Performance Validation and Accomodation for Continuous-Time State Feedback Linear Control Systems

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Abstract

This technical report presents a fault detection and accomodation algorithm for continuous-time linear state feedback control systems with bounded input disturbances and bounded structured model uncertainty. Faults are declared when the system's observed state and state derivatives are inconsistent with a specified level of bounded amplitude performance. Faults are accomodated by using the ellipsoid algorithm to search for a set of state feedback gains which recover (if possible) the specified performance level.

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1 Introduction

A system fault is any change in the system which causes the system's performance to degrade below an acceptable level. Fault accomodation refers to algorithms which modify the system's controller in order to recover performance lost due a detected system fault. This paper presents a fault detection and accomodation algorithm for continuous-time linear state feedback control systems with bounded input disturbances and bounded structured model uncertainty. Our algorithms declare a system fault when the system's observed state and state derivative are inconsistent with the hypothesis that the system lies within a specified uncertainty set. Once a fault has been detected, an accomodating set of state feedback gains are determined by using the central cut ellipsoid algorithm.

An important issue in the design of fault detection and accomodation (FDA) algorithms concerns the algorithm's robustness to input disturbances and unmodeled plant dynamics. Early attempts at robust FDA, such as the parity-check [1] or observer-based methods [2] [3], detected fault signatures which had been orthogonalized to known disturbance directions. While providing some insensitivity to disturbances, these methods could exhibit considerable sensitivity to uncertainty in the nominal system model.

Recent work [4] [5] [6] in time-domain model validation provides another approach to fault detection which explicitly accounts for model uncertainty. These methods use observed input/output data from the plant to determine if the data is inconsistent with the hypothesis that the actual plant lies within a class of uncertainty models possessing a specified performance level. Current work in model validation has focused on discrete-time and sampled-data systems. These methods construct large convex programs whose feasibility determine whether or not the input/output data of the system is inconsistent with an assumed performance level as measured by the system's induced gain. While polynomial-time algorithms exist for solving such convex programs, the size of these convex programs makes the method somewhat cumbersome for on-line validation of the system.

This paper presents an approach to on-line fault detection based on techniques found in the model-validation and robust control literature. Our method applies to continuous-time state feedback control systems with bounded disturbances and bounded structured model uncertainties. Our method solves a convex program determining a quadratic functional of system state which is Lyapunov for all plants within the model set. The monotone decreasing nature of this functional for all admissible state trajectories is a necessary condition for the system to achieve a specified bounded-amplitude performance level. The fault detection algorithm monitors this quadratic functional on-line and declares a fault if it fails to decay at the specified rate. Our test therefore provides a sufficient condition for fault detection which is robust to bounded disturbances and structured model uncertainty. Once the fault has been detected we use an accomodation method introduced in [7] [8] to search for state feedback control gains re-establishing the desired performance level.

The remainder of this paper is organized as follows. Section 2 introduces the assumed system model. Section 3 provides a characterization of the system's asymptotic bounded amplitude performance. The characterization forms a linear matrix inequality whose feasible points represent ellipsoidal sets which are invariant for all plants within the model set. Section 4 introduces the fault detection and accomodation method. The significant contributions of this work and future research directions are summarized in section 5

2 System Model

The system under consideration is represented by the linear differential equations,

$$\dot{x} = Ax + BKx + Bw \tag{1}$$

where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, and $K \in \Re^{m\times n}$. The state trajectory, $x(t; x_0, t_0)$, for all $t > t_0$ is the solution of the above differential equation assuming an initial condition $x(t_0) = x_0$. The input, $w(t)$, is treated as an external input disturbance which is a

Lebesgue measurable function taking values in a compact subset of $\Re^m$ such that

$$\sup_{t \geq t_0} w(t)'w(t) < w_m^2 \tag{2}$$

where w_m is a positive real number. The set of all such bounded disturbance signals will be referred to as $\mathcal{BL}_\infty$.

It is further assumed that the system matrices A and B possess a structured additive uncertainty of the form

$$\begin{aligned} A &= A_0 + B_p \Delta C_q \\ B &= B_0 + B_p \Delta D_q \end{aligned} \tag{3}$$

where $A_0 \in \Re^{n \times n}$, $B_0 \in \Re^{n \times m}$, $B_p \in \Re^{n \times q}$, $C_q \in \Re^{q \times n}$, $D_q \in \Re^{q \times m}$, and $\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_s)$. The elements on the diagonal of Δ represent variables parametrizing the system's uncertainty. These variables are assumed to be bounded as

$$\sup_i |\delta_i| < 1 \tag{4}$$

The system defined by equation 1, 2, 3, and 4 will simply be referred to as the system G in the remainder of this paper.

3 Asymptotic Bounded Amplitude Performance

A subset S of $\Re^n$ will be said to be *invariant* with respect to system G if for all $x_0 \in S$ then $x(t; x_0, t_0) \in E(Q)$ for all $t > t_0$. Now let Q be a symmetric positive definite matrix in $\Re^{n \times n}$. System G will be said to possess *bounded amplitude* performance if the ellipsoidal set

$$E(Q, 0) = \{x \in \Re^n \ : \ x'Qx < 1\} \tag{5}$$

is invariant with respect to system G.

Consider an ellipsoidal invariant set, $E(Q)$, and define the positive definite functional,

$$V(x) = x'Qx \tag{6}$$

Let $\dot{V}(x(t))$ be the time derivative of $V(x)$ evaluated along the state trajectory $x(t)$. Let β be a positive real number. System G will be said to be *uniformly attracted* to set $E(Q)$ with a rate β, if and only if for all x such that $x'Qx > 1$, that $\dot{V}(x(t)) \leq -\beta V(x)$.

System G will be said to exhibit asymptotic bounded amplitude performance if it is uniformly attracted at a rate β to an invariant set $E(Q)$. We're interested in characterizing the set of Q and β with respect to which system G exhibits asymptotic bounded amplitude performance. The following proposition establishes that sufficient for an ellipsoidal set to be an attracting invariant set.

Proposition 1 *Consider system G and assume that P is a symmetric positive definite matrix in $\Re^{n\times n}$ satisfying the matrix inequality*

$$A'P + PA + (w_m^2 + \beta)P + PBB'P \leq 0 \tag{7}$$

then $E(P)$ is an invariant set and system G is uniformly attracted to $E(P)$ at rate β.

Proof: By Schur complements, equation 7 holds if and only if

$$\begin{bmatrix} A'P + PA + (w_m^2 + \beta)P & PB \\ B'P & -I \end{bmatrix} \leq 0 \tag{8}$$

This inequality holds if and only if for all $\eta \in \Re^n$ and $\nu \in \Re^m$

$$\begin{bmatrix} \eta \\ \nu \end{bmatrix}' \begin{bmatrix} A'P + PA + (w_m^2 + \beta)P & PB \\ B'P & -I \end{bmatrix} \begin{bmatrix} \eta \\ \nu \end{bmatrix} \leq 0 \tag{9}$$

Letting $\bar{\beta} = w_m^2 + \beta$ and expanding the inequality yields,

$$\eta'(A'P + PA + \bar{\beta})\eta + \nu'B'P\eta + \eta'PB\nu \leq \nu'\nu \tag{10}$$

adding and subtracting w_m^2 from this inequality yields,

$$\eta'(A'P + PA)\eta + \nu'B'P\eta + \eta'PB\nu \leq -\beta\eta'P\eta + (\nu'\nu - w_m^2) + w_m^2(1 - \eta'P\eta) \tag{11}$$

Note that if we define the functional, $V(x) = x'Px$, then the time derivative of V along the state trajectory is

$$\dot{V}(\eta, \nu) = \eta'(A'P + PA)\eta + \nu'B'P\eta + \eta'PB\nu \tag{12}$$

For all $\eta' P \eta < 1$ and $\nu'\nu < w_m^2$, it can be concluded that

$$\dot{V}(\eta, \nu) \leq -\beta V(\eta) \leq 0 \tag{13}$$

Since $\dot{V}$ is nonincreasing at all points outside of $E(P)$ we can therefore conclude that $E(P)$ is an invariant set. •

Assume that $E(P)$ is an attracting invariant set. If $E(P)$ is contained within another ellipsoidal set $E(Q)$ where Q is a symmetric positive definite matrix in $\Re^{n\times n}$, then we expect $E(Q)$ to be an attracting invariant set whose entry time is bounded above by the entry times of $E(P)$. The following proposition formally states and proves this result.

Proposition 2 *Let $E(P)$ be an attracting invariant set of system G. If Q is a symmetric positive definite matrix such that $P \geq Q$, then $E(Q)$ is also an attracting invariant set of system G.*

Proof: Let $\lambda_{\min} P$ and $\lambda_{\max} P$ denote the minimum and maximum eigenevalues of matrix P, respectively. If $P \geq Q$, then $\lambda_{\min} P \geq \lambda_{\max} Q$. But $E(P)$ is invariant so we can assume that $x_t = x(t) \in E(P)$ for all time. These assumptions imply that

$$\begin{aligned} 1 &\geq x_t' P x_t \\ &\geq \lambda_{\min} P \|x_t\|^2 \\ &\geq \lambda_{\max} Q \|x_t\|^2 \\ &\geq x_t' Q x_t \end{aligned} \tag{14}$$

which is sufficient to imply $E(Q)$ is attracting and invariant. •

Proposition 1 and 2 provide sufficient conditions for a given ellispoidal set $E(Q)$ to be attracting and invariant. These result suggest that we must find a matrix P which satisfies the matrix inequality in proposition 1 and 2 for all possible plants within the uncertainty set. The following proposition derives a linear matrix inequality whose feasibility is sufficient to infer that the matrix inequality in equation 7 is satisfied for all plants in the uncertainty model.

Proposition 3 *Consider the system G as specified by equations 1, 2, 3, and 4. For a given symmetric positive definite matrix R in $\Re^{n\times n}$ and real constant $\bar\beta$, the matrix inequality*

$$A'R + RA + \bar\beta R + BB' \leq 0 \tag{15}$$

holds for all A and B characterized by equations 3 and 4 if there exist positive definite matrices Λ and Π in $\Re^{s\times s}$ such that

$$\begin{bmatrix} RA_0' + A_0R + \bar\beta R + B_p\Lambda B_p' & RC_p' & B_0 & B_p \\ C_pR & -\Lambda & 0 & 0 \\ B_0' & 0 & D_q'\Pi D_q - I & 0 \\ B_p' & 0 & 0 & -\Pi \end{bmatrix} \leq 0 \tag{16}$$

Proof: By Schur complements, the Riccati inequality 15 holds if and only if

$$\begin{bmatrix} RA' + AR + \bar\beta R & B \\ B' & -I \end{bmatrix} \leq 0 \tag{17}$$

This inequality holds for al $\xi \in \Re^n$ and $\nu \in \Re^m$ such that

$$\xi'(RA' + AR + \bar\beta R)\xi + \nu' B'\xi + \xi' B\nu - \nu'\nu \leq 0 \tag{18}$$

We now replace A and B by their associated uncertainty models to obtain

$$\begin{aligned} 0 \ \geq\ & \xi'(RA_0' + A_0R + \bar\beta R)\xi + \nu' B_0'\xi + \xi' B_0\nu - \nu'\nu + \\ & \xi'(RC_q'\Delta' B_p' + B_p\Delta C_qR)\xi + \nu' D_q'\Delta B_p'\xi + \xi' B_p\Delta D_q\nu \end{aligned} \tag{19}$$

Redefine the variables $\eta = \Delta' B_p'\xi$ and $\mu = \Delta D_q\nu$ to obtain

$$\begin{aligned} 0 \ \geq\ & \xi'(RA_0' + A_0R + \bar\beta R)\xi + \nu' B_0'\xi + \xi' B_0\nu - \nu'\nu + \\ & \xi' RC_q'\eta + \eta' RC_q'\xi + \mu' B_p'\xi + \xi' B_p\mu \end{aligned} \tag{20}$$

Since Δ is a diagonal matrix, we also require that

$$\eta' e_j e_j' \eta = \delta_j \xi' B_p e_j e_j' B_p' \xi' \tag{21}$$

$$\mu' e_j e_j' \mu = \delta_j \nu' D_q' e_j e_j' D_q \nu \tag{22}$$

where e_j is the jth elementary basis vector.

Inequality 20 can be rewritten as

$$\begin{bmatrix} \xi \\ \eta \\ \nu \\ \mu \end{bmatrix}' \begin{bmatrix} RA_0' + A_0R + \bar{\beta}R & C_qR & B_0 & B_p \\ RC_q' & 0 & 0 & 0 \\ B_0' & 0 & -I & 0 \\ B_p' & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \nu \\ \mu \end{bmatrix} = z'F_0z \leq 0 \tag{23}$$

where $z = \begin{bmatrix} \xi & \eta & \nu & \mu \end{bmatrix}'$. This inequality is subject to the conditions in inequalities 21 and 22. These last two inequalities can be written as

$$z'F_{1j}z \leq 0 \tag{24}$$

$$z'F_{2j}z \leq 0 \tag{25}$$

for all j where

$$F_{1j} = \begin{bmatrix} -B_pe_je_j'B_p' & 0 & 0 & 0 \\ 0 & e_je_j' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{26}$$

$$F_{2j} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -D_q'e_je_j'D_q & 0 \\ 0 & 0 & 0 & e_je_j' \end{bmatrix} \tag{27}$$

Using the $\mathcal{S}$-procedure, we see that a sufficient condition for matrix inequalities 24, 25, and 26 to hold is that there exist constant λ_{1j} and λ_{2j} such that

$$F_0 - \sum_{j=1}^{m} \lambda_{1j} - \sum_{j=1}^{m} \lambda_{2j}F_{2j} \leq 0 \tag{28}$$

This linear matrix inequality can be readily rewritten as shown in equation 16 where we let $\Lambda = \sum_j \lambda_{1j}e_je_j'$ and $\Pi = \sum_j \lambda_{2j}e_je_j'$. •

4 Performance Invalidation and Accomodation

This section uses the results of section 3 to introduce a performance invalidation (fault detection) algorithm and associated invalidation procedure.

Assume that the system's state, $x(t)$, and state derivative $\dot{x}(t)$ are measureable. In this case, we can directly construct $\dot{V}$ using the equation

$$\dot{V}(x, \dot{x}) = x' P \dot{x} + \dot{x}' P x \tag{29}$$

where P has been determined for system G using the matrix inequalities introduced in proposition 2 and 3. In this case, we can clearly see that if the actual system lies in the uncertainty set then

$$\dot{V}(x, \dot{x}) < -\beta V(x) \tag{30}$$

for all observed data such that $V(x) > 1$. Conversely, if there exists a single data point outside of $E(P)$ such that the above inequality fails, then the system will not lie in the uncertainty set we assumed in section 2.

The simple threshold test given above is sufficient to invalidate the modeling assumption. Now assume that the system matrix can actually be written as

$$A_k = A_0 + B_p \Delta C_q + Bk \tag{31}$$

where k is a set of state feedback control gains. This is the nominal system G. Note that if the current system has been invalidated, then

$$\begin{aligned} \dot{V}(x, \dot{x}) &= x'(A_k' P + P A_k) x + w' B' P x + x' P B w && (32) \\ &\geq -\beta V(x) && (33) \end{aligned}$$

Any other gain $\bar{k}$ for which

$$x'(A_{\bar{k}}' P + P A_{\bar{k}}) x > x'(A_k' P + P A_k) x \tag{34}$$

will also invalidate the model. Since the nominal part of A_k and $A_{\bar{k}}$ are the same, this inequality reduces to

$$x'[\bar{k}' B' P + P B \bar{k}] > x'[k' B' P + P B k] \tag{35}$$

Note that the previous inequality is linear in $\bar{k}$ so that the set of infeasible gains declared inconsistent by the current observed data is represented by the linear inequality constraint.

The linear inequality constraint is convenient rewritten using Kronecker products. Let $Vec(X)$ be the vector formed by stacking columns of matrix X onto one long vector. Let $A \otimes B$ denote the Kronecker product of matrices A and B such that

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \tag{36}$$

The inequality constraint is then rewritten as

$$c'\mathrm{Vec}(\bar{k}) > c'\mathrm{Vec}(k) \tag{37}$$

where

$$c = x' \otimes x'PB \tag{38}$$

The preceding inequality now serves as the basis for the updating algorithm. Let's consider an elliposidal set in the gain space

$$E(Q^{-1}, l) = \left\{ k \in K : (k - l)'Q^{-1}(k - l) \right\} \tag{39}$$

where l is the ellispoid's center and Q is a symmetric positive definite matrix characterizing the ellispoid's shape. By assumption, we assume $E(Q_i^{-1}, k_i)$ contains the set of accomodating gains. The halfspace of infeasbile gains identified in equation 37 form a half space which cuts through the center of $E(Q_i^{-1}, k_i)$. The interesection of $E(Q_i, k_i)$ and the complemented halfspace forms a convex body which also contains the accomodating gains. This new convex body can be contained within another minimal volume ellipsoid $E(Q_{i+1}^{-1}, k_{i+1})$. This updated ellipsoid is the smallest ellipse containing the convex body and is easily computed using the central-cut ellipsoid algorithm

$$d = \frac{Q_i c}{\sqrt{c'Q_i c}} \tag{40}$$

$$k_{i+1} = k_i - \frac{d}{l+1} \tag{41}$$

$$Q_{i+1} = \frac{l^2}{l^2-1}\left(Q_i - \frac{2dd'}{l+1}\right) \tag{42}$$

The updated ellipsoid, $E(Q_{i+1}^{-1}, k_{i+1})$ has a center which is either in the accomodating set or not. If it isn't then another fault will be detected and the system will update once again. This goes on until the accomdating gain is found.

The updating algorithm can be shown to terminate after a finite number of updates. A stopping result can be derived using results on the finite time convergence of the ellipsoid method. As noted above, the ellipsoid method generates a sequence of ellipsoids whose volumes are monotone decreasing. It can be shown that the ratio of the volumes of any two successive ellispoidal updates is given by

$$\frac{Vol(E_{i+1})}{Vol(E_i)} < e^{-\frac{1}{2m}} \tag{43}$$

where l is the number of control gains to be learned. Equation 43 shows that the rate at which the ellipsoid volume decrases is independent of the iteration index, i. Therefore it can be seen that the learning process must converge to an accomodating gain (if it exists) after no more than $2m \ln \frac{\text{vol} E_0}{\text{vol} K}$ updates [7]. In other words, the proposed accomodation algorithm converges after a *finite* number of updates.

5 Summary

The preceding discussion has extended previous work examined in [8] proposing an on-line method for fault detection and accomodation. This paper has, essentially, shown how to choose the P matrices characterizing the fault detection algorithm. The method for choosing P was presented in an earlier technical report [9] on bounded amplitude performance. The key contribution of this paper is to show how those bounded amplitude results in [9] apply to our earlier work in [8].

There are still, however, some important issues remaining to be addressed. In the first case, it is clear that accessibility of the state and state derivative is unreasonable.

Fault detection and accomodation must be accomplished solely with output feedback information. Relaxing the detection method to output feedback is easy since our test is already only based on testing a scalar Lyapunov functional. Relaxing the accomodation method to output feedback, however, is non-trivial and requires further study. Removing the requirement for state derivative information needs to be studied in more detail. In practice, we expect the actual test to be done by a computer and this means that $V(x)$ will be sampled. We therefore need to develop a parallel set of conditions that apply to discrete-time and sampled data systems.

In order of importance, the most pressing immediate work involves extending the matrix inequalities and detection methods to discrete time systems. This should be relatively easy. We then need to establish conditions under which the invalidation of the discrete-time problem is sufficient to invalidate the continuous-time problem. These results would remove the requirement for state derivative information. The next most pressing task involves fault accomodation solely on the basis of output data, rather than full state information.

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