Finite-Horizon Bounded Amplitude Control of Linear Parameter Varying Systems

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Abstract

This paper considers the problem of controlling continuous-time linear parameter varying (LPV) systems when performance is measured in terms of prespecified, finite-time bounds on plant signal amplitude. Such problems arise, for example, in finite time tasks where signal amplitude is more important than signal energy. The notion of finite horizon behaviour of LPV system is formally defined. Sufficient conditions for bounded amplitude performance of such systems are proven and used to derive synthesis conditions for state feedback controllers. These conditions are useful in that they can be reformulated as linear matrix inequality (LMI) conditions amenable to efficient numerical techniques. An illustrative numerical example is included.

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1 Introduction

In any control system design problem, there is a performance objective which is to be achieved by the design. In the continuous-time $\mathcal{H}_\infty$ design paradigm, for example, performance is measured with respect to the induced $\mathcal{L}_2$ norm of the closed loop system which corresponds to the gain that the system applies to the *energy* of the input signal. Another measure of system performance is the *peak value* of the output signal, especially in the presence of input signals which have bounded peak value but unbounded energy. Such control problems are prevalent in the literature: motor control problems with electrical (voltage or current) or mechanical (motion) restrictions and process control problems with chemical concentration restrictions are two examples. In each of these examples, violation of the restrictions could lead to performance degradation and possibly catastrophic failure of the system. In these problems, therefore, it is more important to control the peak value of the plant signals than the energy contained in the signals. In recent years, significant effort has been directed towards the problem of designing control systems which guarantee such prescribed bounds on peak signal values.

For discrete-time systems, the problem of controlling signal amplitudes is known as the ℓ_1-optimal control problem. First stated in [19], this problem is to design a controller which minimizes the effects of persistent (bounded amplitude) disturbances as measured by the induced ℓ_∞-norm (the peak-to-peak gain). A solution to the linear MIMO ℓ_1-optimal control problem appeared in [9], where optimal controllers are synthesized by solving a linear programming problem. Results for the discrete-time case have been extended to handle system uncertainty [7] [18] [15]. As with the solution to the deterministic case, controller synthesis amounts to solving linear programming problems.

While there have been significant advances in the discrete-time problem, the continuous-time problem has not enjoyed the same success. For continuous-time systems, the problem of controlling signal amplitude is denoted the $\mathcal{L}_1$ or induced-$\mathcal{L}_\infty$ optimal control problem. In [8], it is shown that optimal solutions to this problem are irrational or infinite-dimensional, even for rational and finite-dimensional plants. Furthermore, solving the problem exactly involves solving a nonlinear programming problem and while approximate solutions, which are rational and finite dimensional, may be found via linear programming problems, the solutions are still of arbitrarily high order.

An alternative approach for the $\mathcal{L}_1$ optimal control problem was recently proposed in [13]. There, instead of synthesizing a controller to explicitly minimize the induced-$\mathcal{L}_\infty$ norm of the system, an upper bound on the induced-$\mathcal{L}_\infty$ norm is minimized. Although the solution is suboptimal, the solution method involves only the solution of a Riccati matrix inequality. This, in turn, may be expressed as a linear matrix inequality (LMI) problem [6], for which computationally efficient algorithms are available. Furthermore, the synthesized controller is guaranteed to have order no higher than the plant order. The results in [13] concern linear, time-invariant systems only.

In this paper, some of the results presented in [13] are extended to deal with linear parameter varying (LPV) systems [16]. LPV systems are linear dynamical systems whose coefficient matrices depend on a time-varying parameter in a (piecewise-)continuous fashion. The parameter varies over some known compact subset of Euclidean space, called the parameter set. LPV systems have been studied extensively, especially in the context of gain scheduling ([17],[14], [3],[2]). The extension of techniques from [13] to LPV systems also results in a solution method which involves solving a LMI problem with structure similar to the LMIs seen in [14], [3] and [2]. Thus, extending the results in this fashion not only extends the solution method to a larger class of systems; it also implies that the LMI-based self-scheduled controller design techniques of [14], [3] and [2] might be used for bounded amplitude problems as well.

This paper also generalizes the notion of LPV system performance by introducing the notion of finite-horizon LPV system behaviour. In many applications, it is only necessary to consider system behaviour over finite time intervals. In these instances, parameter variations are often restricted to compact subsets of the parameter set. Tighter performance can often be guaranteed for finite time intervals if it is known that the parameter variations can be restricted in this way.

Motivation for the study of finite-horizon behaviour arises from the study of gain scheduled controllers which switch between control agents in a discrete fashion. Another important behavioural characterization arising in such problems concerns initial conditions of the plant state; results presented in [13] consider bounding a system norm which assumes zero initial conditions. This paper generalizes the induced norm conditions to include bounds on decay time for systems with initial conditions that violate performance constraints.

The remainder of this paper is organized as follows. Section 2 formally defines the notion of LPV systems, parameter variations and associated performance measures used in this paper. (Section 2 also presents the notation used throughout the paper.) Section 3 presents the bounded amplitude performance results related to a class of LPV systems. This section contains the main results of the paper (theorem 3.1) which states sufficient conditions for bounded amplitude performance of LPV systems. This result is proven and discussed with respect to practical implementation of the conditions. The section concludes with a reformulation of the condition as an LMI (theorem 3.2). Section 4 considers the problem of linear state-feedback controller synthesis. Theorem 4.1 states how the results of section 3 may be used to characterize a set of state-feedback controllers satisfying performance constraints. Practical implementation of the synthesis result is discussed around theorem 4.2 which shows the synthesis conditions may be expressed as LMIs. Section 5 presents a numerical example to illustrate the theoretical results.

Many of the definitions for LPV systems presented in this section are generalized from [4]; the induced-$\mathcal{L}_\infty$ norm results are generalizations of the LTI results reported in [13].

2 Mathematical Preliminaries

The section begins by formally defining a linear parameter varying (LPV) system and discussing various notions of parameter variation, specifically the notion of finite horizon behaviour. The performance measure considered in this section, the induced-$\mathcal{L}_\infty$ norm, is defined for a LPV system with a special emphasis on finite horizon behaviour. Many of the definitions for LPV systems presented in this section are generalized from [4].

Definition 2.1 establishes the notation for $\mathcal{L}_\infty$ signal norms (finite and infinite horizon) used throughout the paper.

Definition 2.1 *For a finite constant $T > 0$, the finite-horizon infinity norm of a signal $f : \mathbb{R}^+ \to \mathbb{R}^n$ is defined as*

$$\|f\|_{\infty,[0,T]} := \sup_{t\in[0,T]} \|f(t)\|$$

where $\|\cdot\|$ denotes the Euclidean l_2 vector norm. The linear space $\mathcal{L}^n_\infty[0,T]$ is defined by

$$\mathcal{L}^n_\infty[0,T] := \left\{ f : \mathbb{R}^+ \to \mathbb{R}^n \,\middle|\, \|f\|_{\infty,[0,T]} < \infty \right\}$$

The subset $\left\{ f : \mathbb{R}^+ \to \mathbb{R}^n \,\middle|\, \|f\|_{\infty,[0,T]} \le 1 \right\} \subset \mathcal{L}^n_\infty[0,T]$ is denoted $\mathcal{BL}^n_\infty[0,T]$.

The infinite-horizon infinity norm of a signal $f : \mathbb{R}^+ \to \mathbb{R}^n$ is defined as

$$\|f\|_{\infty,[0,\infty)} := \sup_{t\in[0,\infty)} \|f(t)\|$$

where $\|\cdot\|$ denotes the Euclidean l_2 vector norm. The linear space $\mathcal{L}^n_\infty[0,\infty)$ is defined by

$$\mathcal{L}^n_\infty[0,\infty) := \left\{ f : \mathbb{R}^+ \to \mathbb{R}^n \,\middle|\, \|f\|_{\infty,[0,\infty)} < \infty \right\}$$

The subset $\left\{ f : \mathbb{R}^+ \to \mathbb{R}^n \,\middle|\, \|f\|_{\infty,[0,\infty)} \le 1 \right\} \subset \mathcal{L}^n_\infty[0,\infty)$ is denoted $\mathcal{BL}^n_\infty[0,\infty)$. The spaces $\mathcal{L}^n_\infty[0,\infty)$ and $\mathcal{BL}^n_\infty[0,\infty)$ will often be denoted, respectively, $\mathcal{L}^n_\infty$ and $\mathcal{BL}^n_\infty$.

Definitions 2.2-2.3 formally establish the notion of a linear parameter varying system used in this section.

Definition 2.2 (Parameter Variation Set) *Given a compact subset $\tilde{\Theta} \subset \mathbb{R}^s$, the* parameter variation set *$\mathcal{F}_{\tilde{\Theta}}$ denotes the set of all continuous functions mapping $\mathbb{R}^+$ into $\tilde{\Theta}$. For a finite $T > 0$ and a compact subset $\Theta \subseteq \tilde{\Theta}$, the set $\mathcal{F}_{\Theta}[0, T]$ denotes the set of all continuous functions $\theta \in \mathcal{F}_{\tilde{\Theta}}$ which map $[0, T]$ into Θ.*

The notation $\theta \in \mathcal{F}_{\tilde{\Theta}}$ denotes a function in the parameter variation set; $\theta \in \tilde{\Theta}$ denotes a vector in a compact subset of $\mathbb{R}^s$. These definitions are extended to functions which map finite time intervals to compact subsets of $\tilde{\Theta}$. It is clear that $\mathcal{F}_{\Theta} \subset \mathcal{F}_{\Theta}[0, T] \subset \mathcal{F}_{\tilde{\Theta}}$.

Note that both $\mathcal{F}_{\tilde{\Theta}}$ and, for instance, $\mathcal{L}_{\infty}^n$ represent signal spaces. Technically, it can easily be argued that $\mathcal{F}_{\tilde{\Theta}} \subset \mathcal{L}_{\infty}^s$ since $\mathcal{F}_{\tilde{\Theta}}$ consists of supremum bounded s-dimensional vectors which vary continuously in time. In this paper, the following convention is followed: $\mathcal{F}_{\tilde{\Theta}}$ will always refer to parameter signals or parameter variations; $\mathcal{L}_{\infty}^n$ will refer to signals in the plant input, output or state space.

Definition 2.3 (Linear Parameter Varying (LPV) System) *Given a compact set $\tilde{\Theta} \subset \mathbb{R}^s$, and continuous functions $A : \mathbb{R}^s \to \mathbb{R}^{n \times n}$, $B : \mathbb{R}^s \to \mathbb{R}^{n \times n_w}$, $C : \mathbb{R}^s \to \mathbb{R}^{n_z \times n}$, and $D : \mathbb{R}^s \to \mathbb{R}^{n_z \times n_w}$, an n^{th} order linear parameter varying (LPV) system is a dynamical system whose dynamics evolve as*

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A(\theta(t)) & B(\theta(t)) \\ C(\theta(t)) & D(\theta(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \tag{1}$$

where $\theta \in \mathcal{F}_{\tilde{\Theta}}$.

Note that the LPV system is defined over the parameter variation set $\mathcal{F}_{\tilde{\Theta}}$. Certain properties of the LPV system may only be guaranteed while the parameter variations are confined to a subset of $\mathcal{F}_{\tilde{\Theta}}$. Previous results on LPV systems (see, for example, [3], [16], [11], [14]) restrict consideration to properties which can be guaranteed over the entire parameter set $\tilde{\Theta}$ and the associated parameter variation set. This restricts the class of systems for which the linear design techniques may be applied to systems where $\tilde{\Theta}$ is small or posesses sufficient structure in terms of the way that parameter variations enter the system dynamics.

In this paper, the concern is on the situation where specific properties of the LPV system can only be guaranteed for a compact subset $\Theta \subseteq \tilde{\Theta}$ of the parameter set. This presents the possibility that a given parameter variation may evolve out of Θ, causing the system state to evolve out of the performance specification envelope. As noted earlier, such situations may arise, for example, in gain scheduling applications employing *switched* control systems. In those instances, finite-horizon models may be more appropriate. Hence, there is a specific need for the study of finite horizon performance of LPV systems.

To the authors' knowledge, finite horizon peformance of LPV systems in the above context has not been studied. This is not to say that the results which follow are not specific cases of finite horizon nonlinear system behaviour. However, there are apparently no results available in the LPV framework.

The performance measure used in this paper is the induced $\mathcal{L}_\infty$ norm which, as defined below, is related to the peak-to-peak gain of the system. Definition 2.4 formally defines the the induced-$\mathcal{L}_\infty$ norm for finite and infinite system behaviour.

Definition 2.4 *Let the LPV system of definition 2.3 be denoted by $\Sigma(\tilde{\Theta}, A, B, C, D)$, or simply, $\Sigma_{\tilde{\Theta}}$, if the state-space data is clear from the context. Then for any $\theta \in \mathcal{F}_{\tilde{\Theta}}$,*

- *the linear time-varying system described in equation 1 is denoted Σ_θ,*
- *$\Phi_\theta(t, t_0)$ is the state-transition matrix of of Σ_θ*
- *for $x(t_0) = 0$, the causal linear input/output mapping, $H_\theta : \mathcal{L}_\infty^{n_w} \to \mathcal{L}_\infty^{n_z}$, of Σ_θ is defined*

as

$$H_\theta w(t) = \int_{t_0}^{t} C(\theta(\tau))\Phi_\theta(t,\tau)B(\theta(\tau))w(\tau)d\tau + D(\theta(t))w(t)$$

- *for a finite $T > 0$, the finite-horizon induced-$\mathcal{L}_\infty$ norm of H_θ is given by*

$$\|H_\theta\|_{i\infty,[0,T]} := \sup_{w \in \mathcal{BL}_\infty^{n_w}[0,T]} \|H_\theta w\|_{\infty,[0,T]} \tag{2}$$

- *the infinite-horizon induced-$\mathcal{L}_\infty$ norm of H_θ is given by*

$$\|H_\theta\|_{i\infty} := \sup_{w \in \mathcal{BL}_\infty^{n_w}} \|H_\theta w\|_\infty \tag{3}$$

The set $H_{\mathcal{F}_{\tilde{\Theta}}}$ is defined as

$$H_{\mathcal{F}_{\tilde{\Theta}}} := \{H_\theta : \theta \in \mathcal{F}_{\tilde{\Theta}}\}$$

The shorthand $\|H_{\mathcal{F}_{\tilde{\Theta}}}\|_{i\infty} \leq \gamma$ for some $\gamma > 0$ means that for all $\theta \in \mathcal{F}_{\tilde{\Theta}}$, $\|H_\theta\|_{i\infty} \leq \gamma$.

3 Performance Analysis

In many applications, it is only necessary to consider system behaviour for a finite time interval. In these instances, one may often restrict attention to characteristics which can be established for compact subsets $\Theta \subseteq \tilde{\Theta}$ and obtain less restrictive conditions for system performance.

In this section, LPV systems of the form

$$\begin{aligned} \dot{x} &= A(\theta)x + B(\theta)w \\ z &= Cx, \end{aligned} \tag{4}$$

with $\theta \in \mathcal{F}_\Theta$ and where $C \in \mathbb{R}^{n_z \times n}$ is a constant matrix, are considered. Bounded amplitude performance of these systems is considered for finite time intervals and for parameter variations over compact subsets $\Theta \subseteq \tilde{\Theta}$. Sufficient conditions are presented for a general LPV system to satisfy bounded amplitude performance specifications, and numerical techniques for evaluating these conditions are discussed.

The development begins with the following lemma which is useful for relating the size of a system output matrix, C, to a given induced-$\mathcal{L}_\infty$ performance level.

Lemma 3.1 *Given constant $\gamma > 0$, a positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a matrix $C \in \mathbb{R}^{n_z \times n}$, then $x'C'Cx \leq \gamma^2$ for all x such that $x'Px \leq 1$ if and only if $\gamma^2 P \geq C'C$.*

When the output matrix, C, of a LPV system is constant, then lemma 3.1 relates the performance level, γ, to the size of C through a positive definite matrix P. The following theorem applies this result to specify conditions for finite-horizon bounded amplitude performance of the class of LPV systems characterized by equation 4. The theorem represents the main result of the paper.

Theorem 3.1 (Finite Horizon Performance) *Given constant $\gamma > 0$ and the LPV system, $\Sigma(\tilde{\Theta}, A, B, C, 0)$ with $C \in \mathbb{R}^{n_z \times n}$ a constant matrix. Let $\Theta \subset \tilde{\Theta}$ be a compact subset. If there exist constants $\alpha > 0$ and $\beta \geq 0$ and a positive definite matrix $P \in \mathbb{R}^{n \times n}$ satisfying*

$$P \geq \frac{1}{\gamma^2} C'C \tag{5}$$

and

$$A'(\theta)P + PA(\theta) + (\alpha + \beta)\,P + \frac{1}{\alpha} PB(\theta)B'(\theta)P \leq 0 \tag{6}$$

for all $\theta \in \Theta$, then for any $T \in [0, \infty)$,

1. *for $\theta \in \mathcal{F}_\Theta[0,T]$, the conditions $w \in \mathcal{BL}_\infty^{n_w}[0,T]$ and $x'(0)Px(0) \leq 1$ imply that for all $t \in [0,T]$,*

$$x'(t)Px(t) \leq 1 \text{ and } z'(t)z(t) \leq \gamma^2 \tag{7}$$

2.

$$\|H_{\mathcal{F}_\Theta[0,T]}\|_{i\infty,[0,T]} \leq \gamma$$

Additionally, if $\beta > 0$, then for any $T \in [0, \infty)$,

3. *for $\theta \in \mathcal{F}_\Theta[0,T]$, if $w \in \mathcal{BL}_\infty^{n_w}[0,T]$ and $x'(0)Px(0) = r_0 > 1$ and if*

$$t_d := -\frac{1}{\beta}\log\left(\frac{1}{r_0}\right) \leq T$$

then

$$x'(t)Px(t) \leq 1 \text{ for all } t \in [t_d, T]$$

Proof: Let $\gamma > 0$. Given constants $\alpha > 0$ and $\beta \geq 0$ and a positive definite matrix $P \in \mathbb{R}^{n\times n}$, then for any $\theta \in \Theta$,

$$\frac{1}{\alpha}PB(\theta)B'(\theta)P \geq 0$$

so that if equation 6 holds for all $\theta \in \Theta$, then

$$A'(\theta)P + PA(\theta) + (\alpha+\beta)\,P \leq -\frac{1}{\alpha}PB(\theta)B'(\theta)P \leq 0$$

for all $\theta \in \Theta$. Thus, from the theory of Schur complements, equation 6 is true for all $\theta \in \Theta$ if and only if

$$\begin{bmatrix} A'(\theta)P + PA(\theta) + (\alpha+\beta)\,P & PB(\theta) \\ B'(\theta)P & -\alpha I \end{bmatrix} \leq 0 \quad \text{for all } \theta \in \Theta. \tag{8}$$

But equation 8 is satisfied if and only if for all $\theta \in \Theta$ and all $\xi \in \mathbb{R}^n$, $\nu \in \mathbb{R}^{n_w}$

$$\begin{bmatrix} \xi \\ \nu \end{bmatrix}' \begin{bmatrix} A'(\theta)P + PA(\theta) + (\alpha+\beta)\,P & PB(\theta) \\ B'(\theta)P & -\alpha I \end{bmatrix} \begin{bmatrix} \xi \\ \nu \end{bmatrix} \leq 0 \tag{9}$$

Equation 9 is true if and only if

$$\xi'(A'(\theta)P + PA(\theta) + \beta P)\xi + \nu' B'(\theta)P\xi + \xi' PB(\theta)\nu + \alpha(\xi' P\xi - 1) + \alpha(1 - \nu'\nu) \leq 0 \tag{10}$$

for all $\theta \in \Theta$ and all $\xi \in \mathbb{R}^n$, $\nu \in \mathbb{R}^{n_w}$. Equation 10, in turn, implies

$$\xi'(A'(\theta)P + PA(\theta))\xi + \nu' B'(\theta)P\xi + \xi' PB(\theta)\nu \leq -\beta\xi' P\xi \leq 0 \tag{11}$$

for all $\theta \in \Theta$ and all $\xi \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^{n_w}$ such that

$$\xi' P \xi \geq 1 \text{ and } \nu'\nu \leq 1$$

To prove the first part of the theorem, consider any time $T \in [0, \infty)$ and any parameter variation $\theta \in \mathcal{F}_\Theta[0,T]$. Now, define the function $V : \mathbb{R}^n \to \mathbb{R}$ by $V(\xi) := \xi' P \xi$. Along the trajectories of Σ_θ, the time derivative of $V(x(t))$ is given by

$$\frac{d}{dt}V(x(t)) = x(t)'(A'(\theta(t))P + PA(\theta(t)))x(t) + w'(t)B'(\theta(t))Px(t) + x'(t)PB(\theta(t))w(t) \quad (12)$$

From the above argument, equation 6 and equation 11 imply

$$\frac{d}{dt}V(x(t)) \leq -\beta V(x(t)) \leq 0 \quad (13)$$

for any $x(t)$ and $w(t)$ satisfying $x(t)'Px(t) \geq 1$ and $w(t)'w(t) \leq 1$ with $t \in [0,T]$.

Now, suppose that for some $w \in \mathcal{BL}_\infty^{n_w}[0,T]$ that there is a trajectory of Σ_θ with initial state $x(0)$ satisfying $V(x(0)) = x'(0)Px(0) \leq 1$ and $V(x(t_f)) = x'(t_f)Px(t_f) > 1$ for some $t_f \in (0,T]$. Since $V(x(t))$ is differentiable in t, the Mean Value Theorem can be used to imply the existence of a time $\tau \in (0, t_f)$ for which $V(x(\tau)) = x'(\tau)Px(\tau) \geq 1$ and $\dot{V}(x(\tau)) > 0$. This is a contradiction of equation 13, so $x'(t)Px(t) \leq 1$ for all $t \in [0,T]$. By lemma 3.1 and equation 5, this is true if and only if

$$x'(t)C'Cx(t) = z'(t)z(t) \leq \gamma^2$$

for all $t \in [0,T]$ proving the first part of the theorem.

To prove the second part of the theorem, write

$$\begin{aligned}
\|H_{\mathcal{F}_\Theta[0,T]}\|_{i\infty,[0,T]} &\leq \sup\nolimits_{\theta \in \mathcal{F}_\Theta[0,T]} \|H_\theta\|_{i\infty,[0,T]} \\
&= \sup\nolimits_{\theta \in \mathcal{F}_\Theta[0,T]} \left(\sup\nolimits_{w \in \mathcal{BL}_\infty^{n_w}[0,T]} \|H_\theta w\|_{\infty,[0,T]} \right) \\
&= \sup\nolimits_{\theta \in \mathcal{F}_\Theta[0,T]} \left(\sup\nolimits_{w \in \mathcal{BL}_\infty^{n_w}[0,T], t \in [0,T]} \|H_\theta w(t)\| \right) \\
&= \sup\nolimits_{\theta \in \mathcal{F}_\Theta[0,T]} \left(\sup\nolimits_{w \in \mathcal{BL}_\infty^{n_w}[0,T], t \in [0,T]} \|z(t)\| \right)
\end{aligned}$$

But, from part 1, $\|z(t)\| = (z'(t)z(t))^{1/2} \leq \gamma$ for all $t \in [0,T]$ so that

$$\sup_{\theta \in \mathcal{F}_\Theta[0,T]} \left(\sup_{w \in \mathcal{BL}_\infty^{n_w}[0,T], t \in [0,T]} \|z(t)\| \right) \leq \gamma$$

yielding the desired result.

To prove the last part of the theorem, assume that $\beta > 0$. For $\theta \in \mathcal{F}_\Theta[0,T]$ and $w \in \mathcal{BL}_\infty^{n_w}[0,T]$, equation 13 then implies that along any trajectory of Σ_θ for $t \in [0,T]$,

$$V(x(t)) \leq V(x(0)) + \int_0^t -\beta V(x(\tau)) d\tau$$

so that from the Bellman-Gronwall Lemma,

$$V(x(t)) \leq V(x(0)) \exp(-\beta t) \tag{14}$$

Suppose that $V(x(0)) = x'(0)Px(0) = r_0 > 1$ and let

$$t_d := -\frac{1}{\beta} \log\left(\frac{1}{r_0}\right)$$

If $t_d \leq T$, then for all $t \in [t_d, T]$, from equation 14

$$V(x(t)) \leq r_0 \exp(-\beta \left(-\frac{1}{\beta} \log\left(\frac{1}{r_0}\right)\right)) = 1$$

Equivalently,

$$x'(t)Px(t) \leq 1$$

for all $t \in [t_d, T]$. □

Theorem 3.1 demonstrates that sufficient conditions for finite-horizon bounded amplitude performance of a class of LPV systems may be characterized by a Riccati matrix inequality (equation 6) coupled with a constraint on the size of the output matrix (equation 5). These results are stated in parts 1 and 2 of the theorem.

As can be seen from parts 1 and 2 of the theorem, the initial state of the LPV system is an important condition for induced-$\mathcal{L}_\infty$ performance. In previous work [13], the initial state of the system is assumed to be zero, so that transient effects are neglected. In part 3 of the theorem, it is seen that under appropriate conditions, the system state can be guaranteed to decay at a specific rate, β.

The constants α and β are related to the speed of the system and the ability of the system to reject disturbances which enter through the matrix $B(\theta)$. To see these relationships, first consider inequality 6, rewritten as

$$P^{-1/2}(A(\theta)+\frac{\alpha+\beta}{2}I)'P^{1/2}+P^{1/2}(A(\theta)+\frac{\alpha+\beta}{2}I)P^{-1/2}+\frac{1}{\alpha}P^{1/2}B(\theta)B'(\theta)P^{1/2}\leq 0 \quad (15)$$

It is clear from eigenvalue perturbation theory that if

$$\alpha+\beta>\max_{\theta\in\Theta}\{-2\max\mathrm{Re}(\mathrm{eig}(A(\theta)))\}. \quad (16)$$

then $A(\theta)+\frac{1}{2}(\alpha+\beta)I$ will have at least one eigenvalue in the open right half-plane for some value $\theta\in\Theta$. This implies that for some $\theta\in\Theta$, the matrix

$$P^{-1/2}(A(\theta)+\frac{\alpha+\beta}{2}I)'P^{1/2}+P^{1/2}(A(\theta)+\frac{\alpha+\beta}{2}I)P^{-1/2}$$

will be indefinite for any positive definite matrix P. Since $\frac{1}{\alpha}P^{1/2}B(\theta)B'(\theta)P^{1/2}\geq 0$ for any $\alpha>0$, the inequality in equation 15 can have no positive definite solution P is equation 16 is satisfied. Thus, α and β is related to the slowest eigenvalue of the matrix $A(\theta)$ (over Θ). This observation is similar to that found in [13] and related work and agrees with the final part of theorem 3.1 where $\beta>0$ yields an exponential bounding function on the trajectory of the system, as seen from equation 14.

The constant α also reflects the sensitivity of the system to disturbances which enter through the matrix $B(\theta)$. To see this, note that as α decreases, the term $\frac{1}{\alpha}P^{1/2}B(\theta)B'(\theta)P^{1/2}$ becomes larger, making inequality 15 more difficult to solve. A larger α diminishes the influence of this term.

The observations point to an inherent tradeoff in the choice of constants α and β which yield a solution P to inequalities 5 and 6. Since α and β are restricted by equation 16, whenever α is bounded away from zero due to the sensitivity of the system, a corresponding bound is seen in β being bounded below the slowest eigenvalue of the system; the more sensitive the system, the less that can be guaranteed in terms of the transient decay under disturbances.

Theorem 3.1 presents a set of simple conditions for LPV system performance, but it does not provide an efficient computational means of verifying the condition given by equation 6. Under appropriate assumptions on the nature of the parameter dependence and on the structure of the parameter set Θ, however, sufficient conditions for satisfying equation 6 may be expressed as matrix inequalities which are linear in the variable P and which are independent of the parameter θ. This is extremely useful because it allows the use of commercially avaliable convex optimization techniques for the evaluation of the performance conditions. Theorem 3.2 provides an example of these conditions under the assumption that the elements of the system coefficient matrices are rational functions of the elements of θ and that the parameter set Θ is a hypercube in $\mathbb{R}^s$. The theorem states a sufficient condition for satisfying equation 6 for all $\theta \in \Theta$. The sufficient condition is a matrix inequality independent of θ and linear in P.

Theorem 3.2 *Given $\vartheta > 0$ and a compact set*

$$\Theta := \left\{ \theta \,\middle|\, \sup_{1 \le i \le s} |\theta_i| \le \vartheta \right\} \subset \mathbb{R}^s$$

Let $\Delta(\theta) := diag(\delta_1, \ldots, \delta_m)$ where for each $j = 1, \ldots, m$, $\delta_j = \theta_i$ for some $1 \le i \le s$. Suppose that the elements of A and B are rational functions of the elements of θ so that

$$\begin{aligned} A(\theta) &= A_s + B_p(I - \Delta(\theta)D_{qp})^{-1}\Delta(\theta)C_q \\ B(\theta) &= B_w + B_p(I - \Delta(\theta)D_{qp})^{-1}\Delta(\theta)D_{qw} \end{aligned} \tag{17}$$

for constant matrices $A_s \in \mathbb{R}^{n \times n}$, $B_p \in \mathbb{R}^{n \times m}$, $B_w \in \mathbb{R}^{n \times n_w}$, $C_q \in \mathbb{R}^{m \times n}$, $D_{qp} \in \mathbb{R}^{m \times m}$ and $D_{qw} \in \mathbb{R}^{m \times n_w}$.

If there exist constants $\alpha > 0$ and $\beta \geq 0$, a positive definite matrix $P \in \mathbb{R}^{n\times n}$ and diagonal positive semi-definite matrices Λ and Π such that

$$\begin{bmatrix} A_s'P + PA_s + (\alpha+\beta)P + C_q'\Lambda C_q & PB_p + C_q'\Lambda D_{qp} & PB_w & PB_p \\ B_p'P + D_{qp}'\Lambda C_q & D_{qp}'\Lambda D_{qp} - \frac{1}{\vartheta^2}\Lambda & 0 & 0 \\ B_w'P & 0 & D_{qw}'\Pi D_{qw} - \alpha I & D_{qw}'\Pi D_{qp} \\ B_p'P & 0 & D_{qp}'\Pi D_{qw} & D_{qp}'\Pi D_{qp} - \frac{1}{\vartheta^2}\Pi \end{bmatrix} \leq 0 \tag{18}$$

then for all $\theta \in \Theta$,

$$A'(\theta)P + PA(\theta) + (\alpha+\beta)P + \frac{1}{\alpha}PB(\theta)B'(\theta)P \leq 0 \tag{19}$$

The proof of this theorem is a straightforward application of the results presented in [6]. The proof is rather long and not very interesting; it is included in the appendix for completeness.

Note that inequality 18, coupled with inequality 5 are LMIs in P (for fixed α and β), $1/\gamma$ and the slack variables Λ and Π. This suggests that the performance of the LPV system may be characterized by fixing α and β at various operating points and solving the corresponding sequence of convex optimization problems

$$\begin{aligned} \Gamma_{\alpha,\beta} &= \min\nolimits_{P=P'>0,\Lambda\geq 0,\Pi\geq 0} \gamma \\ s.t. & \quad (18),\ (6) \end{aligned}$$

The results may be used to construct a family of 'level curves' which plot performance versus α for various levels of β. These curves may then be used to characterize the tradeoff between amplitude and decay rate. Note, however, that solutions $\Gamma_{\alpha,\beta}$ are not guaranteed to be convex functions of α or β. Similar observations were reported in [13]. As in those results, empirical evidence suggests that the function is unimodal. An example illustrating this approach will be found in section 5.

4 State Feedback Synthesis

This section concerns the synthesis of constant gain state feedback controllers so that a controlled LPV system will meet bounded amplitude performance specifications. The open loop LPV system is defined for a parameter set $\tilde{\Theta}$ of the form

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(\theta(t)) & B_1(\theta(t)) & B_2(\theta(t)) \\ C_1 & 0 & D_{12} \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix} \tag{20}$$

where $\theta \in \mathcal{F}_{\tilde{\Theta}}$. Here, $A : \mathbb{R}^s \to \mathbb{R}^{n \times n}$, $B_1 : \mathbb{R}^s \to \mathbb{R}^{n \times n_u}$ and $B_2 : \mathbb{R}^s \to \mathbb{R}^{n \times n_w}$ are continuous mappings, $C_1 \in \mathbb{R}^{n_z \times n}$, I is an $n \times n$ identity matrix and $D_{12} \in \mathbb{R}^{n_z \times n_u}$.

The finite-horizon performance objective is to synthesize a constant gain state feedback controller $K \in \mathbb{R}^{n_u \times n}$ such that for any $\theta \in \mathcal{F}_{\Theta}[0, T]$, the LPV system under feedback

$$u(t) = Kx(t) \tag{21}$$

satisfies

$$z'(t)z(t) \leq \gamma^2 \tag{22}$$

for all $t \in [0, T]$.

Under the control specified in equation 21, the closed loop takes the form

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A(\theta(t)) + B_2(\theta(t))K & B_1(\theta(t)) \\ C_1 + D_{12}K & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \tag{23}$$

where $\theta \in \mathcal{F}_{\tilde{\Theta}}$, and so falls into the class of LPV systems considered in the previous section. The approach taken to derive appropriate state feedback controllers is to find controllers $K \in \mathbb{R}^{n_u \times n}$ so that the LPV system in equation 23 satisfies the performance conditions of theorem 3.1.

Simple application of those results, however, results in synthesis conditions which are nonlinear in P and K; these nonlinearities cannot be removed through simple use of Schur complements as before. The primary result of this section is to show that a standard reparameterization

$$P = Q^{-1} \qquad \text{and} \qquad K = VQ^{-1}$$

results in synthesis equations linear in Q and V.

Theorem 4.1 *Given $\gamma > 0$ and an LPV system $\Sigma(\tilde{\Theta}, A, B, C, D)$ with*

$$B(\theta) = \begin{bmatrix} B_1(\theta) & B_2(\theta) \end{bmatrix}, \ C(\theta) = \begin{bmatrix} C_1 \\ I \end{bmatrix} \ \text{and} \ D(\theta) = \begin{bmatrix} 0 & D_{12} \\ 0 & 0 \end{bmatrix}$$

where $B_1 : \mathbb{R}^s \to \mathbb{R}^{n \times n_u}$ and $B_2 : \mathbb{R}^s \to \mathbb{R}^{n \times n_w}$ are continuous mappings, $C_1 \in \mathbb{R}^{n_z \times n}$, I is an $n \times n$ identity matrix and $D_{12} \in \mathbb{R}^{n_z \times n_u}$. Let $\Theta \subseteq \tilde{\Theta}$ be a compact subset. Then for constants $\alpha > 0$ and $\beta \geq 0$, there exists a positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a matrix $K \in \mathbb{R}^{n_u \times n}$ such that

$$P \geq \frac{1}{\gamma^2}(C_1 + D_{12}K)'(C_1 + D_{12}K) \tag{24}$$

and

$$[A(\theta) + B_2(\theta)K]' P + P [A(\theta) + B_2(\theta)K] + (\alpha + \beta)P + \frac{1}{\alpha} P B_1(\theta) B_1'(\theta) P \leq 0 \tag{25}$$

for all $\theta \in \Theta$ if and only if there exists a positive definite matrix $Q \in \mathbb{R}^{n \times n}$ and a matrix $V \in \mathbb{R}^{n_u \times n}$ such that

$$\begin{bmatrix} Q & QC_1' + V'D_{12}' \\ C_1 Q + D_{12} V & \gamma^2 I \end{bmatrix} \geq 0 \tag{26}$$

and

$$QA'(\theta) + A(\theta)Q + (\alpha + \beta)Q + \frac{1}{\alpha} B_1(\theta) B_1'(\theta) + B_2(\theta)V + V'B_2'(\theta) \leq 0 \tag{27}$$

for all $\theta \in \Theta$.

Proof: A positive definite matrix $Q \in \mathbb{R}^{n \times n}$ and a matrix $V \in \mathbb{R}^{n_u \times n}$ exist if and only if there exists a positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a matrix $K \in \mathbb{R}^{n_u \times n}$ satisfying

$$P = Q^{-1} \qquad \text{and} \qquad K = VQ^{-1} \tag{28}$$

From the theory of Schur complements, a positive definite matrix $Q \in \mathbb{R}^{n\times n}$ and a matrix $V \in \mathbb{R}^{n_u \times n}$ satisfy

$$\begin{bmatrix} Q & QC_1' + V'D_{12}' \\ C_1Q + D_{12}V & \gamma^2 I \end{bmatrix} \geq 0$$

if and only if

$$Q - \frac{1}{\gamma^2}(QC_1' + V'D_{12}')(C_1Q + D_{12}V) \geq 0 \tag{29}$$

Substituting from equation 28, equation 29 becomes

$$P^{-1} - \frac{1}{\gamma^2}P^{-1}(C_1 + D_{12}K)'(C_1 + D_{12}K)P^{-1} \geq 0$$

which holds if and only if

$$P \geq \frac{1}{\gamma^2}(C_1 + D_{12}K)'(C_1 + D_{12}K)$$

Furthermore, using equation 28, for constants $\alpha > 0$ and $\beta \geq 0$,

$$\begin{aligned} & [A(\theta) + B_2(\theta)K]' P + P[A(\theta) + B_2(\theta)K] + (\alpha+\beta)P + \tfrac{1}{\alpha}PB_1(\theta)B_1'(\theta)P \\ = \; & [A(\theta) + B_2(\theta)K]' Q^{-1} + Q^{-1}[A(\theta) + B_2(\theta)K] + (\alpha+\beta)Q^{-1} + \tfrac{1}{\alpha}Q^{-1}B_1(\theta)B_1'(\theta)Q^{-1} \\ = \; & Q^{-1}\left(QA'(\theta) + A(\theta)Q + (\alpha+\beta)Q + \tfrac{1}{\alpha}B_1(\theta)B_1'(\theta) + B_2(\theta)V + V'B_2'(\theta)\right)Q^{-1} \end{aligned}$$

Since $Q^{-1} > 0$,

$$[A(\theta) + B_2(\theta)K]' P + P[A(\theta) + B_2(\theta)K] + (\alpha+\beta)P + \frac{1}{\alpha}PB_1(\theta)B_1'(\theta)P \leq 0$$

if and only if

$$QA'(\theta) + A(\theta)Q + (\alpha+\beta)Q + \frac{1}{\alpha}B_1(\theta)B_1'(\theta) + B_2(\theta)V + V'B_2'(\theta) \leq 0$$

$\square$

Theorem 4.1 presents a set of conditions on the controller gains for closed loop LPV system performance. By reparameterizing P and K according to equation 28, one obtains synthesis constraints which are linear in the unknowns Q and V. However, the constraints are θ-dependent which limits their usefulness from a computational standpoint. As with the analysis results presented earlier, under appropriate assumptions on the nature of the parameter dependence and on the structure of the parameter set Θ, a sufficient condition for satisfying equation 27 may be expressed as a matrix inequality which is independent of the parameter θ and linear in the variables Q and V. As a result, the synthesis of constant gain state-feedback controllers amounts to solving a pair of matrix inequalities, which are independent of θ, for a positive definite matrix Q and a real matrix V. The following theorem states the matrix inequality condition for equation 27.

Theorem 4.2 *Given $\vartheta > 0$ and a compact set*

$$\Theta := \left\{ \theta | \sup_{1 \leq i \leq s} |\theta_i| \leq \vartheta \right\} \subset \mathbb{R}^s$$

Let $\Delta(\theta) := diag(\delta_1, \ldots, \delta_m)$ where for each $j = 1, \ldots, m$, $\delta_j = \theta_i$ for some $1 \leq i \leq s$. Suppose that the elements of A, B_1 and B_2 are rational functions of the elements of θ so that

$$\begin{aligned} A(\theta) &= A_s + B_p(I - \Delta(\theta)D_{qp})^{-1}\Delta(\theta)C_q \\ B_1(\theta) &= B_w + B_p(I - \Delta(\theta)D_{qp})^{-1}\Delta(\theta)D_{qw} \\ B_2(\theta) &= B_u + B_p(I - \Delta(\theta)D_{qp})^{-1}\Delta(\theta)D_{qu} \end{aligned} \tag{30}$$

for constant matrices $A_s \in \mathbb{R}^{n \times n}$, $B_p \in \mathbb{R}^{n \times m}$, $B_w \in \mathbb{R}^{n \times n_w}$, $C_q \in \mathbb{R}^{m \times n}$, $D_{qp} \in \mathbb{R}^{m \times m}$, $D_{qw} \in \mathbb{R}^{m \times n_w}$ and $D_{qu} \in \mathbb{R}^{m \times n_u}$.

If there exist constants $\alpha > 0$ and $\beta \geq 0$, a positive definite matrix $Q \in \mathbb{R}^{n \times n}$, a matrix $V \in \mathbb{R}^{n_u \times n}$ and diagonal positive semi-definite matrices Λ and Π such that

$$\begin{bmatrix} \begin{array}{c} QA_s' + A_sQ + (\alpha+\beta)Q \\ +V'B_u' + B_uV + B_p\Lambda B_p' \end{array} & QC_p' + V'D_{qu}' + B_p\Lambda D_{qp}' & B_w & B_p \\ D_{qu}V + C_qQ + D_{qp}\Lambda B_p' & D_{qp}\Lambda D_{qp}' - \frac{1}{\vartheta^2}\Lambda & 0 & 0 \\ B_w' & 0 & D_{qw}'\Pi D_{qw} - \alpha I & D_{qw}'\Pi D_{qp} \\ B_p' & 0 & D_{qp}'\Pi D_{qw} & D_{qp}'\Pi D_{qp} - \frac{1}{\vartheta^2}\Pi \end{bmatrix} \leq 0 \tag{31}$$

then for all $\theta \in \Theta$,

$$QA'(\theta) + A(\theta)Q + (\alpha+\beta)Q + \frac{1}{\alpha}B_1(\theta)B_1'(\theta) + B_2(\theta)V + V'B_2'(\theta) \leq 0 \tag{32}$$

The proof is similar to the proof of theorem 3.2; it is contained in the appendix.

Observe that the form of the matrix inequality in equation 31 is similar to that of equation 18. For fixed values of α and β, the inequality becomes linear in its variables. One approach to using this expression for control synthesis is to follow a procedure similar to the one outlined after theorem 3.2 and produce a set of performance curves reflecting choices of α and β. These curves reflect the achievable performance for the open loop system under the structural constraints imposed on the controller, e.g. constant gain state feedback. Given a desired performance level γ, the controller corresponding to a pair (α, β) which lies below the performance constraint (if any) is selected. This procedure will be illustrated in section 5.

It is important to note that the controller is chosen to minimize the feasible value of γ for fixed α and β and teh LMI constraints. The achieved performance may actually be better; this is determined using the analysis tools of section 3.

5 Example

In this section, the theoretical results of sections 3 and 4 are used to design a controller for a nonlinear process control model taken from [10]. This example is intended to illustrate the theoretical results developed above from a practical standpoint; the LMIs from theorems 3.2 and 4.2 are used for analysis and synthesis.

The plant dynamics are given by

$$\begin{aligned} \dot{x}_{p1} &= -x_{p1} + u_1 \\ \dot{x}_{p2} &= -x_{p2} + (1 + x_{p1}^2) u_2 \end{aligned} \tag{33}$$

The variables $x_{p1} \in \mathbb{R}$ and $x_{p2} \in \mathbb{R}$ represent the state of the plant; $u_1 \in \mathbb{R}$ and $u_2 \in \mathbb{R}$ are the control inputs. Here, x_{p1} is the normalized temperature of the plant, x_{p2} is the normalized pressure of the plant, u_1 is the rate at which heat is applied to the plant and u_2 is the rate at which a pneumatic piston is displaced.

The plant is to be used in a process in which the plant must be cycled through various operating points in the plant state space. For the illustrative purposes of this example, the control objective under consideration is to move the plant from an initial state near the operating point $(x_{p1}, x_{p2}) = (2.5, 2)$ to a point near $(x_{p1}, x_{p2}) = (1, 3)$ in 1 second according to the reference trajectory generated by the following dynamical reference model by

$$\begin{aligned} \dot{x}_{m1} &= x_{m1} - 1.63 - x_{m1}/(1 + 0.5 \sin 10(x_{m1} - 1.63)) \\ \dot{x}_{m2} &= 1 \end{aligned} \tag{34}$$

This reference model state trajectory is depicted in figure 1.

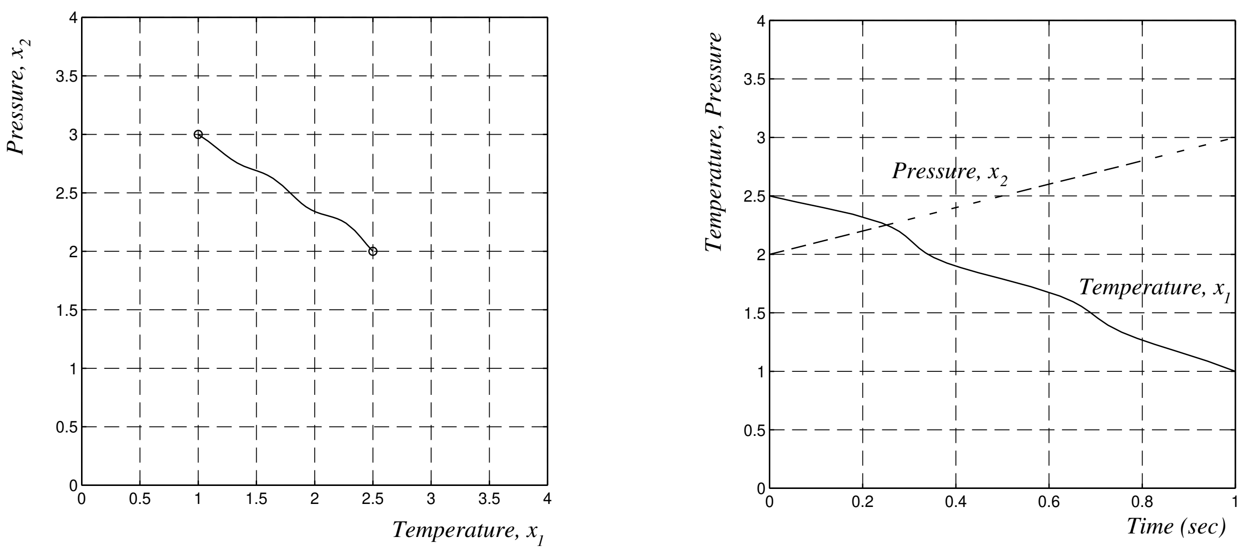

Figure 1: Reference trajectory for process control example

Defining the state error, $x := x_m - x_p$, the control objective is to synthesize an affine control

$$u = Kx + \tilde{u} \tag{35}$$

so that the performance output variable $z = C_1 x + D_{12} u$ satisfies the amplitude bound

$$\sup_{t\in[0,1]} \|z(t)\| \leq \gamma \tag{36}$$

where the performance weights are chosen as $C_1 = I$ and $D_{12} = 0.1I$.

An LPV model of the error dynamics is derived by linearizing the plant at points along the reference trajectory and grouping the state and control dependent elements of the realization matrices into a parameter vector

$$\begin{bmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{bmatrix} = \begin{bmatrix} x_{m1} - x_{m1}/(2 + \sin 10(x_{m1} - 1.63)) \\ x_{m2} + (2x_{m1}x_1 - x_1^2)u_2 \\ x_{m1}^2/3 \end{bmatrix}. \tag{37}$$

The resulting LPV plant model is given by

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(\theta) & B_w(\theta) & B_u(\theta) \\ C_1 & 0 & D_{12} \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{38}$$

where $w = 1$ is introduced as a fictitous disturbance and

$$A(\theta) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B_w(\theta) = \begin{bmatrix} \theta_1 - 1.63 \\ \theta_2 + 1 \end{bmatrix} \text{ and } B_u(\theta) = \begin{bmatrix} -1 & 0 \\ 0 & -\theta_3 - 1 \end{bmatrix}. \tag{39}$$

The bias term, $\tilde{u}$, is fixed by choosing a nominal design point $\tilde{\theta}$ and examining the desired behaviour of the controlled plant and using

$$\tilde{u} = -B_u(\tilde{\theta})^{-1} B_w(\tilde{\theta}) w \tag{40}$$

Defining

$$\tilde{B}_w(\theta) = B_w(\theta) - B_u(\theta) B_u(\tilde{\theta})^{-1} B_w(\tilde{\theta}) \tag{41}$$

and $\hat{\theta} := \theta - \tilde{\theta}$, the LPV system dynamics may be rewritten as

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(\tilde{\theta}) + A(\hat{\theta}) & \tilde{B}_w(\tilde{\theta}) + \tilde{B}_w(\hat{\theta}) & B_u(\tilde{\theta}) + B_u(\hat{\theta}) \\ C_1 & 0 & D_{12} \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u - \tilde{u} \end{bmatrix} \tag{42}$$

Because the error system may be written as in equation 42, the synthesis problem and analysis problem are reduced to the state feedback problem discussed in section 4. The LMIs of theorems 3.2 and 4.2 may be used by rewriting the dynamics one last time as

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \left[\left[\begin{array}{c|c|c} A_s & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ \hline C_y & D_{yw} & D_{yu} \end{array} \right] + \left[\begin{array}{c} B_p \\ \hline D_{zp} \\ \hline D_{yp} \end{array} \right] [I - \Delta D_{qp}]^{-1} \Delta \left[\begin{array}{c|c|c} C_q & D_{qw} & D_{qu} \end{array} \right] \right] \begin{bmatrix} x \\ w \\ u - \tilde{u} \end{bmatrix}$$

where

$$\left[\begin{array}{c|c|c} A_s & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ \hline C_y & D_{yw} & D_{yu} \end{array} \right] = \left[\begin{array}{c|c|c} A(\tilde{\theta}) & \tilde{B}_w(\tilde{\theta}) & B_u(\tilde{\theta}) \\ \hline C_1 & 0 & D_{12} \\ \hline I & 0 & 0 \end{array} \right], \quad D_{qp} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{43}$$

$$\left[\begin{array}{c} B_p \\ \hline D_{zp} \\ \hline D_{yp} \end{array} \right] = \left[\begin{array}{ccc} \sqrt{2} & 0 & 0 \\ 0 & 1 & k_\theta \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \text{ and } \left[\begin{array}{c|c|c} C_q & D_{qw} & D_{qu} \end{array} \right] = \left[\begin{array}{cc|c|cc} 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -k_\theta & 0 & -\sqrt{3} \end{array} \right] \tag{44}$$

with $\Delta = diag(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ and $k_\theta = \sqrt{3(\tilde{\theta}_2 + 1)/(\tilde{\theta}_3 + 1)}$

The derived state space data presented above was used in the LMI of theorem 4.2 to study the achievable performance of state feedback controllers. Using the parameter vector of equation 37, a nominal parameter variation was examined graphically and used to determine that $\vartheta = 1$ would be sufficient for the parameter set, Θ, assumed for the theorem. The LMI problems were solved using the Matlab with LMI Control Toolbox.

Figure 2 shows the guaranteed performance level for values of α and β. These curves were computed according to the procedure outlined at the end of section 4. β was fixed and the matrices Q, V, Λ and Π were found which corresponded to the minimum feasible value of γ subject to the LMI constraints of theorem 4.2 for various values of α. The resulting Q and V were used to compute a controller $K = VQ^{-1}$ for each solution.

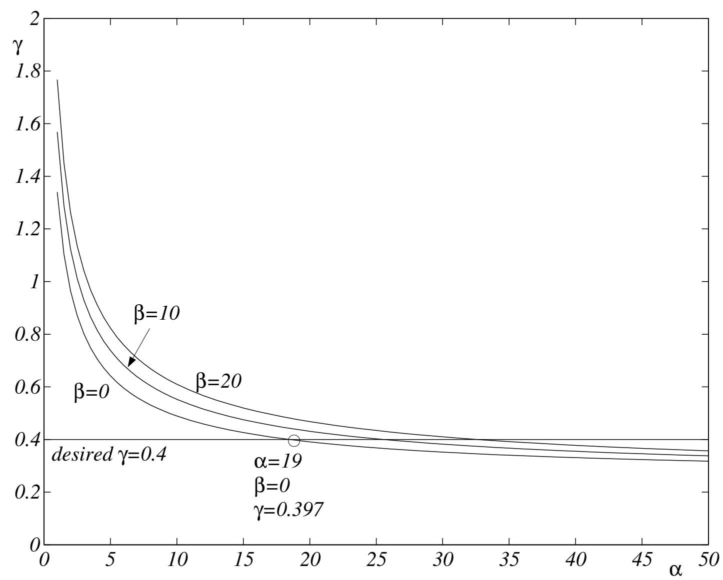

Figure 2: Guaranteed performance curves for open loop system.

For the purposes of this example, the $\beta = 0$ curve was used to choose a controller. For a desired performance level of $\gamma = 0.4$, the minimum value of α which guaranteed performance was read from figure 2 to be $\alpha = 19$. This choice is indicated on the plot. The corresponding controller was determined to be

$$K = \begin{bmatrix} 26.99 & 0 \\ 0 & 18.42 \end{bmatrix}$$

As noted at the end of section 4, while this controller is optimal with respect to the LMI synthesis constraints for the particular choice of $\alpha = 19$ and $\beta = 0$, the achieved performance may be even better than indicated by the synthesis curves in figure 2. Figure 3 depicts the closed loop curves described at the end of section 3 for the analysis LMIs of theorem 3.2.

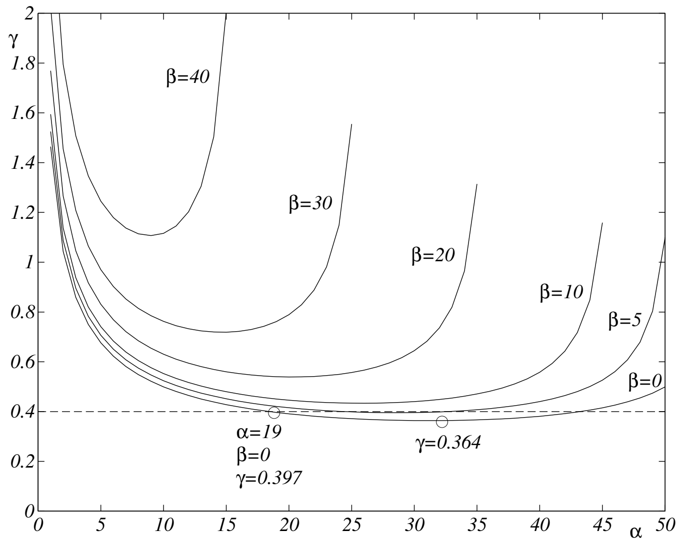

Figure 3: Achieved performance curves for closed loop system.

Two important quantities may be determined from the level curves of figure 3. The first is that the actual achieved amplitude bound is $\gamma = 0.364$; this is the bound which will be observed whenever the initial state satisfies the performance bounds. The second observation concerns initial states which violate the performance constraint $\gamma = 0.4$; in such cases, the norm of the performance variable will be bounded by a decaying exponential with rate $\beta \geq 5$. This second observation is made by noting that the performance curves for $\beta \leq 5$ dip below the $\gamma = 0.4$ level, while the curves for higher levels of β do not. Experiments with other designs show similar results. Note that the level curves appear to be convex. Also not that the the point corresponding to $(\alpha, \beta) = (19, 0)$ corresponds to $\gamma = 0.3950$ in both figures 2 and 3, as expected.

Finally, the controller was simulated in the closed loop system with performance violating initial conditions. The resulting performance variable norm, $\|z(t)\|$, is shown in figure 4 along with the bounding functions specified by theorem 3.1. The exponential decay rate used for the bounding exponential corresponded to $\beta = 5$. Note that once the performance level of $\gamma = 0.4$ is achieved, it is maintained for the remainder of the trajectory.

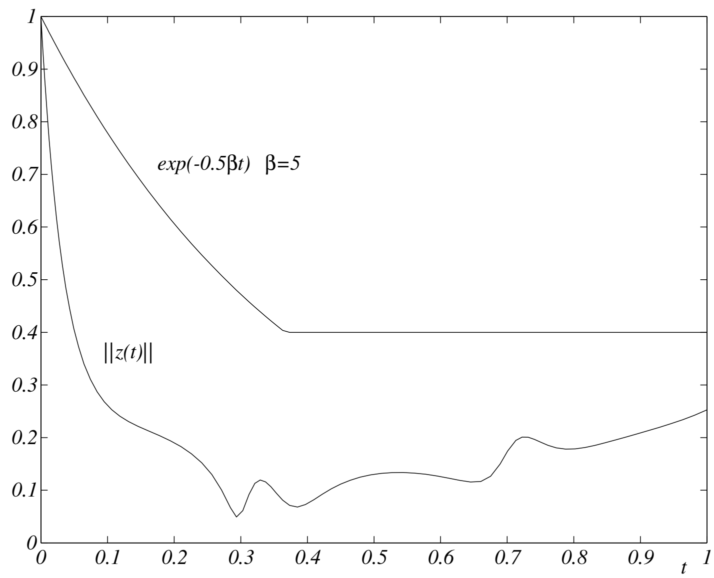

Figure 4: Simulated closed loop performance

6 Conclusions

This paper has presented results for finite horizon bounded amplitude performance of LPV systems. The first set of main results, concerning the *analysis* of LPV system performance, culminated in theorems 3.1 and 3.2; the second set of main results concerning the *synthesis* of LPV system performance, culminated in theorems 4.1 and 4.2. These results demonstrate that verification of induced-$\mathcal{L}_\infty$ performance for LPV systems amounts to checking the feasibility of a set of matrix inequalities. Synthesis of state feedback controllers for induced-$\mathcal{L}_\infty$ performance of a specific class of LPV systems amounts to finding a feasible point of a similarly structured set of matrix inequalities.

As noted in section 1, the induced-$\mathcal{L}_\infty$ results are extensions of results reported in [13] for

LTI systems. One limitation of those results was that the controller optimization problem had not been proven convex. However, the authors report that numerical evidence supports such a conjecture, so this drawback does not appear to be serious.

Continued work along these lines involves extensions to output feedback controllers, along the lines presented in [1] and self-scheduled controllers long the same lines as [14], [3] and [2]. Because these results are derived for finite horizon problems, the results also apply to multiple control agent switching systems [12].

A Proofs of Results

Proof of Lemma 3.1: $x'C'Cx \leq \gamma^2$ for all $x \in \mathbb{R}^n$ such that $x'Px \leq 1$ if and only if $\gamma^2 - x'C'Cx \geq 0$ for all $x \in \mathbb{R}^n$ such that $1 - x'Px \geq 0$. Applying the $\mathcal{S}$-procedure [20], which is lossless in this case, this assertion is true if and only if there exists an $\alpha \geq 0$ such that

$$1 - \frac{1}{\gamma^2}x'C'Cx - \alpha(1 - x'Px) \geq 0 \text{ for all } x \in \mathbb{R}^n$$

which simplifies to

$$x' \left[\alpha P - \frac{1}{\gamma^2}C'C\right] x + (1-\alpha) \geq 0 \text{ for all } x \in \mathbb{R}^n \tag{45}$$

Since equation 45 must be true even for $x = 0$, it can be true if and only if $\alpha \leq 1$. Equation 45 is true if and only if there exists a constant $\alpha \in [0, 1]$ such that

$$\alpha P \geq \frac{1}{\gamma^2}C'C$$

But this is true if and only if $\gamma^2 P \geq C'C$. □

Proof of Theorem 3.2: Suppose that there exist constants $\alpha > 0$ and $\beta \geq 0$ and a positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that for all $\theta \in \Theta$, equation 19 is satisfied. Since

$PB(\theta)B'(\theta)P \geq 0$ for all $\theta \in \Theta$, Schur complements implies that equation 19 is satisfied for all $\theta \in \Theta$ if and only if

$$\begin{bmatrix} A'(\theta)P + PA(\theta) + (\alpha+\beta)P & PB(\theta) \\ B'(\theta)P & -\alpha I \end{bmatrix} \leq 0 \tag{46}$$

for all $\theta \in \Theta$. But equation 46 is true if and only if

$$\xi'(A'(\theta)P + PA(\theta) + (\alpha+\beta)P)\xi + \nu' B'(\theta)P\xi + \xi' PB(\theta)\nu - \alpha\nu'\nu \leq 0 \tag{47}$$

for all $\theta \in \Theta$ and all $\xi \in \mathbb{R}^n$, $\nu \in \mathbb{R}^{n_w}$. Substituting from equation 17, one concludes that equation 19 is satisfies for all $\theta \in \Theta$ if and only if

$$\begin{aligned} \xi'(A_s'P + PA_s + (\alpha+\beta)P)\xi \quad &+ \quad \nu' B_w' P\xi + \xi' PB_w\nu - \alpha\nu'\nu \\ &+ \quad \xi' C_q' \Delta'(\theta)M'(\theta)B_p'P\xi + \xi' PB_pM(\theta)\Delta(\theta)C_q\xi \\ &+ \quad \nu' D_{qw}'\Delta'(\theta)M'(\theta)B_p'P\xi + \xi' PB_pM(\theta)\Delta(\theta)D_{qw}\nu \leq 0 \end{aligned} \tag{48}$$

for all $\theta \in \Theta$, $\xi \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^{n_w}$, where $M(\theta) := (I - \Delta(\theta)D_{qp})^{-1}$.

Now introduce the variables

$$\eta := M(\theta)\Delta(\theta)C_q\xi \quad \text{and} \quad \mu := M(\theta)\Delta(\theta)D_{qw}\nu$$

which may be rewritten, via algebra, as

$$\eta = \Delta(\theta)(C_q\xi + D_{qp}\eta) \quad \text{and} \quad \mu = \Delta(\theta)(D_{qw}\nu + D_{qp}\mu)$$

Since $\Delta(\theta)$ is diagonal, one may write

$$\begin{aligned} \eta' e_j e_j' \eta &= \delta_j^2 (C_q\xi + D_{qp}\eta)' e_j e_j' (C_q\xi + D_{qp}\eta) \qquad & j = 1, \ldots, m \\ \mu' e_j e_j' \mu &= \delta_j^2 (D_{qw}\nu + D_{qp}\mu)' e_j e_j' (D_{qw}\nu + D_{qp}\mu) \qquad & j = 1, \ldots, m \end{aligned}$$

where e_j ($j = 1, \ldots, m$) denotes the jth standard basis vector in $\mathbb{R}^m$. Since $\theta \in \Theta$ if and only if

$$\sup_{1 \leq i \leq s} |\theta_i| \leq \vartheta$$

and since for any $j = 1, \ldots, m$, $\delta_j = \theta_i$ for some $1 \leq i \leq s$ equation 48 is satisfied for all $\theta \in \Theta$, $\xi \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^{n_w}$ if and only if

$$\begin{aligned} \xi'(A_s'P + PA_s + (\alpha+\beta)P)\xi &+ \nu' B_w' P\xi + \xi' P B_w \nu - \alpha \nu'\nu \\ &+ \eta' B_p' P \xi + \xi' P B_p \eta + \mu' B_p' P \xi + \xi' P B_p \mu \leq 0 \end{aligned} \tag{49}$$

for all $\theta \in \Theta$, $\xi \in \mathbb{R}^n$, $\nu \in \mathbb{R}^{n_w}$, $\eta \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^m$ such that

$$\eta' e_j e_j' \eta \leq \vartheta^2 (C_q \xi + D_{qp}\eta)' e_j e_j' (C_q \xi + D_{qp} \eta) \quad j = 1, \ldots, m \tag{50}$$

$$\mu' e_j e_j' \mu \leq \vartheta^2 (D_{qw} \nu + D_{qp}\mu)' e_j e_j' (D_{qw} \nu + D_{qp} \mu) \quad j = 1, \ldots, m \tag{51}$$

Defining $\zeta := (\xi', \eta', \nu', \mu')'$, and

$$F_0 := \begin{bmatrix} A_s'P + PA_s + (\alpha+\beta)P & PB_p & PB_w & PB_p \\ B_p'P & 0 & 0 & 0 \\ B_w'P & 0 & -\alpha I & 0 \\ B_p'P & 0 & 0 & 0 \end{bmatrix}$$

$$F_{\eta j} := \begin{bmatrix} -C_q' e_j e_j' C_q & -C_q' e_j e_j' D_{qp} & 0 & 0 \\ -D_{qp}' e_j e_j' C_q & \frac{1}{\vartheta^2} e_j e_j' - D_{qp}' e_j e_j' D_{qp} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad j = 1, \ldots, m$$

$$F_{\mu j} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -D_{qw}' e_j e_j' D_{qw} & -D_{qw}' e_j e_j' D_{qp} \\ 0 & 0 & -D_{qp}' e_j e_j' D_{qw} & \frac{1}{\vartheta^2} e_j e_j' - D_{qp}' e_j e_j' D_{qp} \end{bmatrix} \quad j = 1, \ldots, m$$

Then the conditions given by equations 49-51 become

$$\zeta' F_0 \zeta \leq 0 \tag{52}$$

for all $\zeta \in \mathbb{R}^{2m+n+n_w}$ such that

$$\zeta' F_{\eta j} \zeta \leq 0 \quad j = 1, \ldots, m \tag{53}$$

$$\zeta' F_{\mu j} \zeta \leq 0 \quad j = 1, \ldots, m \tag{54}$$

A sufficient condition for equations 52-54 is the existence of nonnegative constants $\lambda_1, \ldots, \lambda_m$ and $\pi_1, \ldots, \pi_m$ such that

$$\zeta' \left(F_0 - \sum_{j=1}^{m} \lambda_j F_{\eta j} - \sum_{j=1}^{m} \pi_j F_{\mu j} \right) \zeta \leq 0$$

for all $\zeta \in \mathbb{R}^{2m+n+n_w}$ (this is just an application of the $\mathcal{S}$-procedure) or, equivalently,

$$F_0 - \sum_{j=1}^{m} \lambda_j F_{\eta j} - \sum_{j=1}^{m} \pi_j F_{\mu j} \leq 0 \tag{55}$$

Defining $\Lambda = diag(\lambda_1, \ldots, \lambda_m)$ and $\Pi = diag(\pi_1, \ldots, \pi_m)$, it is easily seen that equation 55 is equivalent to the matrix inequality in equation 18, concluding the proof. $\square$

Proof of Theorem 4.2: Suppose that there exist constants $\alpha > 0$ and $\beta \geq 0$, a positive definite matrix $Q \in \mathbb{R}^{n \times n}$ and a matrix $V \in \mathbb{R}^{n_u \times n}$ such that for all $\theta \in \Theta$, equation 32 is satisfied. Since $B(\theta)B'(\theta) \geq 0$ for all $\theta \in \Theta$, Schur complements implies that equation 32 is satisfied for all $\theta \in \Theta$ if and only if

$$\begin{bmatrix} QA'(\theta) + A(\theta)Q + (\alpha + \beta)Q + V'B_2'(\theta) + B_2(\theta)V & B_1(\theta) \\ B_1'(\theta) & -\alpha I \end{bmatrix} \leq 0 \tag{56}$$

for all $\theta \in \Theta$. But equation 56 is true if and only if

$$\xi'(QA'(\theta) + A(\theta)Q + (\alpha+\beta)Q + V'B_2'(\theta) + B_2(\theta)V)\xi + \nu' B_1'(\theta)\xi + \xi' B_1(\theta)\nu - \alpha\nu'\nu \leq 0 \tag{57}$$

for all $\theta \in \Theta$ and all $\xi \in \mathbb{R}^n$, $\nu \in \mathbb{R}^{n_w}$. Substituting from equation 30, one concludes that equation 32 is satisfies for all $\theta \in \Theta$ if and only if

$$
\begin{aligned}
\xi'(QA_s' + A_sQ + (\alpha+\beta)Q \quad &+ \quad V'B_u' + B_uV)\xi + \nu'B_w'\xi + \xi'B_w\nu - \alpha\nu'\nu \\
&+ \quad \xi'QC_q'\Delta'(\theta)M'(\theta)B_p'\xi + \xi'B_pM(\theta)\Delta(\theta)C_qQ\xi \\
&+ \quad \xi'V'D_{qu}'\Delta'(\theta)M'(\theta)B_p'\xi + \xi'B_pM(\theta)\Delta(\theta)D_{qu}V\xi \\
&+ \quad \nu'D_{qw}'\Delta'(\theta)M'(\theta)B_p'\xi + \xi'B_pM(\theta)\Delta(\theta)D_{qw}\nu \leq 0
\end{aligned}
\tag{58}
$$

for all $\theta \in \Theta$, $\xi \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^{n_w}$, where $M(\theta) := (I - \Delta(\theta)D_{qp})^{-1}$.

Now introduce the variables

$$\eta := \Delta'(\theta)M'(\theta)B_p'\xi \quad \text{and} \quad \mu := M(\theta)\Delta(\theta)D_{qw}\nu$$

which may be rewritten, via algebra, as

$$\eta = \Delta'(\theta)(B_p'\xi + D_{qp}'\eta) \quad \text{and} \quad \mu = \Delta(\theta)(D_{qw}\nu + D_{qp}\mu)$$

Since $\Delta(\theta)$ is diagonal, one may write

$$
\begin{aligned}
\eta'e_je_j'\eta &= \delta_j^2(B_p'\xi + D_{qp}'\eta)'e_je_j'(B_p'\xi + D_{qp}'\eta) \qquad & j = 1, \ldots, m \\
\mu'e_je_j'\mu &= \delta_j^2(D_{qw}\nu + D_{qp}\mu)'e_je_j'(D_{qw}\nu + D_{qp}\mu) \qquad & j = 1, \ldots, m
\end{aligned}
$$

where e_j $(j = 1, \ldots, m)$ denotes the jth standard basis vector in $\mathbb{R}^m$. Since $\theta \in \Theta$ if and only if

$$\sup_{1 \leq i \leq s} |\theta_i| \leq \vartheta$$

and since for any $j = 1, \ldots, m$, $\delta_j = \theta_i$ for some $1 \leq i \leq s$ equation 58 is satisfied for all $\theta \in \Theta$, $\xi \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^{n_w}$ if and only if

$$
\begin{aligned}
\xi'(QA_s + A_sQ + (\alpha+\beta)Q \quad &+ \quad V'B_u' + B_uV)\xi + \nu'B_w'\xi + \xi'B_w\nu - \alpha\nu'\nu \\
&+ \quad \eta'(C_qQ + D_{qu}V)\xi + \xi'(QC_q' + V'D_{qu}')\eta + \mu'B_p'\xi + \xi'B_p\mu \leq 0
\end{aligned}
\tag{59}
$$

for all $\theta \in \Theta$, $\xi \in \mathbb{R}^n$, $\nu \in \mathbb{R}^{n_w}$, $\eta \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^m$ such that

$$\eta'e_je_j'\eta \leq \vartheta^2(B_p'\xi + D_{qp}'\eta)'e_je_j'(B_p'\xi + D_{qp}'\eta) \quad j = 1, \ldots, m \tag{60}$$

$$\mu'e_je_j'\mu \leq \vartheta^2(D_{qw}\nu + D_{qp}\mu)'e_je_j'(D_{qw}\nu + D_{qp}\mu) \quad j = 1, \ldots, m \tag{61}$$

Defining $\zeta := (\xi', \eta', \nu', \mu')'$, and

$$F_0 := \begin{bmatrix} QA_s' + A_sQ + (\alpha+\beta)Q + V'B_u' + B_uV & QC_q' + V'D_{qu}' & B_w & B_p \\ D_{qu}V + C_qQ & 0 & 0 & 0 \\ B_w' & 0 & -\alpha I & 0 \\ B_p' & 0 & 0 & 0 \end{bmatrix}$$

$$F_{\eta j} := \begin{bmatrix} -B_pe_je_j'B_p' & -B_pe_je_j'D_{qp}' & 0 & 0 \\ -D_{qp}e_je_j'B_p' & \frac{1}{\vartheta^2}e_je_j' - D_{qp}e_je_j'D_{qp}' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad j = 1, \ldots, m$$

$$F_{\mu j} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -D_{qw}'e_je_j'D_{qw} & -D_{qw}'e_je_j'D_{qp} \\ 0 & 0 & -D_{qp}'e_je_j'D_{qw} & \frac{1}{\vartheta^2}e_je_j' - D_{qp}'e_je_j'D_{qp} \end{bmatrix} \quad j = 1, \ldots, m$$

Then the conditions given by equations 59-61 become

$$\zeta' F_0 \zeta \leq 0 \tag{62}$$

for all $\zeta \in \mathbb{R}^{2m+n+n_w}$ such that

$$\zeta' F_{\eta j} \zeta \leq 0 \quad j = 1, \ldots, m \tag{63}$$

$$\zeta' F_{\mu j} \zeta \leq 0 \quad j = 1, \ldots, m \tag{64}$$

A sufficient condition for equations 62-64 is the existence of nonnegative constants $\lambda_1, \ldots, \lambda_m$ and $\pi_1, \ldots, \pi_m$ such that

$$\zeta' \left(F_0 - \sum_{j=1}^{m} \lambda_j F_{\eta j} - \sum_{j=1}^{m} \pi_j F_{\mu j} \right) \zeta \leq 0$$

for all $\zeta \in \mathbb{R}^{2m+n+n_w}$ (this is just an application of the $\mathcal{S}$-procedure) or, equivalently,

$$F_0 - \sum_{j=1}^{m} \lambda_j F_{\eta j} - \sum_{j=1}^{m} \pi_j F_{\mu j} \leq 0 \tag{65}$$

Defining $\Lambda = diag(\lambda_1, \ldots, \lambda_m)$ and $\Pi = diag(\pi_1, \ldots, \pi_m)$, it is easily seen that equation 65 is equivalent to the matrix inequality in equation 31, concluding the proof. $\square$

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