Stability of Periodically Switched Systems of First Order Integrators

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Abstract

This paper examines the recurrent behaviour of a class of hybrid systems which periodically switch between different first order integrators. The analysis is based on studying the Poincare map of the hybrid system. For switched integrator systems, the Poincare maps are linear mappings thereby allowing easy characterization of the system's stability. Extending these results to more general classes of linear systems or even higher order integrators, however, does not appear to be analytically tractable.

1 Introduction

A dynamical system can be represented by the triple, (X, Φ, T) where X is a state space, T is a semigroup, and $\Phi : X \times T \rightarrow X$ is a mapping satisfying the identity and semigroup properties (with respect to T). Hybrid dynamical systems arise when either the state space, X, or semigroup, T can be represented as the Cartesian product of two topological spaces with structurally distinct topologies. If, for example $T = \Re \times Z$ where $\Re$ is the set of reals and Z is the set of integers, then the hybrid system is a sampled data system which has been well studied in the context of discrete time control. More recently, considerable attention has been focused on hybrid dynamical systems in which $X = \Re^N \times M$ where M is a set of *events*. This system combines a discrete-event dynamical system with a smooth dynamical system. Such systems arise when computers are used to supervise the high-level or symbolic behaviour of a system.

This paper focuses on this second type of event-driven or asynchronous hybrid system. In particular, we focus on systems in which the structure of the plant switches whenever the continuous-state crosses a known boundary in the state space. We ask a simple question concerning the boundedness of the resulting trajectories in the plant's state space. The "stability" of these systems is analyzed using Poincare maps. As will be seen, the behaviour of such switched systems can be extremely complicated. It only appears that for switched systems of first order integrators that specific results can be determined.

Related work in this area will be found in [1] where the behaviour of so-called "linear" hybrid systems was studied. Similar work will be found in [2]. The analytical techniques found in this paper follow those suggested in [3]. Related work on the stability of switched systems will be found in [4].

2 Hybrid System

Consider a family of R linear time-invariant systems or *plants*. Let the jth plant be parameterized by the ordered pair, $(A^{(j)}, b^{(j)})$ where $A^{(j)} \in \Re^{N \times N}$ and $b^{(j)} \in \Re^N$. Let the state of the jth plant be denoted as $x^{(j)} \in \Re^N$. Associated with this ordered pair is the ordinary differential equation describing the evolution of $x^{(j)}$.

$$\dot{x}^{(j)} = A^{(j)} x^{(j)} + b^{(j)} \tag{1}$$

Consider a family of linear varieties which are called *triggers*. A linear variety is an affine translation of linear subspace in $\Re^N$. The jth trigger is represented by the ordered pair, $(c^{(j)}, w^{(j)})$ where $c^{(j)} \in \Re^N$ and

$w^{(j)} \in \Re$. Associated with $c^{(j)}$ is a linear subspace formed by the range space of a matrix in $\Re^{N\times N}$, $C^{(j)}$, of rank $N-1$ and where $c^{(j)}$ is in the kernel of $C^{(j)}$. The jth trigger is the set

$$T^{(j)} = \left\{ x \in \Re^N \; : \; w^{(j)} = [c^{(j)}]' x \right\} \tag{2}$$

Let X denote the state space of the plants. This is taken to be a linear subspace of $\Re^N$. Let Z_M denote the set of integers from 1 to M, inclusive. Define the map, $\tilde{\Phi} : X \times Z_M \to Z_M$ by the equation

$$\tilde{\Phi}(x,j) = \begin{cases} \mathrm{mod}_M(j+1) & x \in T_j \\ j & \text{otherwise} \end{cases} \tag{3}$$

where $x \in X$ and $j \in Z_M$. Define an indexed family of mappings $\bar{\Phi}_\tau : X \times Z_M$ where the index $\tau \in \Re$ by the following equation

$$\bar{\Phi}_\tau(x,j) = e^{A^{(j)}\tau} x + \int_0^\tau e^{A^{(j)}(\tau-s)} b^{(j)} ds \tag{4}$$

where $x \in X$ and $j \in Z_M$.

A periodically switched linear dynamical system is the ordered pair, $\mathcal{S} = (X \times Z_M, \bar{\Phi}_\tau \times \tilde{\Phi})$ where $\bar{\Phi}_\tau \times \tilde{\Phi}$ is an automorphism over $X \times Z_M$ indexed with respect to $\tau \in \Re$.

3 Simple Integrator Systems

The switched dynamical system, $\mathcal{S}$, will be studied by examining the behaviour of the Poincare map where system's continuous state in the jth trigger set $T^{(j)}$ are mapped back into $T^{(j)}$ by the action of $\mathcal{S}$. In general, this recurrent behaviour will be extremely difficult to analyze. In this section, we first consider the case that occurs where the $A^{(j)} = 0$. In this case, the dynamics for the jth plant state is represented as

$$\dot{x}^{(j)} = b^{(j)} \tag{5}$$

These types of plants consist of a set of N first-order integrator systems. These plants have been examined in detailed by a variety of researchers and represent what some refer to as "linear" hybrid systems [1] [2]. This is, of course, a very restricted class of "linear" system and a more appropriate term would be to call these systems first-order integrator hybrid systems. This is the terminology adopted throughout this section.

The Poincare maps associated with the jth trigger set are particular easy to derive for integrator hybrid systems. Let $x^{(j)}$ denote the continuous part of the hybrid state which lies on the jth trigger set, $T^{(j)}$. Let the continuous-state at some time τ after the plant state lies in $T^{(j)}$ be denoted as $x^{(j)}(\tau)$. The system state between switches can be expressed as

$$\begin{aligned} x^{(j)}(\tau) &= e^{A^{(j)}\tau} x^{(j)} + \int_0^\tau e^{A^{(j)}(\tau-s)} b^{(j)} ds & (6) \\ &= x^{(j)} + \tau b^{(j)} & (7) \end{aligned}$$

The last equation is a consequence of the fact that we're dealing with a first order integrator hybrid system. We're interested in determining the time τ when $x^{(j)}(\tau) = x^{(j+1)}$. In other words, we're interested in determining the time τ when the plant state lies in the linear variety $T^{(j+1)}$; the next switch. Clearly this occurs when

$$\left[c^{(j+1)}\right]' x^{(j)}(\tau) = w^{(j)} \tag{8}$$

Combining this constraint with the preceding equation yields the scalar equation

$$w^{(j)} = \left[c^{(j+1)}\right]' x^{(j)} + \tau \left[c^{(j+1)}\right]' b^{(j)} \tag{9}$$

Solving for the time τ, yields

$$\tau = \frac{w^{(j)} - \left[c^{(j+1)}\right]' x^{(j)}}{\left[c^{(j+1)}\right]' b^{(j)}} \tag{10}$$

Inserting this relationship into our evolution equation yields the map from $x^{(j)}$ to $x^{(j+1)}$,

$$x^{(j+1)} = \left(I - \frac{b^{(j)} \left[c^{(j+1)}\right]'}{\left[c^{(j+1)}\right]' b^{(j)}} \right) x^{(j)} + \frac{w^{(j)}}{\left[c^{(j+1)}\right]' b^{(j)}} b^{(j)} \tag{11}$$

The preceding map is a linear map and we can by iterating this for $j = 1, \ldots, N$, get the iterated map from $T^{(j)}$ back onto itself. This iterated map, $\phi : T^{(j)} \to T^{(j)}$ is a linear map of the form

$$x^{(j)} \to \Phi x^{(j)} + \Gamma b^{(j)} \tag{12}$$

Clearly if the spectral radius of Φ is less than unity, then we obtain a stable mapping whose steady state recurrent behaviour is a periodic orbit or stable fixed point.

Conditions for the stability and existence of such orbits are trivially apparent.

4 General Systems

While the first order integrator systems are relatively trivial to analyze, the behaviour of more general dynamical systems appears to be nearly intractable using the techniques of this paper. In fact, it appears that the conclusions alluded to in [5] are indeed true; that techniques of dynamical systems theory may only provide clear answers for planar hybrid systems.

The principal difficulty in extending these results to more complex systems lies in the determination of the Poincare map. Consider the following example, for instance, where we have a 3rd order integrator (relative degree of 3). Such systems might arise in the feedback linearization of nonlinear systems. The associated state equation for the jth plant will be of the form (we will drop the j super-index to ease the notational burden).

$$\dot{x}_1 = x_2 \tag{13}$$

$$\dot{x}_2 = x_3 \tag{14}$$

$$\dot{x}_3 = b \tag{15}$$

The state evolution equation has the form

$$x_1(\tau) = x_1 + \tau x_2 + \frac{\tau^2}{2!} x_3 + \frac{\tau^3}{3!} b \tag{16}$$

$$x_2(\tau) = x_2 + \tau x_3 + \frac{\tau^2}{2!} b \tag{17}$$

$$x_3(\tau) = x_3 + \tau b \tag{18}$$

If we now use our condition for the linear variety, we obtain a third order polynomial that must be solved for τ. For higher order integrator systems, the associated polynomial we need to solve for τ has a higher degree. Inverting these equations to obtain an analytic expression for τ as a function of x is not realistic. Therefore the exact iterated map cannot be obtained.

A similar situation appears if we let the system be a general linear system. Let's assume that $A^{(j)}$ is a diagonal system of non-repeated roots. A simple calculation now obtains a formula for τ which is a sum of exponentials. Inverting this expression is not easily accomplished and once again the iterated map cannot be easily obtained.

5 Conclusions

The preceding discussion indicated that the exact analysis of switched hybrid systems using iterated maps will probably be impractical for all be first order integrator systems. The maps obtained from this type of analysis will generally be nonlinear (even if the systems are linear) and N-dimensional. The resultant behaviour of these maps can exhibit complex recurrent behaviours including limit cycles of mixed periodicites and turbulent or chaotic trajectories.

Even though formal analysis of these maps is not feasible, it may be possible to use these methods to obtain a qualitative understanding of the system's behaviour. In the linear system cases, we expect to be able to simulate the map relatively easily. This means that it may be possible to generate a number of input-output examples of the Poincare map and then approximate the structure of this map using a neural network function approximator. Whether or not this approach is indeed practical will depend on the nature of the resulting maps. N-d iterated maps can prove to be extremely difficult to analyze. 1-d iterated maps, however, are well-understood [6]. If it is possible to obtain approximate 1-d iterated maps, it may be possible to complete the analysis of the switched system.

Figure 1 provides an example of this approach. Introduce an energy functional $V(x)$ which is positive definite. Let F be the Poincare map. We can plot the input/output pairs as a scatter plot in figure 1 and identify an upper and lower bound on the resulting system. By the comparison principal, we expect the actual system to exhibit a behaviour which is bounded by the two bounding maps. Clearly, if both bounding systems are stable, then we expect the actual system to be stable also. Whether or not this approach would work well depends on how tight the bounding maps will be.

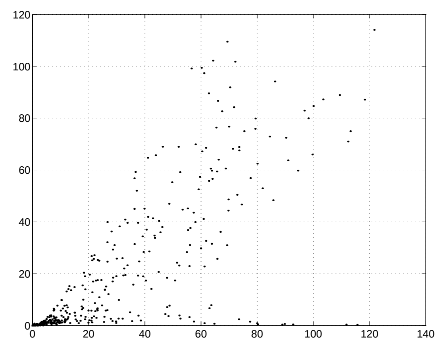

Figure 1: Scatter Plot of Input/Output Energies

References

[1] R. Alur, C. Courcoubetis, T. Henzinger, N. Halbwachs, Pei-Hsin Ho, X. Nicollin, A. Olivero, J. Sifakis, and S. Yovine. "The algorithmic analysis of hybrid systems", *Theoretical Computer Science*, 138:3-34, 1994.

[2] M. Tittus, B. Egardt, "Control-Law Synthesis for Linear Hybrid Systems", IEEE 33rd Conference on Decision and Control, Florida, Dec. 1994.

[3] M. Branicky, "Stability of Switched and Hybrid Systems", *Proc. 33rd IEEE Conf. Decision and Control*, Florida, Dec. 1994.

[4] M. Wicks, P. Peleties, and R. DeCarlo, "Construction of Piecewise Lyapunov Functions for Stabilizing Switched Systems", *Proceedings of the IEEE 33rd Conference on Decision and Control*, Florida, Dec. 1994.

[5] J. Guckenheimer, "a robust hybrid stabilization strategy for equilibria", *IEEE Trans. on Automatic Control*, Vol 40, No. 2, Feb. 1995.

[6] L.S. Block and W.A. Coppel, *Dynamics in One Dimension*, Springer Verlag, 1992.

Appendix

The enclosed matlab code was used to generate figure 1. This code computes input and output samples for the mixed 2nd and 1st order integrator example.

```
%1-2 integrator hybrid system
%specific example computes the Poincare map
%of a chaotic system
%
%triggers
C=[.8680 .7802 -.2713;.3327 .9867 -.3628;.5967 .9544 .5969];
C=rand(3,3);
C=[.2053 .9397 .0703;.1714 .9649 .1238; .3725 .2550 .5263];
Cpart=C(:,[1 2]);

%plants
B=[.8228 .6864 .6280; .7770 .8419 .7684];
B=rand(2,3);
B=[.1601 .1052 .3686;.5177 .8416 .4239];

poincare=[];
e1=[];
e2=[];

for k=1:500;

 %initial state
 xp0=10.*(rand(3,1)-ones(3,1)./2);
 xn=[0;0;0];
 xp=xp0;

 point=[];
 time=[];
 ttp=0;

 jmax=2;
 for j=1:jmax;
  tt=0;
  for i=1:3;
    %tau polynomial
    c=C(:,i);
    b=B(:,i);
    p0=c(1)*xp(1)+c(2)*xp(2)+c(3)*xp(3);
    p1=c(1)*b(1)+c(2)*xp(3)+c(3)*b(2);
    p2=c(2)*b(2)/2;
    p=[p2 p1 p0];
    z=roots(p);
    zz=z(find(imag(z)==0),:);
    t=max(zz);

    if(isempty(t))
      test=1;
    elseif (t<0)
      test=1;
    else
      test=0;
    end;

    if(test==1)
```

```
      b=-B(:,i);
      p0=c(1)*xp(1)+c(2)*xp(2)+c(3)*xp(3);
      p1=c(1)*b(1)+c(2)*xp(3)+c(3)*b(2);
      p2=c(2)*b(2)/2;
      p=[p2 p1 p0];
    z=roots(p);
    zz=z(find(imag(z)==0),:);
    t=max(zz);

    if(isempty(t))
      test=1;
    elseif (t<0)
      test=1;
    else
      test=0;
    end;

      if(test==1);
        disp('negative transit time!');
        break;
      end;
    end;

      xn(1)=xp(1)+b(1)*t;
      xn(2)=xp(2)+xp(3)*t+b(2)*t*t/2;
      xn(3)=xp(3)+b(2)*t;
      xp=xn;
  tt=tt+t;
  end;

  ttp=tt;

  time=[time ; tt];
  point=[point  xn];
  if(j==1);
   e1=[e1 xn'*Cpart*Cpart'*xn];
  end;
  if(j == 2);

    e2=[e2 xn'*Cpart*Cpart'*xn];
  end;

end;
  pointxy=Cpart'*point;

end;


plot(e1,e2,'.');
grid on;
hold off;
```