with the results of this report: the LFT representation of a given multidimensional rational matrix mapping is not unique. Finding the 'best' representation, however, is a difficult question and a current research issue (see, e.g., [3][4][8][7][6][10][14][16][19][20][21]. While the question of a best representation will not be discussed in any detail, references to current research are provided for the interested reader. It is intended that this report serve as a brief tutorial in that respect.

The remainder of this paper is organized as follows. Section 2 defines a linear fractional transformation (LFT) and states some useful properties. Section 3 states results pertaining to linear fractional representations of scalar multidimensional rational functions; section 4 builds upon the scalar results to derive the general results for multidimensional rational matrix mappings. The problem of obtaining minimal realizations is discussed in section 5.

2 Preliminaries

This section presents some key results from the theory of linear fractional transformations (LFTs) and defines the notion of multidimensional rational functions used throughout the paper.

2.1 Properties of linear fractional transformations

Definition 2.1 (Linear Fractional Transformation [21]) *Let M be a complex matrix partitioned as*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{C}^{(p_1+p_2)\times(q_1+q_2)},$$

and let $\Delta_\ell \in \mathbb{C}^{q_2\times p_2}$ and $\Delta_u \in \mathbb{C}^{q_1\times p_1}$ be two other complex matrices. M is called the coefficient matrix.

The lower linear fractional transformation with respect to Δ_ℓ is the map

$$\mathcal{F}_\ell(M,\cdot) : \mathbb{C}^{q_2\times p_2} \to \mathbb{C}^{p_1\times q_1}$$

with

$$\mathcal{F}_\ell(M,\Delta_\ell) := M_{11} + M_{12}\Delta_\ell(I - M_{22}\Delta_\ell)^{-1}M_{21}$$

provided $(I - M_{22}\Delta_\ell)^{-1}$ *exists. When the inverse exists, the LFT is called* *well-defined* *or* *well-posed.*

The *upper linear fractional transformation* *with respect to* Δ_u *is the map*

$$\mathcal{F}_u(M,\cdot) : \mathbb{C}^{q_1 \times p_1} \to \mathbb{C}^{p_2 \times q_2}$$

with

$$\mathcal{F}_u(M,\Delta_u) := M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}$$

provided $(I - M_{11}\Delta_u)^{-1}$ *exists.*

The following theorem states some useful properties of LFTs. The proof is a straightforward (though somewhat tedious) application of definition 2.1 and matrix algebra. It is included for completeness, but relegated to the appendix.

Theorem 2.1 ([21]) *Let* M*,* Q *and* G *be suitably partitioned matrices:*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, G = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}$$

1. $\mathcal{F}_u(M,\Delta) = \mathcal{F}_\ell(N,\Delta)$ *with*

$$N = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} M \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} M_{22} & M_{21} \\ M_{12} & M_{11} \end{bmatrix}$$

   where the dimensions of the identity matrices are compatible with the dimensions of M *and* N*.*

2. *Let* T *be a nonsingular matrix with dimensions compatible with* Δ*. Then*

$$\mathcal{F}_u(M, T\Delta T^{-1}) = \mathcal{F}_u(N,\Delta)$$

   with

$$N = \begin{bmatrix} T^{-1}M_{11}T & T^{-1}M_{12} \\ M_{21}T & M_{22} \end{bmatrix}.$$

3. *Suppose* $\mathcal{F}_u(M,\Delta)$ *is square and well-defined and* M_{22} *is nonsingular. Then the inverse of* $\mathcal{F}_u(M,\Delta)$ *exists and is also an LFT with respect to* Δ*:*

$$(\mathcal{F}_u(M,\Delta))^{-1} = \mathcal{F}_u(N,\Delta)$$

with N given by

$$N = \begin{bmatrix} M_{11} - M_{12}M_{22}^{-1}M_{21} & -M_{12}M_{22}^{-1} \\ M_{22}^{-1}M_{21} & M_{22}^{-1} \end{bmatrix}.$$

4. $\mathcal{F}_u(M, \Delta_1) + \mathcal{F}_u(Q, \Delta_2) = \mathcal{F}_u(N, \Delta)$ *with*

$$N = \left[\begin{array}{cc|c} M_{11} & 0 & M_{12} \\ 0 & Q_{11} & Q_{12} \\ \hline M_{21} & Q_{21} & M_{22} + Q_{22} \end{array}\right], \qquad \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}.$$

5. $\mathcal{F}_u(M, \Delta_1)\mathcal{F}_u(Q, \Delta_2) = \mathcal{F}_u(N, \Delta)$ *with*

$$N = \left[\begin{array}{cc|c} M_{11} & M_{12}Q_{21} & M_{12}Q_{22} \\ 0 & M_{11} & Q_{12} \\ \hline M_{21} & M_{22}Q_{21} & M_{22}Q_{22} \end{array}\right], \qquad \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}.$$

6. *Suppose*

$$\begin{bmatrix} z \\ z_l \end{bmatrix} = \mathcal{F}_u(G, \Delta_1) \begin{bmatrix} w \\ w_l \end{bmatrix}, \qquad x = \mathcal{F}_u(Q, \Delta_2)y$$

where the dimensions of Δ_1 are compatible with the dimensions of A. Then the closed loop mapping from w to z under feedback connection $w_l = x$ and $z_l = y$ is given by

$$\mathcal{F}_\ell(\mathcal{F}_u(G, \Delta_1), \mathcal{F}_u(Q, \Delta_2)) = \mathcal{F}_u(\mathcal{F}_\ell(G, \mathcal{F}_u(Q, \Delta_2)), \Delta_1) = \mathcal{F}_u(N, \Delta)$$

$$N = \left[\begin{array}{cc|c} A + B_2L_2Q_{22}C_2 & B_2L_2Q_{21} & B_1 + B_2L_2Q_{22}D_{21} \\ Q_{12}L_1C_2 & Q_{11} + Q_{12}D_{22}L_2Q_{21} & Q_{12}L_1D_{21} \\ \hline C_1 + D_{12}L_2Q_{22}C_2 & D_{12}L_2Q_{21} & D_{11} + D_{12}L_2Q_{22}D_{21} \end{array}\right]$$

where $L_1 := (I - D_{22}Q_{22})^{-1}$, $L_2 := (I - Q_{22}D_{22})^{-1}$ and $\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$.

The following definitions of multidimensional rational matrix functions depend the following definition of a polynomial function. This definition is not entirely formal; rather, it is entirely practical.

Definition 2.2 *Let $E \subseteq \mathbb{R}$ and take n to be a nonnegative integer. A polynomial function $p : E \to \mathbb{R}$ is a function defined by*

$$p(\delta) := a_0 + a_1\delta + a_2\delta^2 + \cdots + a_n\delta^n, \qquad \delta \in E$$

for real numbers $a_0, a_1, \ldots, a_n$. A rational function $f : E \to \mathbb{R}$ is a function defined by

$$f(\delta) := \frac{p(\delta)}{q(\delta)}, \qquad \delta \in E$$

where $p : E \to \mathbb{R}$ and $q : E \to \mathbb{R}$ are polynomial functions with $q(\delta) \neq 0$ for $\delta \in E$.

Let $E \subseteq \mathbb{R}^s$ and take n and $r_{j,i}$ (for $j = 1, \ldots, s$ and $i = 1, \ldots, n$) to be a nonnegative integers. A s-dimensional polynomial function $p_s : E \to \mathbb{R}$ is a function defined by

$$p_s(\delta_1, \delta_2, \ldots, \delta_s) := a_0 + \sum_{i=1}^{n} a_i \delta_1^{r_{1,i}} \delta_2^{r_{2,i}} \cdots \delta_s^{r_{s,i}} \qquad (\delta_1, \delta_2, \ldots, \delta_s) \in E$$

for real numbers $a_0, a_1, \ldots, a_n$. A s-dimensional rational function $f_s : E \to \mathbb{R}$ is a function defined by

$$f_s(\delta_1, \delta_2, \ldots, \delta_s) := \frac{p_s(\delta_1, \delta_2, \ldots, \delta_s)}{q_s(\delta_1, \delta_2, \ldots, \delta_s)}, \qquad (\delta_1, \delta_2, \ldots, \delta_s) \in E$$

where $p_s : E \to \mathbb{R}$ and $q_s : E \to \mathbb{R}$ are s-dimensional polynomial functions with

$$q_s(\delta_1, \delta_2, \ldots, \delta_s) \neq 0$$

for $(\delta_1, \delta_2, \ldots, \delta_s) \in E$.

A s-dimensional rational matrix mapping is a matrix mapping whose elements are s-dimensional rational functions.

When it is understood, a s-dimensional polynomial will be abbreviated

$$f_s(\delta) = f_s(\delta_1, \delta_2, \ldots, \delta_s)$$

Finally, a *linear fractional representation* of any of the entities defined above will be denoted LFR.

3 Scalar multidimensional polynomials

In this section, the existence question is answered for scalar multidimensional polynomials. The primary results of this section are lemmas 3.1 and 3.2 which serve as building blocks for rewriting the multidimensional rational matrix mappings in the next section.

The following lemma states that any polynomial function has an LFR.

Lemma 3.1 *Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial defined by*

$$p(\delta) = a_0 + a_1\delta + a_2\delta^2 + \cdots + a_n\delta^n$$

for some nonnegative integer n. Then

$$p(\delta) = \mathcal{F}_u(M, \Delta)$$

with

$$M = \left[\begin{array}{cccc|c} 0 & \cdots & 0 & 0 & 1 \\ 1 & \cdots & 0 & 0 & 0 \\ 0 & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ \hline a_1 & \cdots & a_{n-1} & a_n & a_0 \end{array}\right], \qquad \textit{and} \quad \Delta = \delta I \tag{2}$$

Proof: Let M and Δ be as defined in equation 2. Then

$$\begin{aligned}
\mathcal{F}_u(M, \Delta) &= a_0 + \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \delta I \left[\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \delta & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \delta & 0 \end{bmatrix} \right]^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= a_0 + \delta \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \delta & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{n-1} & \delta^{n-2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= a_0 + \delta \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} 1 \\ \delta \\ \vdots \\ \delta^{n-1} \end{bmatrix} \\
&= a_0 + a_1\delta + a_2\delta^2 + \cdots + a_n\delta^n \\
&= p(\delta)
\end{aligned}$$

□

Remark: Note that if $\delta := \frac{1}{s}$, $\mathcal{F}_u(M, \Delta)$ as defined in the lemma evaluates to

$$\mathcal{F}_u(M, \Delta) = \frac{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}{s^n}. \tag{3}$$

Thus, the LFR of the polynomial function as specified in the lemma is just the controllability canonical state-space realization (see, e.g., [13]) of the associated transfer function in equation 3. This fact underscores the previously stated fact that LFRs are non-unique since any transfer function has an infinite number of realizations. It also provides an intuitive notion of minimality of an LFR, that is, a minimal LFR is the one with the smallest dimension. Finally, note that because polynomial functions have this analogy to linear time-invariant systems, computing linear fractional representations of polynomial functions has a straightforward computer implementation.

The next lemma generalizes the previous result to s-dimensional polynomial functions.

Lemma 3.2 *Let $p : \mathbb{R}^s \to \mathbb{R}$ be a s-dimensional polynomial defined by*

$$p(\delta_1, \delta_2, \ldots, \delta_s) = \delta_1^{r_1} \delta_2^{r_2} \cdots \delta_s^{r_s}$$

for nonnegative integers r_j, $j = 1, \ldots, s$. Then there is a real matrix $M \in \mathbb{R}^{(r+1)\times(r+1)}$ with $r = \sum_{j=1}^{s} r_j$ such that

$$p(\delta_1, \delta_2, \ldots, \delta_s) = \mathcal{F}_u(M, \Delta)$$

with $\Delta = diag(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_s I_{r_s})$.

Proof: By lemma 3.1, for each $j = 1, \ldots, s$, there is an $M_j \in \mathbb{R}^{(r_j+1)\times(r_j+1)}$ such that

$$\delta_j^{r_j} = \mathcal{F}_u(M_j, \delta_j I_{r_j})$$

so that

$$p(\delta_1, \delta_2, \ldots, \delta_s) = \mathcal{F}_u(M_1, \delta_1 I_{r_1}) \mathcal{F}_u(M_2, \delta_2 I_{r_2}) \cdots \mathcal{F}_u(M_s, \delta_s I_{r_s}) \tag{4}$$

By repeated direct application of theorem 2.1, equation 4 a matrix $M \in \mathbb{R}^{(r+1)\times(r+1)}$ with $r = \sum_{j=1}^{s} r_j$ may be formed from M_j, $j = 1, \ldots, s$, such that

$$\mathcal{F}_u(M_1, \delta_1 I_{r_1}) \mathcal{F}_u(M_2, \delta_2 I_{r_2}) \cdots \mathcal{F}_u(M_s, \delta_s I_{r_s}) = \mathcal{F}_u(M, \Delta) \tag{5}$$

with $\Delta = diag(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_s I_{r_s})$. Combining equations 4 and 5 completes the proof. $\square$

The proof of lemma 3.2 suggests that computing the LFR of the s-dimensional polynomial function can be accomplished by writing the function as the product of LFRs of 1-dimensional polynomial functions, which may then be combined using the properties of LFTs from section 2. For the s-dimensional polynomial defined in the lemma, this results in a minimal LFR. However, for more general s-dimensional polynomials such a procedure will result in LFR with unnecessarily high order. Developing algorithms for finding linear fractional representations of minimal or small order is still an open research area. This problem is further discussed in section 5.

4 Multidimensional rational matrix mappings

This section uses the scalar results of section 3 to establish the more general and useful results on linear fractional representations of s-dimensional rational matrix mappings.

The following lemma states that any s-dimensional polynomial matrix mapping has a linear fractional representation.

Lemma 4.1 *Let $F : \mathbb{R}^s \to \mathbb{R}^{m\times q}$ be a s-dimensional polynomial matrix mapping defined by*

$$F(\delta_1, \delta_2, \ldots, \delta_s) = A_0 + \sum_{i=1}^{n} A_i \delta_1^{r_{1,i}} \delta_2^{r_{2,i}} \cdots \delta_s^{r_{s,i}}$$

for a positive integer n, constant matrices $A_i \in \mathbb{R}^{m\times q}$, $i = 0, 1, \ldots, n$, and nonnegative integers $r_{j,i}$ (for $j = 1, \ldots, s$ and $i = 1, \ldots, n$). Then there is a real matrix M and nonnegative integers $n_1, n_2, \ldots, n_s$ such that

$$F(\delta_1, \delta_2, \ldots, \delta_s) = \mathcal{F}_u(M, \Delta)$$

with $\Delta = diag(\delta_1 I_{n_1}, \delta_2 I_{n_2}, \ldots, \delta_s I_{n_s})$.

Proof: By lemma 3.2, for each $i = 1, \ldots, n$,

$$A_i \delta_1^{r_{1,i}} \delta_2^{r_{2,i}} \cdots \delta_s^{r_{s,i}} = A_i \mathcal{F}_u(M_i, \Delta_i)$$

where $\Delta_i = diag(\delta_1 I_{r_{1,i}}, \delta_2 I_{r_{2,i}}, \ldots, \delta_s I_{r_{s,i}})$ and M_i are appropriately dimensioned real matrices which are partitioned according to

$$M_i = \left[\begin{array}{c|c} M_{11,i} & M_{12,i} \\ \hline M_{21,i} & M_{22,i} \end{array}\right]$$

yielding

$$F(\delta_1, \delta_2, \ldots, \delta_s) = A_0 + \sum_{i=1}^{n} A_i \mathcal{F}_u(M_i, \Delta_i) \tag{6}$$

For each $i = 1, \ldots, n$, $A_i \mathcal{F}_u(M_i, \Delta_i)$ is the LFT $\mathcal{F}_u(N_i, \Delta_i)$ with

$$N_i = \left[\begin{array}{c|c} N_{11,i} & N_{12,i} \\ \hline N_{21,i} & N_{22,i} \end{array}\right] = \left[\begin{array}{c|c} M_{11,i} & M_{12,i} \\ \hline A_i M_{21,i} & A_i M_{22,i} \end{array}\right]$$

so that $\sum_{i=1}^{n} A_i \mathcal{F}_u(M_i, \Delta_i) = \sum_{i=1}^{n} \mathcal{F}_u(N_i, \Delta_i)$. By repeated application of of theorem 2.1, $\sum_{i=1}^{n} \mathcal{F}_u(N_i, \Delta_i)$

may be written as $\mathcal{F}_u(N,\tilde{\Delta})$ with

$$N = \left[\begin{array}{c|c} N_{11} & N_{12} \\ \hline N_{21} & N_{22} \end{array}\right] = \left[\begin{array}{cccc|c} M_{11,1} & 0 & \cdots & 0 & M_{12,1} \\ 0 & M_{11,2} & \cdots & 0 & M_{12,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M_{11,n} & M_{12,n} \\ \hline A_1 M_{21,1} & A_2 M_{21,2} & \cdots & A_n M_{21,n} & \sum_{i=1}^{n} A_i M_{22,i} \end{array}\right] \tag{7}$$

and

$$\tilde{\Delta} = diag(\Delta_1, \Delta_2, \ldots, \Delta_n). \tag{8}$$

Equation 6 then becomes

$$F(\delta_1, \delta_2, \ldots, \delta_s) = A_0 + \mathcal{F}_u(N, \tilde{\Delta}) \tag{9}$$

Since Δ_i is diagonal for each $i = 1, \ldots, n$, $\tilde{\Delta}$ is also diagonal with $n_j = \sum_{i=1}^{n} r_{j,i}$ diagonal elements equal to δ_j for each $j = 1, \ldots, s$. Thus, $\tilde{\Delta}$ is related to

$$\Delta = diag(\delta_1 I_{n_1}, \delta_2 I_{n_2}, \ldots, \delta_s I_{n_s})$$

by a similarity transformation given by a real matrix T so that $\tilde{\Delta} = T \Delta T^{-1}$. Using this fact and theorem 2.1,

$$\mathcal{F}_u(N, \tilde{\Delta}) = \mathcal{F}_u(\tilde{M}, \Delta)$$

with

$$\tilde{M} = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & \tilde{M}_{22} \end{array}\right] = \left[\begin{array}{c|c} T^{-1} N_{11} T & T^{-1} N_{12} \\ \hline N_{21} T & N_{22} \end{array}\right],$$

and defining $M_{22} = \tilde{M}_{22} + A_0$ and

$$M = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right]$$

one obtains

$$A_0 + \mathcal{F}_u(N, \tilde{\Delta}) = \mathcal{F}_u(M, \Delta) \tag{10}$$

with M and Δ as defined above. Combining equations 9 and 10 yields the desired result. $\square$

This result was also proven in [19]. In a manner similar to the scalar s-dimensional problem, the proof of lemma 4.1 suggests a means of computing the LFR of multidimensional matrix polynomial mappings: write the LFR of each of the *scalar* s-dimensional polynomials, scale the coefficient matrices appropriately and add the resulting LFTs using the properties stated in section 2. Unfortunately, this procedure will, in general, lead to LFT representations of unnecessarily high order because the procedure ignores cross coupling and common factors.

In the case where $s = 1$, the multidimensional matrix polynomials reduce to affine matrix mappings of the indeterminants. In this case, well-defined procedures exist for computing the LFT representations. Basically, these procedures derive the coefficient matrix of the LFT representation by computing the singular value decomposition of the coefficient matrices A_i in the polynomial representation. The procedure is detailed in [21]. Minor variations of the approach are found in the control literature as early as [16] and also in [15][10].

The following lemma is the final result of the report. It states that any multidimensional rational matrix mapping has a linear fractional representations.

Lemma 4.2 *Let $E \subseteq \mathbb{R}^s$ and let $F : E \to \mathbb{R}^{m\times q}$ be a s-dimensional rational matrix function. Then there is a real matrix M and nonnegative integers $n_1, n_2, \ldots, n_s$ such that*

$$F(\delta_1, \delta_2, \ldots, \delta_s) = \mathcal{F}_u(M, \Delta)$$

with $\Delta = diag(\delta_1 I_{n_1}, \delta_2 I_{n_2}, \ldots, \delta_s I_{n_s})$.

Proof: Since $F(\delta_1, \delta_2, \ldots, \delta_s)$ is a rational matrix function, it may be written as

$$F(\delta_1, \delta_2, \ldots, \delta_s) = N(\delta_1, \delta_2, \ldots, \delta_s)\,[D(\delta_1, \delta_2, \ldots, \delta_s)]^{-1} \tag{11}$$

where $N : E \to \mathbb{R}^{m\times q}$ and $D : E \to \mathbb{R}^{q\times q}$ are s-dimensional matrix polynomials with $D(\delta_1, \delta_2, \ldots, \delta_s)$ nonsingular for $(\delta_1, \delta_2, \ldots, \delta_s) \in E$. By lemma 4.1, N and D may be written as LFTs

$$N(\delta_1, \delta_2, \ldots, \delta_s) \quad = \quad \mathcal{F}_u(M_N, \Delta_N) \tag{12}$$

$$D(\delta_1, \delta_2, \ldots, \delta_s) \quad = \quad \mathcal{F}_u(M_{D_{inv}}, \Delta_D) \tag{13}$$

where M_N and $M_{D_{inv}}$ are real matrices,

$$\Delta_N = diag(\delta_1 I_{N_1}, \delta_2 I_{N_2}, \ldots, \delta_s I_{N_s})$$

and

$$\Delta_D = diag(\delta_1 I_{D_1}, \delta_2 I_{D_2}, \ldots, \delta_s I_{D_s}).$$

By theorem 2.1, $[\mathcal{F}_u(M_{D_{inv}}, \Delta_D)]^{-1}$ is also an LFT in Δ_D, so there is an M_D such that

$$[D(\delta_1, \delta_2, \ldots, \delta_s)]^{-1} = \mathcal{F}_u(M_D, \Delta_D) \tag{14}$$

Combining equations 11, 12 and 14,

$$F(\delta_1, \delta_2, \ldots, \delta_s) = \mathcal{F}_u(M_N, \Delta_N)\mathcal{F}_u(M_D, \Delta_D) \tag{15}$$

Thus, F is the product of two LFTs so by theorem 2.1, it has a representation

$$F(\delta_1, \delta_2, \ldots, \delta_s) = \mathcal{F}_u(\tilde{M}, \tilde{\Delta}) \tag{16}$$

with

$$\tilde{M} = \left[\begin{array}{c|c} \tilde{M}_{11} & \tilde{M}_{12} \\ \hline \tilde{M}_{21} & \tilde{M}_{22} \end{array}\right] \quad \text{and} \quad \tilde{\Delta} = \begin{bmatrix} \Delta_N & 0 \\ 0 & \Delta_D \end{bmatrix}$$

Since Δ_N and Δ_D are diagonal, $\tilde{\Delta}$ is also diagonal with $n_j = N_j + D_j$ diagonal elements equal to δ_j for each $j = 1, \ldots, s$. Thus, $\tilde{\Delta}$ is related to

$$\Delta = diag(\delta_1 I_{n_1}, \delta_2 I_{n_2}, \ldots, \delta_s I_{n_s})$$

by a similarity transformation given by a real matrix T so that $\tilde{\Delta} = T\Delta T^{-1}$. Using this fact and theorem 2.1,

$$\mathcal{F}_u(\tilde{M}, \tilde{\Delta}) = \mathcal{F}_u(\tilde{M}, T\Delta T^{-1}) = \mathcal{F}_u(M, \Delta) \tag{17}$$

with

$$M = \left[\begin{array}{c|c} T^{-1}\tilde{M}_{11}T & T^{-1}\tilde{M}_{12} \\ \hline \tilde{M}_{21}T & \tilde{M}_{22} \end{array}\right] \quad \text{and} \quad \Delta = diag(\delta_1 I_{n_1}, \delta_2 I_{n_2}, \ldots, \delta_s I_{n_s})$$

Combining equations 16 and 17 yields the desired result. $\square$

As in the previous problems, the proof of lemma 4.2 suggests a means of computing the LFR of multidimensional rational matrix mappings. Factor the matrix mapping into numerator and denominator multidimensional matrix polynomials. Then write this factorization as the product of two LFTs on the appropriately defined indeterminant matrices by finding LFRs of the numerator and denominator s-dimensional matrix polynomials as outlined previously and applying the LFT inversion property. Finally combine the LFT representations into a single LFT representation using the LFT multiplication property. The resulting realization is referred to as a *direct representation* in [8]. As noted, the procedure for obtaining a direct representation ignores cross coupling and common factors and will generally lead to LFRs of unnecessarily high order. This is compounded by the fact that the procedure must be performed for both the numerator and denominator matrix mappings.

This result could have also been proven by first constructing LFRs of the scalar multidimensional rational functions in each element of the matrix and then combining the overall system of realizations using a feedback and cascade composition property of LFTs, sometimes called the Redheffer star-product (see [10] or [21]). This is the approach taken in [19], [20] and [14] to informally prove the existence of LFRs for arbitrary multidimensional rational matrix functions. The procedure for constructing the realization is called *the basic principle* in [21] and *pulling out the* Δs in other places, such as [10] and [20].

5 Minimal Realizations

While obtaining direct representations of multidimensional rational matrix mappings is straightforward, it generally results in LFRs with unnecessarily high order. One is usually interested in finding a representation with minimal order for obvious reasons.

In the case,when $s = 1$, obtaining the minimal representation is equivalent to finding a minimal state-space realization of a (MIMO) transfer function. This is accomplished by finding any non-minimal state-space realization of the transfer function and removing any uncontrollable/unobservable modes (i.e. a Kalman decomposition [13]). Unfortunately, the technique does not generalize to the multidimensional case which is sometimes referred to as the multivariable multidimensional minimal realization problem [5].

In the multidimensional scalar case, the linear system model reduction techniques are applied in [14] on each repeated scalar block in the uncertainty matrix, generating s 'local' minimal realizations which are then combined in a straightforward fashion to yield the overall representation. The combined representation is not guaranteed to be minimal, however. The approach taken in [4] is to break the polynomial into affine and non-affine parts and reduce each term separately using polynomial factorization and a nilpotency condition. The technique of treating the affine and non-affine parts of general multidimensional scalar polynomials separately is also employed in [6], [19] and [20]. These techniques extend to the multidimensional matrix case. Note, however, that these approaches do not, in general, lead to minimal representations.

A novel approach to obtaining representations of low-order appears in the recent articles [8][7]. (This work also contains a brief, but useful, tutorial of other approached to finding low-order representations.) There, the authors apply recursive sum and product decompositions of multidimensional polynomial matrices. By accounting for the cross-coupling terms in the multidimensional rational matrix mappings, the problem reduces to finding trivial representations of affine polynomial matrices (i.e. LTI transfer function state-space realizations). While this approach leads to low-order representations, the problem of obtaining minimal representations is still an open research issue.

Another important body of work is found in [3] and the references therein. In that work, the authors generalize the notions of balanced realizations, balanced Gramians and balanced truncation model reduction to the linear fractional representations discussed in this paper. That work is currently being extended (see [2] and the references therein) to establish a general system theory for these *multidimensional LFT systems* which parallels the theory for linear time-invariant systems.

6 Conclusions

This report has provided an intuitive proof for the existence of linear fractional representations of multi-dimensional rational matrix mappings. While the results presented here are by no means new, they are often stated without proof or with a cite to references which are not part of the general body of research publications [17]. In addition to providing existence proofs for the general problem, the paper has also listed references for similar results which are contained in several conference and journal papers but which seem to have escaped reference in much of the structured uncertainty research and LPV system publications.

In addition, the proofs included here provide a means for computing the linear fractional representations which is useful in many applications, including μ-analysis/synthesis and LPV system analysis/synthesis. Unfortunately, the procedures outlined here will, in general, lead to linear fractional representations of unnecessarily high order. Algorithms for computing LFT representations of low order are currently under development by other researchers; the problem still presents difficult open research questions.

A Appendix

A.1 Results from Matrix Algebra

The following results from matrix algebra are easily proven using Schur complements. They are stated here without proof, however the necessary tools may be found in, for example, [12].

Lemma A.1 *Given a nonsingular matrix $M \in \mathbb{R}^{n\times n}$ partitioned as*

$$M := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

If A, D, $A - BD^{-1}C$ and $D - CA^{-1}B$ are nonsingular, then

$$M^{-1} = \begin{bmatrix} (A-BD^{-1}C)^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{bmatrix}$$

Lemma A.2 (Matrix Inversion Lemma) *Let A, C and $C^{-1}+DA^{-1}B$ be nonsingular. Then*

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}$$

A.2 Proof of Theorem 2.1

Proof:

1. Let
$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} M \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} M_{22} & M_{21} \\ M_{12} & M_{11} \end{bmatrix}$$
with the obvious relationships between the partitions of M and N. Then
$$\begin{aligned} \mathcal{F}_u(M,\Delta) &= M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12} \\ &= N_{11} + N_{12}\Delta(I - N_{22}\Delta)^{-1}N_{21} \\ &= \mathcal{F}_\ell(N,\Delta) \end{aligned}$$

2. Let
$$N = \left[\begin{array}{c|c} N_{11} & N_{12} \\ \hline N_{21} & N_{22} \end{array}\right] = \left[\begin{array}{c|c} T^{-1}M_{11}T & T^{-1}M_{12} \\ \hline M_{21}T & M_{22} \end{array}\right]$$
Then
$$\begin{aligned} \mathcal{F}_u(M,T\Delta T^{-1}) &= M_{22} + M_{21}T\Delta T^{-1}(I - M_{11}T\Delta T^{-1})^{-1}M_{12} \\ &= M_{22} + M_{21}T\Delta(I - T^{-1}M_{11}T\Delta)^{-1}T^{-1}M_{12} \\ &= N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \\ &= \mathcal{F}_u(N,\Delta) \end{aligned}$$

3. Let
$$N = \left[\begin{array}{c|c} N_{11} & N_{12} \\ \hline N_{21} & N_{22} \end{array}\right] = \left[\begin{array}{c|c} M_{11} - M_{12}M_{22}^{-1}M_{21} & -M_{12}M_{22}^{-1} \\ \hline M_{22}^{-1}M_{21} & M_{22}^{-1} \end{array}\right].$$
Then
$$[\mathcal{F}_u(M,\Delta)]^{-1} = \left[M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}\right]^{-1}$$

Applying the Matrix Inversion Lemma (lemma A.2), this expression becomes

$$\begin{aligned}
[\mathcal{F}_u(M,\Delta)]^{-1} &= M_{22}^{-1} - M_{22}^{-1}M_{21}\Delta(M_{12}M_{22}^{-1}M_{21}\Delta + I - M_{11}\Delta)^{-1}M_{12}M_{22}^{-1} \\
&= M_{22}^{-1} - M_{22}^{-1}M_{21}\Delta(I - (M_{11} - M_{12}M_{22}^{-1}M_{21})\Delta)^{-1}M_{12}M_{22}^{-1} \\
&= N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \\
&= \mathcal{F}_u(N,\Delta)
\end{aligned}$$

4. Let

$$N = \left[\begin{array}{c|c} N_{11} & N_{12} \\ \hline N_{21} & N_{22} \end{array}\right] = \left[\begin{array}{cc|c} M_{11} & 0 & M_{12} \\ 0 & Q_{11} & Q_{12} \\ \hline M_{21} & Q_{21} & M_{22} + Q_{22} \end{array}\right], \quad \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}.$$

Then

$$\begin{aligned}
&\mathcal{F}_u(M,\Delta_1) + \mathcal{F}_u(Q,\Delta_2) = \\
=\ & M_{22} + M_{21}\Delta_1(I - M_{11}\Delta_1)^{-1}M_{12} + Q_{22} + Q_{21}\Delta_2(I - Q_{11}\Delta_2)^{-1}Q_{12} \\
=\ & M_{22} + Q_{22} \\
& + \begin{bmatrix} M_{21} & Q_{21} \end{bmatrix} \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \begin{bmatrix} (I - M_{11}\Delta_1)^{-1} & 0 \\ 0 & (I - Q_{11}\Delta_2)^{-1} \end{bmatrix} \begin{bmatrix} M_{12} \\ Q_{12} \end{bmatrix} \\
=\ & M_{22} + Q_{22} \\
& + \begin{bmatrix} M_{21} & Q_{21} \end{bmatrix} \Delta \left[\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} M_{11} & 0 \\ 0 & Q_{11} \end{bmatrix} \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \right]^{-1} \begin{bmatrix} M_{12} \\ Q_{12} \end{bmatrix} \\
=\ & N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \\
=\ & \mathcal{F}_u(N,\Delta)
\end{aligned}$$

5. Let

$$N = \left[\begin{array}{c|c} N_{11} & N_{12} \\ \hline N_{21} & N_{22} \end{array}\right] = \left[\begin{array}{cc|c} M_{11} & M_{12}Q_{21} & M_{12}Q_{22} \\ 0 & Q_{11} & Q_{12} \\ \hline M_{21} & M_{22}Q_{21} & M_{22}Q_{22} \end{array}\right], \quad \Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}.$$

Then

$$
\begin{aligned}
&\mathcal{F}_u(M,\Delta_1)\mathcal{F}_u(Q,\Delta_2) = \\
= \;& M_{22}Q_{22} + M_{22}Q_{21}\Delta_2(I-Q_{11}\Delta_2)^{-1}Q_{12} + M_{21}\Delta_1(I-M_{11}\Delta_1)^{-1}M_{12}Q_{22} \\
& + M_{21}\Delta_1(I-M_{11}\Delta_1)^{-1}M_{12}Q_{21}\Delta_2(I-Q_{11}\Delta_2)^{-1}Q_{12} \\
= \;& M_{22}Q_{22} + \begin{bmatrix} M_{21} & M_{22}Q_{21} \end{bmatrix} \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \times \\
& \begin{bmatrix} (I-M_{11}\Delta_1)^{-1} & (I-M_{11}\Delta_1)^{-1}M_{12}Q_{21}\Delta_2(I-Q_{11}\Delta_2)^{-1} \\ 0 & (I-Q_{11}\Delta_2)^{-1} \end{bmatrix} \begin{bmatrix} M_{12}Q_{22} \\ Q_{12} \end{bmatrix}
\end{aligned}
$$

By the Schur complement inverse formula (corollary A.1), this expression may be rewritten as

$$
\begin{aligned}
&\mathcal{F}_u(M,\Delta_1)\mathcal{F}_u(Q,\Delta_2) = \\
= \;& M_{22}Q_{22} + \begin{bmatrix} M_{21} & M_{22}Q_{21} \end{bmatrix} \Delta \begin{bmatrix} I-M_{11}\Delta_1 & -M_{12}Q_{21}\Delta_2 \\ 0 & I-Q_{11}\Delta_2 \end{bmatrix}^{-1} \begin{bmatrix} M_{12}Q_{22} \\ Q_{12} \end{bmatrix} \\
= \;& M_{22}Q_{22} + \begin{bmatrix} M_{21} & M_{22}Q_{21} \end{bmatrix} \Delta \times \\
& \left[\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12}Q_{21} \\ 0 & Q_{11} \end{bmatrix} \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \right]^{-1} \begin{bmatrix} M_{12}Q_{22} \\ Q_{12} \end{bmatrix} \\
= \;& N_{22} + N_{21}\Delta(I-N_{11}\Delta)^{-1}N_{12} \\
= \;& \mathcal{F}_u(N,\Delta)
\end{aligned}
$$

6. Let T_{zw} denote the closed loop mapping from w to z. Define auxiliary variables w_u, x_u, y_u and z_u so that

$$
\begin{bmatrix} z_u \\ z \\ z_l \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} w_u \\ w \\ w_l \end{bmatrix} = G \begin{bmatrix} w_u \\ w \\ w_l \end{bmatrix} \tag{18}
$$

and

$$
\begin{bmatrix} x_u \\ x \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} y_u \\ y \end{bmatrix} = Q \begin{bmatrix} y_u \\ y \end{bmatrix} \tag{19}
$$

Then

$$
\begin{bmatrix} z \\ z_l \end{bmatrix} = \mathcal{F}_u(G,\Delta_1) \begin{bmatrix} w \\ w_l \end{bmatrix} \tag{20}
$$

is obtained by eliminating the variables w_u and z_u from the above expression using the feedback $w_u = \Delta_1 z_u$ and

$$
x = \mathcal{F}_u(Q,\Delta_2)y \tag{21}
$$

is obtained by eliminating the variables x_u and y_u using the feedback $y_u = \Delta_2 x_u$.

$\left(T_{zw} = \mathcal{F}_\ell(\mathcal{F}_u(G,\Delta_1),\mathcal{F}_u(Q,\Delta_2))\right)$

Let

$$G_{11} = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix},\ G_{12} = \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix},$$
$$G_{21} = \begin{bmatrix} A & B_1 \\ C_2 & D_{21} \end{bmatrix},\ G_{22} = \begin{bmatrix} A & B_2 \\ C_2 & D_{22} \end{bmatrix}.$$

Then it is easily verified that

$$\begin{bmatrix} z \\ z_l \end{bmatrix} = \begin{bmatrix} \mathcal{F}_u(G_{11},\Delta_1) & \mathcal{F}_u(G_{12},\Delta_1) \\ \mathcal{F}_u(G_{21},\Delta_1) & \mathcal{F}_u(G_{22},\Delta_1) \end{bmatrix} \begin{bmatrix} w \\ w_l \end{bmatrix} = \mathcal{F}_u(G,\Delta_1) \begin{bmatrix} w \\ w_l \end{bmatrix} \tag{22}$$

so that eliminating the variables w_l and z_l from this expression using the feedback connection $y = z_l$ and $w_l = x$ with equation 20 yields

$$\begin{aligned} z &= \mathcal{F}_u(G_{11},\Delta_1)w \\ &\quad +\mathcal{F}_u(G_{12},\Delta_1)\mathcal{F}_u(Q,\Delta_2)(I-\mathcal{F}_u(G_{22},\Delta_1)\mathcal{F}_u(Q,\Delta_2))^{-1}\mathcal{F}_u(G_{21},\Delta_1)w \\ &= \mathcal{F}_\ell(\mathcal{F}_u(G,\Delta_1),\mathcal{F}_u(Q,\Delta_2))w \end{aligned}$$

where $\mathcal{F}_u(G,\Delta_1)$ is partitioned as in equation 22. Thus,

$$T_{zw} = \mathcal{F}_\ell(\mathcal{F}_u(G,\Delta_1),\mathcal{F}_u(Q,\Delta_2))$$

$\left(T_{zw} = \mathcal{F}_u(\mathcal{F}_\ell(G,\mathcal{F}_u(Q,\Delta_2)),\Delta_1)\right)$

Eliminate z_l and w_l from equation 18 using the feedback connection $w_l = x$ and $z_l = y$ and equation 21 to obtain

$$\begin{aligned} \begin{bmatrix} z_u \\ z \end{bmatrix} &= \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} \begin{bmatrix} w_u \\ w \end{bmatrix} \\ &\quad + \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} \mathcal{F}_u(Q,\Delta_2)(I - D_{22}\mathcal{F}_u(Q,\Delta_2))^{-1} \begin{bmatrix} C_2 & D_{21} \end{bmatrix} \begin{bmatrix} w_u \\ w \end{bmatrix} \\ &= \mathcal{F}_\ell(G,\mathcal{F}_u(Q,\Delta_2)) \begin{bmatrix} w_u \\ w \end{bmatrix} \end{aligned}$$

Then eliminate z_u and w_u from this expression using $w_u = \Delta_1 z_u$ to obtain

$$z = \mathcal{F}_u(\mathcal{F}_\ell(G,\mathcal{F}_u(Q,\Delta_2)),\Delta_1)w$$

so that

$$T_{zw} = \mathcal{F}_u(\mathcal{F}_\ell(G, \mathcal{F}_u(Q, \Delta_2)), \Delta_1)$$

$(T_{zw} = \mathcal{F}_u(N, \Delta))$

The last expression is obtained by combining equations 18 and 19 as

$$\begin{bmatrix} z_u \\ x_u \\ z \\ z_l \\ x \end{bmatrix} = \begin{bmatrix} A & 0 & B_1 & B_2 & 0 \\ 0 & Q_{11} & 0 & 0 & Q_{12} \\ C_1 & 0 & D_{11} & D_{12} & 0 \\ C_2 & 0 & D_{21} & D_{22} & 0 \\ 0 & Q_{21} & 0 & 0 & Q_{22} \end{bmatrix} \begin{bmatrix} w_u \\ y_u \\ w \\ w_l \\ y \end{bmatrix} \tag{23}$$

with

$$\begin{bmatrix} w_u \\ y_u \end{bmatrix} = \Delta \begin{bmatrix} z_u \\ x_u \end{bmatrix} = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} \begin{bmatrix} z_u \\ x_u \end{bmatrix}. \tag{24}$$

From this expression, one has

$$\begin{bmatrix} z_l \\ x \end{bmatrix} = \begin{bmatrix} I & -D_{22} \\ -Q_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} C_2 & 0 & D_{21} \\ 0 & Q_{21} & 0 \end{bmatrix} \begin{bmatrix} w_u \\ y_u \\ w \end{bmatrix}$$

Using the Schur complement inverse formula (corollary A.1), this becomes

$$\begin{aligned} \begin{bmatrix} z_l \\ x \end{bmatrix} &= \begin{bmatrix} L_1 & D_{22}L_2 \\ L_2Q_{22} & L_2 \end{bmatrix} \begin{bmatrix} C_2 & 0 & D_{21} \\ 0 & Q_{21} & 0 \end{bmatrix} \begin{bmatrix} w_u \\ y_u \\ w \end{bmatrix} \\ &= \begin{bmatrix} L_1C_2 & D_{22}L_2Q_{21} & L_1D_{21} \\ L_2Q_{22}C_2 & L_2Q_{21} & L_2Q_{22}D_{21} \end{bmatrix} \begin{bmatrix} w_u \\ y_u \\ w \end{bmatrix} \end{aligned}$$

where $L_1 := (I - D_{22}Q_{22})^{-1}$ and $L_2 := (I - Q_{22}D_{22})^{-1}$. Substituting this expression into equation 23 to eliminate z_l, x, w_l and y, one obtains

$$\begin{bmatrix} z_u \\ x_u \\ z \end{bmatrix} = \begin{bmatrix} A + B_2L_2Q_{22}C_2 & B_2L_2Q_{21} & B_1 + B_2L_2Q_{22}D_{21} \\ Q_{12}L_1C_2 & Q_{11} + Q_{12}D_{22}L_2Q_{21} & Q_{12}L_1D_{21} \\ C_1 + D_{12}L_2Q_{22}C_2 & D_{12}L_2Q_{21} & D_{11} + D_{12}L_2Q_{22}D_{21} \end{bmatrix} \begin{bmatrix} w_u \\ y_u \\ w \end{bmatrix}$$

so that when the variables w_u, x_u, y_u and z_u are eliminated using equation 24,

$$z = \mathcal{F}_u(N, \Delta)w = T_{zw}w$$

where N and Δ are as defined in the theorem.

$\square$

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