Design of Output Feedback Controllers for Robust Stability and Optimal Performance of Discrete-Time Systems

Technical Report of the ISIS Group
at the University of Notre Dame
ISIS-94-009
September, 1994

Ioannis K. Konstantopoulos and Panos J. Antsaklis
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556
email: antsakli@saturn.ee.nd.edu

Interdisciplinary Studies of Intelligent Systems

DESIGN OF OUTPUT FEEDBACK CONTROLLERS FOR ROBUST STABILITY AND OPTIMAL PERFORMANCE OF DISCRETE-TIME SYSTEMS

Ioannis K. Konstantopoulos and Panos J. Antsaklis
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556, U.S.A.
email: antsakli@saturn.ee.nd.edu

Abstract

This paper presents an algorithm for the design of robust output feedback controllers for linear uncertain discrete-time systems. The algorithm utilizes a version of the BFGS method of conjugate directions and minimizes a performance index that includes an LQR term to optimize performance and a robustness term which is based on recently developed bounds. The minimization of only the robustness term which corresponds to the maximization of the uncertainty bound is also studied. The case of unstructured perturbations in A has been the only one studied in the literature; the present algorithm not only introduces a unified approach to both unstructured and structured perturbations in A but also is shown to improve considerably the unstructured uncertainty bound for the system matrix A found in the literature. Additionally, several other cases involving unstructured/structured perturbations in all the state-space matrices are exploited and numerous examples are provided to illustrate the results.

1 Introduction

The problem of determining a linear feedback control law for uncertain linear systems has drawn considerable attention recently. A number of criteria have been used to characterize the system uncertainties, so that the stability (asymptotic, quadratic, exponential) of the uncertain systems is guaranteed if these criteria are satisfied, and several robust controller design methods have been developed.

In [14], [18], the Guaranteed Cost Control approach is used for the design of robust feedback controllers that guarantee both the robust stability and performance of continuous systems. In [8], a two-level optimization process that guarantees quadratic stabilizability of continuous systems is presented and in [9], an algorithm consisting of a strictly quasiconvex minimization is applied to the design of quadratically stabilizing output feedback controllers for both continuous and discrete-time systems. In [25], algorithms are proposed, based on the Lyapunov stability criterion, to choose a set of weighting matrices for the quadratic cost function; these matrices are then used in the standard Riccati equation to give the linear quadratic optimal control law for the nominal continuous system, which is shown to quadratically stabilize the uncertain system. A similar combination of the Lyapunov stability criterion and the Riccati equation is used in [22], where a noniterative procedure is presented for the design of a robust state feedback controller that ensures the exponential stabilizability of uncertain continuous systems.

The LQR formulation for continuous systems is used in [27], where an upper bound on the cost incurred by state feedback law and parameter uncertainties is derived and the control law that minimizes this upper bound is found; conditions are presented, under which the feedback system is stable for all admissible parameter variations. Another LQR based control design which is robust to parametric uncertainties is developed in [5] for continuous systems, where the resulting full-state controller is designed by solving a single Riccati-type equation. In [17], the robustness of the discrete-time LQG problem is studied, where the system to be controlled is described by a state-space formulation that includes plant parameter perturbations and noise uncertainty.

Numerous synthesis results based on H_∞ techniques have also appeared in the literature. In [7], [4], for instance, conditions for quadratic stability with disturbance attenuation and quadratic stabilization via dynamic output feedback are derived respectively for uncertain continuous and discrete-time systems, whose uncertainty matrices are assumed to be of a specific structure. Then, an H_∞ based approach is described for the design of controllers that satisfy the aformentioned conditions. However, no specific information about the uncertainty bounds that describe the uncertainty matrices is provided. In [28], the authors present a convex programming based approach to the design of H_∞ controllers for uncertain systems. Specifically, they reduce the problem of controller design to a matrix inequalities problem and search for a controller that satisfies the conditions for the strongly robust H_∞ performance criterion they define. Note that the systems they study are the typical H_∞ systems with exogeneous disturbances included in the state-space model. Note also that no explicit way is presented to compute the uncertainty bounds, which are decided experimentally via the ellipsoidal method. Several other papers, some of those included in the references of [4], [7] deal with the problem of robust output feedback controller design in a fashion similar to the one of [4], [7], [28] discussed above.

All the above controller design approaches share the same general objective, which is to find a stabilizing controller that satisfies some kind of stability conditions or is robust in some sense, without considering the maximization of any of the robust stability bounds existing in literature. This has been done, however, in [26] for continuous systems with structured uncertainties in the system matrix A and in [12] for discrete-time systems with unstructured uncertainties in A. Note that the design in the first paper relies on the selection of a weighting matrix not directly associated with the structured uncertainties and in the latter on the bound developed in [11]. In both of these papers, the information about the uncertainty bound is a part of the minimizing quantity, which also includes the classical LQR cost. Therefore, the controller design objective is twofold, that is to minimize the LQR cost and maximize the perturbation bounds. A similar approach was used earlier in [21] for continuous systems with structured uncertainties in all the state-space matrices, under some quite restrictive assumptions imposed on the perturbation matrices. Note that although the maximization of some stability bound is not considered in the design process, the information about the structured uncertainty is directly included in the minimizing quantity.

From the previous paragraph, it is quite obvious that for discrete-time systems, only the case of unstructured perturbations in A has been studied. Here, we present a unified output feedback controller design approach to both unstructured and structured perturbations in A. Our approach is based on some new theorems for both the structured and unstructured case, which were recently developed in [13], where it was also shown that these theorems provide bounds that improve the ones obtained via the methodology suggested in [11]; note that, as mentioned before, the unstructured bound of [11] was the one used in [12]. Our design not only provides a stabilizing static output feedback controller that improves the unstructured bound for A derived in [12] but is also capable of finding another such controller that maximizes the bound for the case that A is perturbed by known uncertainty matrices. In addition, we study several other interesting cases, as it is shown next. In all these cases, the minimizing quantity consists of two terms; one is the robustness term, which is associated with the specific unstructured/structured bound we wish to maximize and the other is the LQR term, which is associated with the specific control performance we wish to establish. Note that our approach is also applied to a minimizing quantity consisted of only the robustness term, in order to find the controller that maximizes the stability bounds, without considering any control specifications. Note that only the case of static output feeback is studied, since it can be easily shown that the case of dynamic output feedback can be reduced to that of static feedback as well. Finally note that our minimization algorithm utilizes a version of the Broyden family method of conjugate directions, which is based on the BFGS rule, [2] and that the case of state feedback can be easily considered as a special case of the output feedback case for $C = I$.

The paper is organized as follows. In section 2, we give a brief review of the new theorems developed in [13], as mentioned before, for the cases of unstructured and structured perturbations in discrete-time systems. In section 3, we study the case of unstructured/structured perturbations in the system matrix A, and present an algorithm based on the BFGS rule that solves the minimization problem. In section 4, we consider unstructured/structured perturbations in either the input matrix B or the outpur matrix C and in section 5, unstructured/structured perturbations in either (A, B) or (A, C). In section 6, we study the case of unstructured perturbations in all state-space matrices. In section 7, we provide several illustrative examples for the cases studied above and finally in section 8, concluding remarks are briefly discussed.

2 Preliminaries

We consider the linear discrete-time system with the state-space description

$$x(k+1) = A\, x(k) \tag{1}$$

where $x \in \Re^n$ is the state vector and A an asymptotically stable matrix. Then, for every symmetric positive definite matrix Q, we can find a symmetric positive definite matrix P, which is the unique solution of the Lyapunov equation

$$A^T P A - P + Q = 0 \tag{2}$$

When A is perturbed by the matrix ΔA, then for the perturbed system

$$y(k+1) = (\ A + \Delta A\ )\ x(k) \tag{3}$$

the following theorem has been proven in [13]. First define

$$\Omega_1 = A^T P Z^{-1} P A \tag{4}$$

Theorem 2.1 *Consider the linear discrete-time system (1) where A is an asymptotically stable matrix that satisfies (2). Suppose that $A \to A + \Delta A$, then the perturbed system (3) remains asymptotically stable, if*

$$(\Delta A)^T\ (\alpha Z + P)\, (\Delta A)\ +\ \frac{1}{\alpha}\, \Omega_1\ <\ Q \tag{5}$$

or

$$\sigma_{max}(\Delta A)\ <\ \sqrt{\frac{\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha}\ \Omega_1)}{\sigma_{max}(\alpha Z + P)}} \tag{6}$$

where P, Q are defined in (2), Ω_1 in (4), Z can be any positive definite matrix of appropriate dimensions, and α any positive number that satisfies

$$\alpha\ >\ \frac{\sigma_{max}(\Omega_1)}{\sigma_{min}(Q)} \tag{7}$$

Next, we consider the case that the perturbation matrix ΔA is described by

$$\Delta A = \sum_{i=1}^{m} \kappa_i A_i = (K \otimes I_n)^T \ \tilde{A} \tag{8}$$

where $\kappa_i, i = 1,..,m$ denote real, uncertain parameters and $A_i, i = 1,..,m$ are constant, known matrices, and the following definitions have been used

$$\begin{aligned} K &= [\ \kappa_1 \ \ \kappa_2 \ \ \cdots \ \ \kappa_m \ ]^T & (9) \\ \tilde{A} &= [\ A_1^T \ \ A_2^T \ \ \cdots \ \ A_m^T \ ]^T & (10) \end{aligned}$$

Then the following theorem has again been proven in [13].

Theorem 2.2 *The linear discrete-time system (3) with structured perturbations of the form of (8) remains asymptotically stable, if the uncertainty parameters satisfy*

$$\sum_{i=1}^{m} \kappa_i^2 \ < \ \frac{\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha}\ \Omega_1)}{\sigma_{max}^2(\tilde{A})\ \sigma_{max}(\alpha Z + P)} \tag{11}$$

where Ω_1, k_i, $\tilde{A}$ are defined in (4), (9), (10) respectively, Z can be any positive definite matrix of appropriate dimensions, and α any positive number that satisfies (7).

Note that, as it has been shown in [13], the above theorems provide bounds that improve the ones obtained via the methodology suggested in [11]. The main point of the approach used for the theorems above is the selection of a positive definite matrix Z and a positive number α that maximize the stability region within which the uncertain parameters vary. Finally note that the above approach has also been extended, in [13], to the case of structured perturbations in all the state-space matrices.

3 Perturbations in A

In this section, we consider the linear discrete-time system with the state-space description

$$\begin{aligned} x(k+1) &= A\ x(k) + B_0\ u(k) & (12) \\ y(k) &= C_0\ x(k) & (13) \end{aligned}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^r$ is the input vector and $y \in \Re^q$ is the output vector. Both unstructured and structured perturbations for the system matrix A are of interest here, that is

$$
\begin{aligned}
A &= A_0 + \Delta A && (14)\\
A &= A_0 + \sum_{i=1}^{m} \kappa_i A_i && (15)
\end{aligned}
$$

With the static output feedback law

$$u = Ky(k) = KC_0\, x(k) \tag{16}$$

the closed-loop system is described by

$$
\begin{aligned}
x(k+1) &= (\ \bar{A}_0 + \Delta A\ )\ x(k) && (17)\\
x(k+1) &= (\ \bar{A}_0 + \sum_{i=1}^{m} \kappa_i A_i\ )\ x(k) && (18)
\end{aligned}
$$

for the unstructured and structured case respectively, where obviously the following definition has been used

$$\bar{A}_0 = A_0 + B_0 K C_0 \tag{19}$$

3.1 Design without performance specifications

Our objective is to find a static stabilizing output feedback gain K that maximizes the bounds given in (6) and (11). Note the similarity between these two relations. Due to this similarity, we present here a unified approach to both the unstructured and structured case. For the closed-loop systems of (17), (18), relations (2) and (4) can be translated into the following

$$\bar{A}_0^T P \bar{A}_0 - P + Q = 0 \tag{20}$$

$$\Omega_1 = \bar{A}_0^T P Z^{-1} P \bar{A}_0 \tag{21}$$

Since Q in (20) is selected beforehand, in order to maximize the bounds of (6), (11), we need to

- *(A.1) : minimize* $\sigma_{max}(\alpha Z + P)$
- *(A.2) : minimize* $\sigma_{max}(\frac{1}{\alpha}\,\Omega_1)$

For any matrices A and B, the following properties hold

$$
\begin{aligned}
\sigma^2_{max}(A) &\leq ||A||^2_F = Tr(A^T A) && (22)\\
\sigma_{max}(AB) &\leq \sigma_{max}(A)\,\sigma_{max}(B) && (23)\\
\sigma_{max}(A+B) &\leq \sigma_{max}(A) + \sigma_{max}(B) && (24)
\end{aligned}
$$

where $||A||_F$ denotes the Frobenius norm and $Tr(A)$ the trace of a matrix A. Hence for *(A.1)*, we choose to minimize the quantity

$$J_{11} = Tr[\,(\alpha Z + P)^T\,(\alpha Z + P)\,] = Tr(\alpha^2 Z^2 + 2\alpha Z P + P^2) \tag{25}$$

where obviously the following property has been used

$$Tr(AB) = Tr(BA) \tag{26}$$

for any appropriately dimensioned matrices A and B, for which AB and BA are defined. In view of (21), (23), we have

$$\sigma_{max}(\frac{1}{\alpha}\,\Omega_1)\; \leq \frac{1}{\alpha}\,\sigma^2_{max}(\bar{A}_0)\,\sigma^2_{max}(P)\,\sigma_{max}(Z^{-1}) \tag{27}$$

Since Z is selected beforehand and an upper bound of $\sigma_{max}(P)$, that is $Tr(P^2)$, is already minimized in (25), for *(A.2)* we simply choose to minimize

$$J_{12} = \frac{1}{\alpha}\,Tr(\bar{A}_0^T\,\bar{A}_0) \tag{28}$$

which is an upper bound of $\frac{1}{\alpha}\sigma^2_{max}(\bar{A}_0)$. Note that the above choice, that is the minimization of the sum of $Tr(\bar{A}_0^T \bar{A}_0)$ and $Tr(P^2)$ is an indirect and harder way to minimize their product; in other words, we impose a more demanding task on the minimizing process. On the other hand, note that α is included in the minimizing quantity, because we need to satisfy the positiveness of the numerator, as indicated in (7). Therefore, in view of (25), (28) the minimizing quantity is given as

$$J_1' \;=\; J_{11} + J_{12} \;=\; Tr(\alpha^2 Z^2 + 2\alpha PZ + P^2 + \frac{1}{\alpha}\, \bar{A}_0^T \bar{A}_0) \tag{29}$$

under the condition that (20) holds. This is clearly a constrained minimization problem. By including (20) in (29), we finally reduce the problem to an unconstrained minimization one, with the minimizing quantity given as

$$J_1 \;=\; Tr[\, \alpha^2 Z^2 + 2\alpha PZ + P^2 + \frac{1}{\alpha}\, \bar{A}_0^T \bar{A}_0 + L_1\,(\bar{A}_0^T P \bar{A}_0 - P + Q)\,] \tag{30}$$

where $L_1 \in \Re^{n\times n}$ is the Lagrange multiplier matrix. Next, we compute the partial derivatives of J_1 with respect to L_1, α, P, and K; these partial derivatives are needed for the algorithm that is presented next for the minimization of J_1. In order to compute them, we need the following properties from [1]

$$\begin{aligned}
\frac{\partial}{\partial X}\, Tr(X^2) &= 2X^T && (31)\\
\frac{\partial}{\partial Y}\, Tr(A_1 Y B_1) &= A_1^T B_1^T && (32)\\
\frac{\partial}{\partial Y}\, Tr(A_2 Y^T B_2) &= B_2 A_2 && (33)\\
\frac{\partial}{\partial Y}\, Tr(A_3 Y B_3 Y^T) &= A_3 Y B_3 + A_3^T Y B_3^T && (34)
\end{aligned}$$

for any $X \in \Re^{n\times n}$, $Y \in \Re^{n\times m}$, $A_1 \in \Re^{l\times n}$, $B_1 \in \Re^{m\times l}$, $A_2 \in \Re^{l\times m}$, $B_2 \in \Re^{n\times l}$, $A_3 \in \Re^{n\times n}$, $B_3 \in \Re^{m\times m}$. With these properties, we have

$$\begin{aligned}
\frac{\partial J_1}{\partial L_1} &= \Delta^1_{L_1} = \bar{A}_0^T P \bar{A}_0 - P + Q && (35)\\
\frac{\partial J_1}{\partial \alpha} &= \Delta^1_{\alpha} = 2\alpha\, Tr(Z^2) + 2\, Tr(PZ) - \frac{1}{\alpha^2}\, Tr(\bar{A}_0^T \bar{A}_0) && (36)\\
\frac{\partial J_1}{\partial P} &= \Delta^1_{P} = 2P + 2\alpha Z + \bar{A}_0 L_1^T \bar{A}_0^T - L_1^T && (37)\\
\frac{\partial J_1}{\partial K} &= \Delta^1_{K} = \frac{2}{\alpha}\, B_0^T B_0 K C_0 C_0^T + \frac{2}{\alpha}\, B_0^T A_0 C_0^T \\
&\quad + B_0^T P B_0 K C_0\,(L_1 + L_1^T)\, C_0^T + B_0^T P A_0\,(L_1 + L_1^T)\, C_0^T && (38)
\end{aligned}$$

In order to minimize (30), we use a version of the Broyden family method of conjugate directions, which is based on the Broyden-Fletcher-Goldfarb-Shanno (BFGS) update rule; details in [2]. The proposed algorithm is presented next.

Initialization Step Let $\epsilon > 0$ be the termination scalar. Choose an initial stabilizing gain

$$K_1 = \begin{pmatrix} (\tau_1^1)^T \\ \vdots \\ (\tau_r^1)^T \end{pmatrix} \tag{39}$$

where $(\tau_l^1)^T, l = 1, .., r$ are the $1 \times q$ rows of K_1, which stabilizes (A_0, B_0, C_0), that is $\bar{A}_0$ stable. Also, choose an initial symmetric positive definite matrix D_1. Let

$$y_1 = x_1 = \begin{pmatrix} (\tau_1^1) \\ \vdots \\ (\tau_r^1) \end{pmatrix} \tag{40}$$

be a column vector consisting of the transposes of the rows of K_1. Also let $k = j = 1$ and go to the *Main Step*.

Main Step

M1. Substitute the gain matrix K_j in the gradients of (35)-(37), set them to zero, that is $\Delta_{L1}^1 = 0$, $\Delta_\alpha^1 = 0$, $\Delta_P^1 = 0$, and solve for P, α, L_1 respectively, in that specific order.

M2. Substitute these parameters in (38) and compute

$$\Delta_{K_j}^1 = \begin{pmatrix} (\sigma_1^j)^T \\ \vdots \\ (\sigma_r^j)^T \end{pmatrix} \tag{41}$$

where $(\sigma_l^j)^T, l = 1, .., r$ are the $1 \times q$ rows of $\Delta_{K_j}^1$.

M3. Define

$$\nabla J_1(y_j) = \begin{pmatrix} \sigma_1^j \\ \vdots \\ \sigma_r^j \end{pmatrix} \tag{42}$$

If $\|\nabla J_1(y_j)\| < \epsilon$, *STOP*. The optimal gain is K_j.

Otherwise, go to M4.

M4. If $j > 1$, update the positive definite matrix D_j as follows:

$$D_j = D_{j-1} + \frac{p_{j-1}p_{j-1}^T}{p_{j-1}^T q_{j-1}}[1 + \frac{q_{j-1}^T D_{j-1} q_{j-1}}{p_{j-1}^T q_{j-1}}] - \frac{[D_{j-1}q_{j-1}p_{j-1}^T + p_{j-1}q_{j-1}^T D_{j-1}]}{p_{j-1}^T q_{j-1}} \tag{43}$$

where

$$\begin{aligned} p_{j-1} &= \lambda_{j-1} d_{j-1} = y_j - y_{j-1} && (44) \\ q_{j-1} &= \nabla J_1(y_j) - \nabla J_1(y_{j-1}) && (45) \end{aligned}$$

M5. Define

$$d_j = -D_j \nabla J_1(y_j) \tag{46}$$

and let λ_j be an optimal solution to the problem of minimizing $J_1(y_j + \lambda d_j)$ subject to $\lambda \geq 0$. Let

$$y_{j+1} = y_j + \lambda_j d_j = \begin{pmatrix} (\tau_1^{j+1}) \\ \vdots \\ (\tau_r^{j+1}) \end{pmatrix} \tag{47}$$

which implies that

$$K_{j+1} = \begin{pmatrix} (\tau_1^{j+1})^T \\ \vdots \\ (\tau_r^{j+1})^T \end{pmatrix} \tag{48}$$

where obviously $(\tau_l^{j+1}), l = 1, .., r$ are $q \times 1$ column vectors.

M6. If $j < qr$, replace j by $j + 1$ and repeat the *Main Step.*

Otherwise, if $j = qr$, then let $y_1 = x_{k+1} = y_{qr+1}$, replace k by $k + 1$, let $j = 1$ and repeat the *Main Step.* □□

There are several issues that need to be discussed here. Since we try to find a gain matrix that not only minimizes the quantity given in (30) but also stabilizes the closed-loop system, we need to check at each step (either of j or k), if the new matrix K_{new} makes $A_0 + B_0 K_{new} C_0$ stable. If this is not true, we try the matrix $K_{alt} = K_{old} + \frac{1}{2}(K_{new} - K_{old})$, as suggested in [20], where K_{old} is the gain matrix computed in the previous step of j or k. Therefore, once we find a stabilizing K_{alt}, we restart the algorithm. Otherwise, we try to find a stabilizing gain matrix from the set of matrices

$$K_{alt}^{\nu} = K_{old} + \frac{1}{2^{\nu}}(K_{new} - K_{old}), \qquad \nu = 1, 2, ... \tag{49}$$

and then we restart the algorithm.

Since our algorithm is an indirect version of the BFGS algorithm, as an alternative to the stopping criterion of *(M3)*, we could use another quite practical criterion. Specifically, we may consider monitoring J_1 and stop when we reach a plateau or when we see that J_1 is sufficiently small and the associated bound derived is acceptably large. Additionally, note that for optimization problems similar to the one we study here, alternative methods based on gradient-type and nongradient-type algorithms have been proposed in [10] and [20] respectively.

From (36), we easily see that we have either one or two positive solutions for α. For our algorithm, we choose to keep the largest value of α, since we also need to satisfy the positiveness of the numerator of (6) and (11), as discussed before. However, in the case of an unsuccessful search for a stabilizing gain matrix, as explained in the previous paragraph, we can also try the other positive solution of α, if any. Finally note that the line search in *(M5)* of the *Main Step* was performed in our examples by the Fibonacci method; details in [2].

3.2 Design with performance specifications

In the previous subsection, we focused on finding an output feedback gain K that maximizes the bounds of (6) and (11). If, in addition to this objective, we also wish to attain a specific control performance, then we need to include in our minimizing quantity a term that evaluates this control performance. Therefore, we consider the familiar LQR cost, [6], which is given as follows

$$J_2' = \sum_{k=0}^{\infty} x^T(k) Q_1 x(k) + u^T(k) R_1 u(k) \tag{50}$$

where Q_1, R_1 are positive definite matrices of appropriate dimensions. For the nominal system (A_0, B_0, C_0) with the output feedback law (16), we rewrite (50) as follows

$$\begin{aligned} J_2' &= \sum_{k=0}^{\infty} x^T(k)\,(Q_1 + C_0^T K^T R_1 K C_0)\,x(k) \\ &= \sum_{k=0}^{\infty} x^T(k)\,\bar{Q}\,x(k) \end{aligned}$$

$$= \sum_{k=0}^{\infty} x^T(0)\, (\bar{A}_0^T)^k\, \bar{Q}\, (\bar{A}_0)^k\, x(0) \tag{51}$$

where obviously

$$\bar{Q} = Q_1 + C_0^T K^T R_1 K C_0 \tag{52}$$

The following equivalence has been shown in [23] for the solution of the discrete-time Lyapunov equation (2)

$$A^T P A - P + Q = 0 \quad \Longleftrightarrow \quad P = \sum_{k=0}^{\infty} (A^T)^k\, Q\, A^k \tag{53}$$

With the above relation, (51) can be rewritten as

$$J_2' = x^T(0)\; P_2\; x(0) \tag{54}$$

where P_2 is the solution of the Lyapunov equation

$$\bar{A}_0^T P_2 \bar{A}_0 - P_2 + \bar{Q} = 0 \tag{55}$$

As we see, J_2' depends on the initial state $x(0)$, which implies that the optimal gain matrix K will also depend on $x(0)$. To eliminate this dependence, we may assume, [16], [24], that $x(0)$ is a random vector with expected value and second-order moment given respectively as

$$\begin{aligned} E[x(0)] &= x_0 && (56) \\ E[x(0)x^T(0)] &= X_0\; > 0 && (57) \end{aligned}$$

The most widely used method, [15], is to consider $x(0)$ uniformly distributed on a sphere of radius σ, that is

$$X_0 = \sigma I_n \tag{58}$$

with $\sigma = 1$ being the obvious choice. Note that alternative methods to deal with this dependence can be found in [3], [19]. We choose the following modified cost

$$J_2 \;=\; E[\, Tr(J_2')\,]$$

$$
\begin{aligned}
&= E\ \{\ Tr[x^T(0)\ P_2\ x(0)]\ \} \\
&= E\ \{\ Tr(P_2\ x(0)\ x^T(0)]\ \} \\
&= Tr\ \{E[P_2\ x(0)\ x^T(0)]\} \\
&= Tr(P_2\ X_0) \qquad (59)
\end{aligned}
$$

In view of (30) and (59), we finally define the overall minimizing quantity, which is associated with both the robustness of the matrix $\bar{A}_0$ and the control performance of the closed-loop system

$$
\begin{aligned}
J_A &= J_1 + J_2 \\
&= Tr[\ \alpha^2 Z^2 + 2\alpha PZ + P^2 + \frac{1}{\alpha}\ \bar{A}_0^T\bar{A}_0 + L_1\ (\bar{A}_0^T P\bar{A}_0 - P + Q) \\
&\qquad + P_2X_0 + L_2\ (\bar{A}_0^T P_2\bar{A}_0 - P_2 + \bar{Q})\ ] \qquad (60)
\end{aligned}
$$

where, similarly to (30), we have reduced the problem to an unconstrained minimization one by including (55) in the minimizing quantity via the Langrange multiplier matrix L_2. It is now obvious that with the introduction of P_2 and L_2 in the new cost J_A, we need to consider its partial derivatives with respect to these new matrix variables as well. On the other hand, again due to P_2 and L_2, we have some additional terms in Δ_K^1 of (38). Therefore, the partial derivatives of the final cost J_A with respect to all the matrix variables entailed have as follows

$$
\begin{aligned}
\frac{\partial\ J_A}{\partial L_1} &= \Delta_{L_1}^A = \bar{A}_0^T P\bar{A}_0 - P + Q \qquad (61) \\
\frac{\partial\ J_A}{\partial L_2} &= \Delta_{L_2}^A = \bar{A}_0^T P_2\bar{A}_0 - P_2 + \bar{Q} \qquad (62) \\
\frac{\partial\ J_A}{\partial \alpha} &= \Delta_{\alpha}^A = 2\alpha\ Tr(Z^2) + 2\ Tr(PZ) - \frac{1}{\alpha^2}\ Tr(\bar{A}_0^T\bar{A}_0) \qquad (63) \\
\frac{\partial\ J_A}{\partial P} &= \Delta_{P}^A = 2P + 2\alpha Z + \bar{A}_0 L_1^T\bar{A}_0^T - L_1^T \qquad (64) \\
\frac{\partial\ J_A}{\partial P_2} &= \Delta_{P_2}^A = X_0^T + \bar{A}_0 L_2^T\bar{A}_0^T - L_2^T \qquad (65) \\
\frac{\partial\ J_A}{\partial K} &= \Delta_{K}^A = \frac{2}{\alpha}\ B_0^T B_0 K C_0 C_0^T + \frac{2}{\alpha}\ B_0^T A_0 C_0^T +\ R_1 K C_0\ (L_2 + L_2^T)\ C_0^T \\
&\qquad + B_0^T P B_0 K C_0\ (L_1 + L_1^T)\ C_0^T + B_0^T P A_0\ (L_1 + L_1^T)\ C_0^T \\
&\qquad + B_0^T P_2 B_0 K C_0\ (L_2 + L_2^T)\ C_0^T + B_0^T P_2 A_0\ (L_2 + L_2^T)\ C_0^T \qquad (66)
\end{aligned}
$$

In order to minimize J_A, the algorithm of the previous subsection can be used again, the only difference being that steps *(M1)*, *(M2)* have to be replaced by the following

M1a. Substitute the gain matrix K_j in the gradients of (61)-(65), set them to zero, that is $\Delta_{L1}^A = 0$, $\Delta_{L2}^A = 0$, $\Delta_{\alpha}^A = 0$, $\Delta_P^1 = 0$, $\Delta_{P_2}^A = 0$ and solve for P, P_2, α, L_1, L_2 respectively, in that specific order.

M2a. Substitute these parameters in (66) and compute

$$\Delta_{K_j}^A = \begin{pmatrix} (\sigma_1^j)^T \\ \vdots \\ (\sigma_r^j)^T \end{pmatrix} \tag{67}$$

where $(\sigma_l^j)^T, l = 1,..,m$ are the $1 \times q$ rows of $\Delta_{K_j}^A$.

All the other steps of the algorithm remain the same, bearing in mind that we now refer to J_A, instead of J_1.

4 Perturbations in B or C

In this section, we consider perturbations in either the input matrix B or the output matrix C. Since both cases are similar, we study the case of perturbations in B. Note that the approach outlined next readily applies to the case of perturbations in C. Therefore, we consider the linear discrete-time system with the state-space description

$$x(k+1) = A_0\, x(k) + B\, u(k) \tag{68}$$
$$y(k) = C_0\, x(k) \tag{69}$$

Both unstructured and structured perturbations for the input matrix B are again of interest here, that is

$$B = B_0 + \Delta B \tag{70}$$
$$B = B_0 + \sum_{i=1}^{m} \lambda_i B_i \tag{71}$$

where again $\lambda_i, i = 1,..,m$ denote real, uncertain parameters and $B_i, i = 1,..,m$ are constant, known matrices.

4.1 Unstructured perturbations

We apply the static output feedback law (16) to the system of (68)-(70). The closed-loop system is then described by

$$\begin{aligned} x(k+1) &= [\, A_0 + (B_0 + \Delta B)\, KC_0 \,]\, x(k) \\ &= [\, \bar{A}_0 + (\Delta B)\, KC_0 \,]\, x(k) \end{aligned} \tag{72}$$

In view of (6), (23), we have

$$\begin{aligned} \sigma_{max}[\, (\Delta B)\, KC_0 \,] &\leq \sigma_{max}(\Delta B)\, \sigma_{max}(KC_0) \\ &< \sqrt{\frac{\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha}\, \Omega_1)}{\sigma_{max}(\alpha Z + P)}} \end{aligned} \tag{73}$$

or finally

$$\sigma_{max}(\Delta B) \;<\; \frac{\sqrt{\frac{\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha}\, \Omega_1)}{\sigma_{max}(\alpha Z + P)}}}{\sigma_{max}(KC_0)} \tag{74}$$

where P, Q are given in (20) and Ω_1 is defined in (21). If this sufficient condition is satisfied by the unstructured perturbations of B, then the stability of (72) is maintained. It is obvious, that in addition to the minimization objectives of (A.1) and (A.2), as studied in the previous section, we also need to

- *(A.3) : minimize* $\sigma_{max}(KC_0)$

Similarly to (25) and (28), instead of *(A.3)*, we choose to minimize its upper bound, that is

$$J_3 = Tr[\, (KC_0)^T (KC_0) \,] \tag{75}$$

Therefore, the minimizing quantity for the case of unstructured perturbations in B is

$$J^r_{B_u} = J_1 + J_3 \tag{76}$$

when no performance specifications are considered, where J_1 is defined in (30). When performance specifications are included in the minimizing quantity, then this quantity is

$$J_{B_u}^{r,p} = J_A + J_3 \tag{77}$$

where J_A is defined in (60). The algorithm of the previous section can be used here for $J_{B_u}^{r}$ and $J_{B_u}^{r,p}$ as well, the only difference being that the term

$$\frac{\partial\, J_3}{\partial K} = 2\; K C_0 C_0^T \tag{78}$$

needs to be added to (38) and (66).

4.2 Structured perturbations

In this part, we assume structured perturbation matrices for the input matrix B, as indicated in (71) above. The closed-loop system, after the output feedback law (16), is then described by

$$\begin{aligned} x(k+1) &= (\; A_0 + B_0 K C_0 + \sum_{i=1}^{m} \lambda_i B_i K C_0 \;) x(k) \\ &= (\; \bar{A}_0 + \sum_{i=1}^{m} \lambda_i B_i K C_0 \;) x(k) \end{aligned} \tag{79}$$

Defining

$$B^\star = \begin{pmatrix} B_1 \\ \vdots \\ B_m \end{pmatrix} \tag{80}$$

we have

$$\begin{pmatrix} B_1 K C_0 \\ \vdots \\ B_m K C_0 \end{pmatrix} = B^\star K C_0 \tag{81}$$

Hence, (11) readily gives

$$\sum_{i=1}^{m} \lambda_i^2 \; < \; \frac{\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha}\,\Omega_1)}{\sigma_{max}^2(B^\star K C_0)\,\sigma_{max}(\alpha Z + P)} \tag{82}$$

If this sufficient condition is satisfied by the structured perturbation matrices of B, then the stability of (79) is maintained. Now, in addition to the minimization objectives of *(A.1)* and *(A.2)*, we also need to

- *(A.4) : minimize* $\sigma_{max}(B^\star K C_0)$

Similarly to (25), (28), (75), instead of *(A.4)*, we choose to minimize its upper bound, that is

$$J_4 = Tr[\,(B^\star K C_0)^T (B^\star K C_0)\,] \tag{83}$$

so that the minimizing quantity for the case of structured perturbations in B is given as

$$J_{B_s}^{r} \;=\; J_1 + J_4 \tag{84}$$

$$J_{B_s}^{r,p} \;=\; J_A + J_4 \tag{85}$$

similarly to (76), (77) before. The algorithm of the previous section applies here for $J_{B_s}^{r}$ and $J_{B_s}^{r,p}$ as well, the only difference now being that the term

$$\frac{\partial\, J_4}{\partial K} = 2\,(B^\star)^T B^\star K C_0 C_0^T \tag{86}$$

needs to be added to (38) and (66).

5 Perturbations in (A, B) or (A, C)

In this section, we consider perturbations in the system matrix A and the input matrix B or in A and the output matrix C. Again, since both cases are similar, we study the case of perturbations in (A, B). Obviously, the approach outlined next applies to the case of perturbations in (A, C) as well.

5.1 Unstructured perturbations

We consider the linear discrete-time system with the state-space description

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) && (87) \\ y(k) &= C_0\,x(k) && (88) \end{aligned}$$

where

$$\begin{aligned} A &= A_0 + \Delta A && (89) \\ B &= B_0 + \Delta B && (90) \end{aligned}$$

With the output feedback law (16), the closed-loop system is given by

$$\begin{aligned} x(k+1) &= [\, A_0 + \Delta A + B_0 K C_0 + (\Delta B) K C_0 \,]\, x(k) \\ &= [\, \bar{A}_0 + \Delta A + (\Delta B) K C_0 \,]\, x(k) && (91) \end{aligned}$$

In view of (23), (24), we have

$$\sigma_{max}[\, \Delta A + (\Delta B) K C_0 \,] \;\leq\; \sigma_{max}(\Delta A) \;+\; \sigma_{max}(\Delta B)\, \sigma_{max}(K C_0) \qquad (92)$$

Therefore, the stability of the closed-loop system (91) is maintained, if the perturbation matrices ΔA and ΔB satisfy the following sufficient condition

$$\sigma_{max}(\Delta A) \;+\; \sigma_{max}(\Delta B)\, \sigma_{max}(K C_0) \;<\; \sqrt{\frac{\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha}\,\Omega_1)}{\sigma_{max}(\alpha Z + P)}} \qquad (93)$$

where again P, Q are defined in (20) and Ω_1 in (21). The region that satisfies the inequality

$$x + \gamma y \;< \delta \qquad (94)$$

for positive x, y, γ, δ is the shaded triangle shown in Fig. 1. Obviously, this region gets larger for larger δ and smaller γ. Therefore, in order to maximize the stability region that is

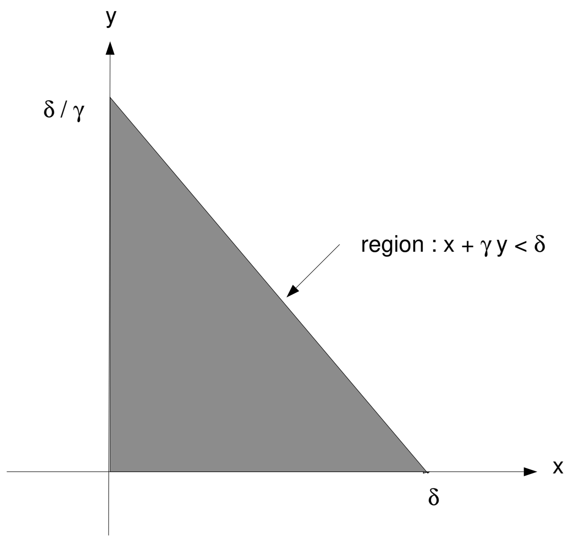

Figure 1: Stability region for unstructured perturbations in (A, B)

defined by (93), we need to maximize the RHS of (93) and minimize $\sigma_{max}(KC_0)$. The first corresponds to objectives *(A.1)*, *(A.2)* and the latter to objective *(A.3)*. Therefore, we see that the present case has the same objectives with the case of unstructured perturbations in B or C we studied before, which implies that (78) needs to be added again to (38) and (66).

5.2 Structured perturbations

We consider again the system of (87), (88), but now A and B are perturbed as in (8) and (71), that is

$$A = A_0 + \sum_{i=1}^{m_A} \kappa_i A_i \tag{95}$$

$$B = B_0 + \sum_{j=1}^{m_B} \lambda_j B_j \tag{96}$$

We define

$$\hat{K} = [\ \kappa_1\ \ \kappa_2\ \ \cdots\ \ \kappa_{m_A}\ ]^T \tag{97}$$

$$\hat{\Lambda} = [\ \lambda_1\ \ \lambda_2\ \ \cdots\ \ \lambda_{m_B}\ ]^T \tag{98}$$

$$\hat{\Theta} = [\ \hat{\theta}_1\ \ \hat{\theta}_2\ \ \cdots\ \ \hat{\theta}_{m_A+m_B}\ ]^T = [\ \hat{K}^T\ \ \hat{\Lambda}^T\ ]^T \tag{99}$$

$$
\begin{aligned}
\hat{A} &= [\ A_1^T \ \ A_2^T \ \ \cdots \ \ A_{m_A}^T\ ]^T && (100) \\
\hat{B} &= [\ (B_1KC_0)^T \ \ (B_2KC_0)^T \ \ \cdots \ \ (B_{m_B}KC_0)^T\ ]^T && (101) \\
\hat{\Pi} &= [\ (\hat{A})^T \ \ (\hat{B})^T\ ]^T && (102)
\end{aligned}
$$

With the output feedback law (16), the closed-loop system is given by

$$
\begin{aligned}
x(k+1) &= (\ A_0 + B_0KC_0 + \sum_{i=1}^{m_A} \kappa_i A_i + \sum_{j=1}^{m_B} \lambda_j B_j K C_0\ )\ x(k) \\
&= [\ \bar{A}_0 + (\hat{\Theta} \otimes I_n)^T\ \hat{\Pi}\ ]\ x(k) && (103)
\end{aligned}
$$

In view of (11), the above system is asymptotically stable, if the uncertain parameters satisfy

$$
\sum_{i=1}^{m_A+m_B} \hat{\theta}_i^2 \ < \ \frac{\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha}\ \Omega_1)}{\sigma_{max}^2(\hat{\Pi})\ \sigma_{max}(\alpha Z + P)} \tag{104}
$$

Therefore, in addition to objectices *(A.1)*, *(A.2)*, we also need to

- *(A.5) : minimize* $\sigma_{max}(\hat{\Pi})$

As before, we choose to minimize $Tr(\hat{\Pi}^T\hat{\Pi})$. Since

$$
\hat{\Pi}^T\hat{\Pi} = \sum_{i=1}^{m_A} A_i^T A_i + \sum_{j=1}^{m_B} (B_jKC_0)^T(B_jKC_0) \tag{105}
$$

we simply need to add the following term to (30) and (60)

$$
J_5 = Tr[\ \sum_{j=1}^{m_B} (B_jKC_0)^T(B_jKC_0)\ ] \tag{106}
$$

so that the minimizing quantities for the case of structured perurbations in A and B are given by

$$
\begin{aligned}
J_{AB}^r &= J_1 + J_5 && (107) \\
J_{AB}^{r,p} &= J_A + J_5 && (108)
\end{aligned}
$$

similarly to (84), (85) before. Therefore, our algorithm can be used again, with the addition of the term

$$\frac{\partial J_5}{\partial K} = \sum_{j=1}^{m_B} 2\, B_j^T B_j K C_0 C_0^T \tag{109}$$

to (38) and (66).

6 Perturbations in (A, B, C)

In this section, we consider unstructured perturbations in the system matrix A, the input matrix B and the output matrix C. Specifically, we consider the linear discrete-time system with the state-space description

$$\begin{aligned} x(k+1) &= A\, x(k) + B\, u(k) \qquad (110) \\ y(k) &= C\, x(k) \qquad (111) \end{aligned}$$

where

$$\begin{aligned} A &= A_0 + \Delta A \qquad (112) \\ B &= B_0 + \Delta B \qquad (113) \\ C &= C_0 + \Delta C \qquad (114) \end{aligned}$$

With the output feedback law (16), the closed-loop system is given by

$$\begin{aligned} x(k+1) &= [\, A_0 + \Delta A + B_0 K C_0 + (\Delta B) K C_0 + B_0 K (\Delta C) + (\Delta B) K (\Delta C)\,]\, x(k) \\ &= [\, \bar{A}_0 + \Delta A + (\Delta B) K C_0 + B_0 K (\Delta C) + (\Delta B) K (\Delta C)\,]\, x(k) \qquad (115) \end{aligned}$$

Defining

$$\Delta_{ABC} = \Delta A + (\Delta B) K C_0 + B_0 K (\Delta C) + (\Delta B) K (\Delta C) \tag{116}$$

we have

$$\begin{aligned}\sigma_{max}(\Delta_{ABC}) \;\leq\; & \sigma_{max}(\Delta A) \;+\; \sigma_{max}(\Delta B)\, \sigma_{max}(K)\, \sigma_{max}(C_0) \\ & + \sigma_{max}(B_0)\, \sigma_{max}(K)\, \sigma_{max}(\Delta C) + \; \sigma_{max}(\Delta B)\, \sigma_{max}(K)\, \sigma_{max}(\Delta C) \qquad (117)\end{aligned}$$

Therefore, the stability of the closed-loop system (115) is maintained, if the perturbation matrices ΔA, ΔB and ΔC satisfy the following sufficient condition

$$\begin{aligned}\sigma_{max}(\Delta A) \;+\; \sigma_{max}(\Delta B)\, \sigma_{max}(K)\, \sigma_{max}(C_0) + \; \sigma_{max}(B_0)\, \sigma_{max}(K)\, \sigma_{max}(\Delta C) & \\ + \sigma_{max}(\Delta B)\, \sigma_{max}(K)\, \sigma_{max}(\Delta C) \;<\; & \sqrt{\frac{\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha}\, \Omega_1)}{\sigma_{max}(\alpha Z + P)}} \qquad (118)\end{aligned}$$

where again P, Q are defined in (20) and Ω_1 in (21). The above inequality defines a region in $\Re^3$ for $\sigma_{max}(\Delta A)$, $\sigma_{max}(\Delta B)$ and $\sigma_{max}(\Delta C)$. As shown in appendix A, in order to maximize the volume of this region, we need to maximize the RHS of (118), which corresponds to objectives *(A.1)*, *(A.2)* above, and also

- *(A.6) : minimize* $\sigma_{max}(K)$

which, similarly to what we did before, corresponds to the minimization of

$$J_6 = Tr(K^T K) \qquad (119)$$

Therefore, the minimizing quantity for the case of unstructured perturbations in A, B, C is given by

$$\begin{aligned} J^r_{ABC} &= J_1 + J_6 && (120)\\ J^{r,p}_{ABC} &= J_A + J_6 && (121)\end{aligned}$$

similarly to (107), (108) before. Therefore, our algorithm can be used again, with the addition of the term

$$\frac{\partial\, J_6}{\partial K} = 2\, K \qquad (122)$$

to (38) and (66). Note that all the above hold under the restriction that

$$\frac{1}{\sigma_{max}\ (C_0)}\ \frac{1}{\sigma_{max}\ (B_0)}\ \frac{1}{\sigma_{max}\ (K)}\ \sqrt{\frac{\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha}\ \Omega_1)}{\sigma_{max}(\alpha Z + P)}}\ >\ 10^{-5} \tag{123}$$

which, as discussed in the appendix, is a condition that practically can not be violated.

7 Illustrative examples

Example 1 Consider the scalar system

$$x(k+1) = 0.5\ x(k) + u(k), \qquad x(0) = 1.0 \tag{124}$$

with state feedback $u(k) = Kx(k)$. This system was studied in [12], where the following LQR cost was used

$$J_2' = \sum_{k=0}^{\infty} x^2(k) + u^2(k) \tag{125}$$

that is $Q_1 = R_1 = 1$. The derived bound for unstructured perturbations in the system matrix A was

$$\sigma_{max}(\Delta A)\ <\ 0.8436 \tag{126}$$

for a gain of $K = -0.3436$. We apply our method for the same LQR term. Choosing $Q = 1.30$, $Z = 0.60$, initial stabilizing gain $K_1 = 0.1$ and positive definite matrix $D_1 = 0.05$, we obtain a stabilizing gain of $K = -0.499984$, which corresponds to

$$\sigma_{max}(\Delta A)\ <\ 1.00 \tag{127}$$

which compares favorably to (126) of [12]. The components of the minimizing quantity (60) that are associated with the robustness and the performance objectives are respectively

$$J_1 \ = \ 1.69 \tag{128}$$

$$J_2 \ = \ 1.25 \tag{129}$$

Now, if we neglect the performance specifications, as indicated by the above LQR cost, and focus on just the maximization of the robustness bound, that is the minimization of (30), we obtain a stabilizing gain of $K = -0.499909$, which corresponds to

$$\begin{aligned} \sigma_{max}(\Delta A) &< 0.9999 \qquad (130) \\ J_1 &= 1.6902 \qquad (131) \end{aligned}$$

Note that the same Q, Z, K_1 and Q_1 have been used. As we see, in this scalar case, we obtain almost the same results for the final stabilizing gain K, the uncertainty bound and the robustness component J_1 of the minimizing quantity, no matter whether the LQR term is included or not in the minimizing quantity. Note, however, that this is not the case, in general, for MIMO systems, as we can clearly see in the example that follows.

Example 2 Consider the discrete-time system of (12)-(13) with

$$A_0 = \begin{pmatrix} -2 & 1.20 \\ 0.10 & -0.10 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 1.2 & 1.5 \\ 1 & 1 \end{pmatrix} \qquad (132)$$

First we find the gain matrix K that maximizes the bounds of (6), (11) for unstructured perturbations in the system matrix A, without considering any performance specifications; therefore we minimize J_1 of (30). We consider $Q = I_2$ and

$$Z = \begin{pmatrix} 0.6631 & -0.0665 \\ -0.0665 & 0.9869 \end{pmatrix} \qquad (133)$$

We also choose as initial stabilizing gain

$$K_1 = \begin{pmatrix} -1.5 & 3.5 \\ 0 & 0 \end{pmatrix} \qquad (134)$$

and as initial positive definite matrix $D_1 = 0.01\ I_2$. The stabilizing gain obtained by our algorithm is

$$K = \begin{pmatrix} -10.6677 & 14.8015 \\ 6.6660 & -9.0005 \end{pmatrix} \qquad (135)$$

which corresponds to

$$\begin{aligned} \sigma_{max}(\Delta A) &< 0.9997 \qquad (136) \\ J_1 &= 2.0011 \qquad (137) \end{aligned}$$

Now, we include the LQR cost of (50) in the minimizing quantity, that is we minimize J_A of (60). We consider $Q_1 = R_1 = X_0 = I_2$ and the same initial positive definite matrix D_1, initial gain matrix K_1, Q, Z as before. The obtained stabilizing gain is now

$$K = \begin{pmatrix} -10.1829 & 14.1263 \\ 0.7532 & -1.0181 \end{pmatrix} \tag{138}$$

which corresponds to

$$\sigma_{max}(\Delta A) \ < \ 0.7575 \tag{139}$$
$$J_A = 9.7996 \tag{140}$$

where the components that are associated with the robustness and the performance objectives are respectively

$$J_1 \ = \ 2.6700 \tag{141}$$
$$J_2 \ = \ 7.1296 \tag{142}$$

Comparing (141) to (136), we see that the robustness component is greater, when we consider performance specifications in the minimization process. This was expected, since in that case we have a harder task, whereas in the case of no control performance, we simply try to minimize the robustness component, that is J_1. In that respect, it is not surprising that the derived bound is smaller in the case of both objectives, since the inclusion of the performance term in the minimizing quantity adds an additional requirement to the minimizing task, whereas in the case of no performance specifications we simply try to maximize the bound.

Example 3 We repeat the previous example for structured perturbations in A; specifically we assume that

$$\Delta A = \kappa_1 \begin{pmatrix} 0.20 & 0.10 \\ 0 & 0 \end{pmatrix} \ + \ \kappa_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{143}$$

Using the results of example 2, for the case of only robustness specifications, we obtain

$$\kappa_1^2 + \kappa_2^2 \ < \ (0.9756)^2 \tag{144}$$

whereas for the case of both robustness and performance specifications, we obtain

$$\kappa_1^2 + \kappa_2^2 \;<\; (0.7392)^2 \tag{145}$$

Example 4 Consider the discrete-time system (132) of the previous two examples with unstructured perurbations in the input matrix B. First, we include performance specifications in the minimizing quantity. We consider Q, Q_1, R_1, X_0, K_1, D_1 as in example 2 and

$$Z = \begin{pmatrix} 1.4183 & -0.5209 \\ -0.5209 & 0.9241 \end{pmatrix} \tag{146}$$

Our algorithm gives a stabilizing gain of

$$K = \begin{pmatrix} -9.5488 & 13.2458 \\ 0.5427 & -0.7376 \end{pmatrix} \tag{147}$$

and

$$\sigma_{max}(\Delta B) \;<\; 0.3141 \tag{148}$$
$$J_{B_u}^{r,p} \;=\; 14.6404 \tag{149}$$

where

$$J_1 \;=\; 3.4616 \tag{150}$$
$$J_2 \;=\; 6.8104 \tag{151}$$
$$J_3 \;=\; 4.3684 \tag{152}$$

Then, we neglect the control specifications. For the same matrices of interest, we obtain a stabilizing gain of

$$K = \begin{pmatrix} -9.9777 & 13.8439 \\ 0.8420 & -1.1372 \end{pmatrix} \tag{153}$$

and

$$\sigma_{max}(\Delta B) \;<\; 0.3554 \tag{154}$$
$$J_{B_u}^{r} \;=\; 7.7531 \tag{155}$$

where

$$J_1 = 2.9616 \tag{156}$$
$$J_3 = 4.7915 \tag{157}$$

Note the improvement of the bound in (154), compared to (148).

Example 5 We repeat the previous example for structured perturbations in B; specifically we assume that

$$\Delta B = \lambda_1 \begin{pmatrix} 1 & -0.1 \\ 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 0.1 \\ -0.1 & 0 \end{pmatrix} \tag{158}$$

Considering Q, Q_1, R_1, X_0, K_1, D_1 as in examples 2-4 and

$$Z = \begin{pmatrix} 1.4458 & -0.3709 \\ -0.3709 & 0.9184 \end{pmatrix} \tag{159}$$

we obtain

$$\lambda_1^2 + \lambda_2^2 < (0.3030)^2 \tag{160}$$
$$J_{B_s}^{r,p} = 14.7122 \tag{161}$$

where

$$J_1 = 3.4898 \tag{162}$$
$$J_2 = 6.8060 \tag{163}$$
$$J_4 = 4.4164 \tag{164}$$

Neglecting the performance specifications and keeping the same matrices of interest, we obtain

$$\lambda_1^2 + \lambda_2^2 < (0.3208)^2 \tag{165}$$
$$J_{B_s}^{r} = 7.8135 \tag{166}$$

where

$$J_1 = 3.2104 \tag{167}$$
$$J_4 = 4.6031 \tag{168}$$

Note again the improvement of the bound in (165), compared to (160).

Example 6 We consider again the nominal system of examples 2-5 and assume unstructured perturbations in both A and B. As discussed before, the minimization problem that we need to solve for this case is the same with the one of the case of unstructured perturbations in B. Therefore, we can use the results of example 4. For the cases of robustness specifications and robustness/peformance specifications, we obtain the following stability regions respectively

$$\sigma_{max}(\Delta A) + 2.1885\ \sigma_{max}(\Delta B) < 0.7778 \tag{169}$$
$$\sigma_{max}(\Delta A) + 2.0900\ \sigma_{max}(\Delta B) < 0.6565 \tag{170}$$

which correspond to the triangle of Fig. 1 for $x = \sigma_{max}(\Delta A)$ and $y = \sigma_{max}(\Delta B)$ and

$$\delta = 0.7778 \quad \frac{\delta}{\gamma} = 0.3554 \tag{171}$$
$$\delta = 0.6565 \quad \frac{\delta}{\gamma} = 0.3141 \tag{172}$$

for the above cases respectively. Note that for the case of only robustness specifications, the stability region is obviously larger than for the case of robustness/performance specifications.

Example 7 We revisit the system of example 5. In addition, we consider structured perturbations in A as well. Therefore, the perturbation matrices for A and B are as indicated in (143), (158) respectively. Using the same matrices of interest, as in example 5, for the case of robustness/control specifications, we obtain a stabilizing gain of

$$K = \begin{pmatrix} -9.5345 & 13.2270 \\ 0.2867 & -0.3677 \end{pmatrix} \tag{173}$$

and

$$\kappa_1^2 + \kappa_2^2 + \lambda_1^2 + \lambda_2^2 < (0.2728)^2 \tag{174}$$
$$J_{AB}^{r,p} = 14.7105 \tag{175}$$

where

$$J_1 = 3.4949 \tag{176}$$
$$J_2 = 6.8071 \tag{177}$$
$$J_5 = 4.4085 \tag{178}$$

For the case of robustness specifications only, we obtain a stabilizing gain of

$$K = \begin{pmatrix} -9.8876 & 13.7204 \\ -1.0071 & 1.6152 \end{pmatrix} \tag{179}$$

and

$$\kappa_1^2 + \kappa_2^2 + \lambda_1^2 + \lambda_2^2 < (0.2899)^2 \tag{180}$$
$$J_{AB}^r = 7.8135 \tag{181}$$

where

$$J_1 = 3.2145 \tag{182}$$
$$J_5 = 4.5990 \tag{183}$$

Note, once more, the improvement of the bound in (180) compared to (174) .

8 Conclusions

An optimization algorithm for the design of robust output feedback controllers for linear uncertain discrete-time systems has been presented. This algorithm utilizes a version of the Broyden family method of conjugate directions which is based on the BFGS rule. The minimizing quantity reflects the twofold optimization objective, which is the simultaneous maximizaton of established uncertainty bounds and the minimization of the typical LQR performance criterion. The first objective is based on recently established improved bounds that were developed in [13]. Note that the algorithm has also been applied to the case that the control specifications implied by the LQR term are not included in the minimizing quantity, so that the only objective is the design of a stabilizing output feedback controller that maximizes the uncertainty bounds.

In that case, the derived stability bounds are, in general, larger than the ones derived in the case of the robustness/LQR minimizing quantity. This was expected, since the inclusion of the LQR term in the minimizing quantity added an additonal requirement to the optimization task.

Previous related work was restricted to the case of unstructured perturbations in the system matix A. Here, a unified approach to both unstructured and structured perturbations in A has been presented. It has been shown that the present design process improves significantly the unstructured bound derived in [12]. Additionally, the cases of unstructured/structured perturbations in B or C, in (A, B) or (A, C), together with the case of unstructured perturbations in (A, B, C) are also studied. Numerous examples have been provided to illustrate the results. A case that remains to be addressed is the one of structured perturbations in all the state-space matrices. The recently developed bound of [13] does not appear very convenient for that case and therefore, alternative bounds need to be investigated. Finally, the continuous counterpart of the present discrete-time case, mostly for the various cases of perturbations involving the state-space matrices that were mentioned above, remains to be further investigated.

A Computation of volume

In this section, we are interested in computing the volume that is confined by the following inequalities

$$\begin{aligned} x, y, z &> 0 && (184) \\ x + ay + bz + cyz &< d \qquad for \;\; a, b, c, d > 0 && (185) \end{aligned}$$

This volume is computed as follows

$$\begin{aligned} V &= \int_0^d \int_0^{\frac{d}{a}-\frac{x}{a}} \frac{d - x - ay}{cy + b} \; dy \; dx \\ &= \int_0^d \int_0^{\frac{d}{a}-\frac{x}{a}} \frac{d - x + \frac{ab}{c} - \frac{a}{c}\,(cy + b)}{cy + b} \; dy \; dx \\ &= \int_0^d \int_0^{\frac{d}{a}-\frac{x}{a}} \left(\frac{d - x + \frac{ab}{c}}{cy + b} - \frac{a}{c} \right) \; dy \; dx \\ &= \int_0^d \left[(d - x + \frac{ab}{c}) \frac{\ln \mid cy + b \mid}{c} - \frac{ay}{c} \right] \Bigg|_0^{\frac{d}{a}-\frac{x}{a}} \; dx \\ &= \int_0^d \left[\frac{d - x + \frac{ab}{c}}{c} \left(\ln \mid \frac{cd}{a} - \frac{cx}{a} + b \mid - \ln b \right) - \frac{d}{c} + \frac{x}{c} \right] \; dx \end{aligned}$$

$$
\begin{aligned}
&= \int_0^d \frac{d - x + \frac{ab}{c}}{c} \ln \mid 1 + \frac{cd}{ab} - \frac{cx}{ab} \mid \quad dx \quad + \quad \int_0^d \frac{x-d}{c} \; dx \\
&= \int_0^d \frac{d - x + \frac{ab}{c}}{c} \ln \mid 1 + \frac{cd}{ab} - \frac{cx}{ab} \mid \quad dx \quad - \quad \frac{d^2}{2c} \qquad (186)
\end{aligned}
$$

Substituting

$$
\begin{aligned}
u &= 1 + \frac{cd}{ab} - \frac{cx}{ab} \qquad (187)\\
du &= -\frac{c}{ab}\, dx \qquad (188)
\end{aligned}
$$

we rewrite the above integral as follows

$$
\begin{aligned}
V &= \frac{a^2 b^2}{c^3} \int_1^{1+\frac{cd}{ab}} u \, \ln \mid u \mid \quad du \quad - \quad \frac{d^2}{2c} \\
&= \frac{a^2 b^2}{c^3} \int_1^{1+\frac{cd}{ab}} u \, \ln \,(u) \;\; du \quad - \quad \frac{d^2}{2c} \\
&= \frac{a^2 b^2}{c^3} \left(\frac{u^2}{2} \ln \,(u) \Big|_1^{1+\frac{cd}{ab}} - \int_1^{1+\frac{cd}{ab}} \frac{u}{2} \;\; du \right) \quad - \quad \frac{d^2}{2c} \\
&= \frac{a^2 b^2}{c^3} \left(\frac{u^2}{2} \ln \,(u) - \frac{u^2}{4} \right) \Bigg|_1^{1+\frac{cd}{ab}} \quad - \quad \frac{d^2}{2c} \\
&= \frac{a^2 b^2}{c^3} \left[\frac{1}{2} \,(1 + \frac{cd}{ab})^2 \;\ln \,(1 + \frac{cd}{ab}) - \frac{1}{4} \,(1 + \frac{cd}{ab})^2 + \frac{1}{4} \right] \quad - \quad \frac{d^2}{2c} \\
&= \frac{a^2 b^2}{4c^3} \left\{ \,(1 + \frac{cd}{ab})^2 \;\left[\ln \,(1 + \frac{cd}{ab})^2 - 1 \right] + 1 \right\} \quad - \quad \frac{d^2}{2c} \qquad (189)
\end{aligned}
$$

If a, b can be written in terms of c, that is

$$
\begin{aligned}
a &= \omega_1 \, c \qquad for \; \omega_1 \; > 0 \qquad (190)\\
b &= \omega_2 \, c \qquad for \; \omega_2 \; > 0 \qquad (191)
\end{aligned}
$$

then, (189) can be written as

$$
V \,(c, d) = \frac{\omega_1^2 \, \omega_2^2 \, c}{4} \left\{ \,(1 + \frac{d}{\omega_1 \, \omega_2 \, c})^2 \;\left[\ln \,(1 + \frac{d}{\omega_1 \, \omega_2 \, c})^2 - 1 \right] + 1 \right\} \quad - \quad \frac{d^2}{2c} \qquad (192)
$$

Since we need to know how c and d affect $V(c,d)$, we compute next the partial derivatives of V with respect to c and d. First, we see how $V(c,d)$ is affected by d

$$
\begin{aligned}
\frac{\partial\, V\,(c,d)}{\partial\, d} &= \omega_1\,\omega_2\,(1+\frac{d}{\omega_1\,\omega_2\,c})\;\ln\,(1+\frac{d}{\omega_1\,\omega_2\,c})+\;\frac{1}{2}\,\omega_1\,\omega_2\,(1+\frac{d}{\omega_1\,\omega_2\,c}) \\
&\quad -\frac{1}{2}\,\omega_1\,\omega_2\,(1+\frac{d}{\omega_1\,\omega_2\,c})-\frac{d}{c} \\
&= \omega_1\,\omega_2\,(1+\frac{d}{\omega_1\,\omega_2\,c})\;\ln\,(1+\frac{d}{\omega_1\,\omega_2\,c})-\;\frac{d}{c} \\
\Longrightarrow\;(\frac{1}{\omega_1\,\omega_2})\,\frac{\partial\, V\,(c,d)}{\partial\, d} &= F_1\,(\frac{d}{\omega_1\,\omega_2\,c}) \qquad (193)
\end{aligned}
$$

where obviously

$$
F_1\,(x)=(1+x)\;\ln\,(1+x)-x \qquad (194)
$$

We can easily verify, numerically, that $F_1\,(x)\;>\;0$ for $x>10^{-7}$. Hence, in view of (193), (194) we have

$$
\frac{d}{\omega_1\,\omega_2\,c}\;>\;10^{-7}\quad\Longrightarrow\quad\frac{\partial\, V\,(c,d)}{\partial\, d}\;>\;0 \qquad (195)
$$

Next, we see how $V\,(c,d)$ is affected by c

$$
\begin{aligned}
\frac{\partial\, V\,(c,d)}{\partial\, c} &= \frac{\omega_1^2\,\omega_2^2}{4}\left\{(1+\frac{d}{\omega_1\,\omega_2\,c})^2\;\left[\ln\,(1+\frac{d}{\omega_1\,\omega_2\,c})^2-1\right]+1\right\} \\
&\quad +\frac{\omega_1^2\,\omega_2^2\,c}{2}\left[-2\,(1+\frac{d}{\omega_1\,\omega_2\,c})\;\ln\,(1+\frac{d}{\omega_1\,\omega_2\,c})\,(\frac{d}{\omega_1\,\omega_2\,c^2})\right. \\
&\quad \left.-\;(1+\frac{d}{\omega_1\,\omega_2\,c})\,(\frac{d}{\omega_1\,\omega_2\,c^2})+\;(1+\frac{d}{\omega_1\,\omega_2\,c})\,(\frac{d}{\omega_1\,\omega_2\,c^2})\right]+\;\frac{d^2}{2\,c^2} \\
\Longrightarrow\;(\frac{1}{\omega_1^2\,\omega_2^2})\,\frac{\partial\, V\,(c,d)}{\partial\, c} &= F_2\,(\frac{d}{\omega_1\,\omega_2\,c}) \qquad (196)
\end{aligned}
$$

where obviously

$$
\begin{aligned}
F_2\,(x) &= \frac{1}{2}\,(1+x)^2\;\ln\,(1+x)-\;\frac{1}{4}\,(1+x)^2+\;\frac{1}{4}-\;x\,(1+x)\;\ln\,(1+x)+\;\frac{x^2}{2} \\
&= \frac{x}{4}\,(x-2)-\;\frac{1}{2}\,(x^2-1)\;\ln\,(1+x) \qquad (197)
\end{aligned}
$$

Again, we can easily verify, numerically, that $F_2(x) < 0$ for $x > 10^{-5}$. Hence, in view of (196), (197) we have

$$\frac{d}{\omega_1\,\omega_2\,c} > 10^{-5} \quad \Longrightarrow \quad \frac{\partial\, V\,(c,d)}{\partial\, c} < 0 \tag{198}$$

Therefore, from (195), (198), we easily conclude that

$$\frac{d}{\omega_1\,\omega_2\,c} > 10^{-5} \quad \Longrightarrow \quad \begin{cases} \frac{\partial\, V\,(c,d)}{\partial\, d} > 0 \\ \frac{\partial\, V\,(c,d)}{\partial\, c} < 0 \end{cases} \tag{199}$$

which implies that in order to maximize the volume $V\,(c,d)$ of (192), we need to maximize d and minimize c, under the restriction, of course, that $\frac{d}{\omega_1\,\omega_2\,c} > 10^{-5}$ holds. Note that since this lower bound is too small, we practically never violate this restriction.

References

[1] M. Athans, " The matrix minimum principle, " *Informat. and Contr.*, vol. 11, no. 5/6, pp. 592-606, 1967.

[2] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty, *Nonlinear Programming: Theory and Applications*. New York: John Wiley & Sons, Inc., 1993.

[3] K. P. Dabke, " Suboptimal linear regulators with incomplete state feedback, " *IEEE Trans. Automat. Contr.*, vol. 15, no. 1, pp. 120-122, 1970.

[4] C. E. de Souza, M. Fu, L. Xie, " H_∞ analysis and synthesis of discrete-time systems with time-varying uncertainty, " *IEEE Trans. Automat. Contr.*, vol. 38, no. 3, pp. 459-462, 1993.

[5] J. Douglas and M. Athans, " Robust linear quadratic designs with real parameter uncertainty, " *IEEE Trans. Automat. Contr.*, vol. 39, no. 1, pp. 107-111, 1994.

[6] G. F. Franklin, J. D. Powell, and M. L. Workman, *Digital Control of Dynamic Systems*. Reading, MA: Addison-Wesley Publishing Company, 1990.

[7] M. Fu, L. Xie, and C. E. de Souza, " H_∞ control for linear systems with time-varying parameter uncertainty, " in *Control of Uncertain Dynamic Systems*, S. P. Bhattacharyya and L. H. Keel, Eds., pp. 63-75, Boca Raton, FL: CRC Press, 1991.

[8] K. Gu, Y. H. Chen, M. A. Zohdy, and N. K. Loh, " Quadratic stabilizability of uncertain systems: a two level optimization setup, " *Automatica*, vol. 27, no. 1, pp. 161-165, 1991.

[9] K. Gu, " Designing stabilizing control of uncertain systems by quasiconvex optimization, " *IEEE Trans. Automat. Contr.*, vol. 39, no. 1, pp. 127-131, 1994.

[10] H. P. Horisberger and P. R. Belanger, " Solution of the optimal constant output feedback problem by conjugate gradients, " *IEEE Trans. Automat. Contr.*, vol. 19, pp. 434-435, 1974.

[11] S. R. Kolla, R. K. Yedavalli, and J. B. Farison, " Robust stability bounds on time-varying perturbations for state-space models of linear discrete-time systems, " *Int. J. Contr.*, vol. 50, no. 1, pp. 151-159, 1989.

[12] S. R. Kolla and J. B. Farison, " Reduced-order dynamic compensator design for stability robustness of linear discrete-time systems, " *IEEE Trans. Automat. Contr.*, vol. 36, no. 9, pp. 1077-1081, 1991.

[13] I. K. Konstantopoulos and P. J. Antsaklis, " Robust stability of linear continuous and discrete-time systems under parametric uncertainty, " *Technical Report of the ISIS Group at the University of Notre Dame, ISIS-94-006*, March 1994.

[14] O. I. Kosmidou, " Robust stability and performance of systems with structured and bounded uncertainties: an extension of the guaranteed cost control approach, " *Int. J. Contr.*, vol. 52, no. 3, pp. 627-640, 1990.

[15] W. S. Levine and M. Athans, " On the determination of optimal constant output feedback gains for linear multivariable systems, " *IEEE Trans. Automat. Contr.*, vol. 15, no. 1, pp. 44-48, 1970.

[16] W. S. Levine, T. L. Johnson, and M. Athans, " Optimal limited state variable feedback controllers for linear systems, " *IEEE Trans. Automat. Contr.*, vol. 16, no. 6, pp. 785-793, 1971.

[17] J. S. Luo and A. Johnson, " Stability robustness of the discrete-time LQG system under plant perturbation and noise uncertainty, " *Int. J. Contr.*, vol. 55, no. 6, pp. 1491-1502, 1992.

[18] J. S. Luo, A. Johnson, and P. P. J. Van Den Bosch, " Minimax guaranteed cost control for linear continuous-time systems with large parameter uncertainty, " *Automatica*, vol. 30, no. 4, pp. 719-722, 1994.

[19] F. T. Man, " Suboptimal control of linear time-invariant systems with incomplete feedback, " *IEEE Trans. Automat. Contr.*, vol. 15, no. 1, pp. 112-114, 1970.

[20] J. M. Mendel and J. Feather, " On the design of optimal time-invariant compensators for linear stochastic time-invariant systems, " *IEEE Trans. Automat. Contr.*, vol. 20, no. 5 , pp. 653-657, 1975.

[21] G. Menga and P. Dorato, " Observer-feedback design for linear systems with large parameter uncertainties, " in *Proc. 1974 IEEE Conf. Decision Contrl.*, Phoenix, AZ, pp. 872-878, 1974.

[22] M.-L. Ni and H.-X. Wu, " A Riccati equation approach to the design of linear robust controllers, " *Automatica*, vol. 29, no. 6, pp. 1603-1605, 1993.

[23] K. Ogata, *Discrete-Time Control Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1987.

[24] J. O'Reilly, " Optimal low-order feedback controllers for linear discrete-time systems, " *Contr. and Dynam. Syst.*, vol. 16, pp. 335-367, 1980.

[25] S.-C. Tsay, I.-K. Fong, and T.-S. Kuo, " Robust linear quadratic optimal control for systems with linear uncertainties, " *Int. J. Contr.*, vol. 53, no. 1, pp. 81-96, 1991.

[26] R. K. Yedavali, " Dynamic compensator design for robust stability of linear uncertain systems, " in *Proc. 25th IEEE Conf. Decision Contrl.*, Athens, Greece, pp. 34-36, 1986.

[27] S.-D. Wang, T.-S. Kuo, Y.-H. Lin, C.-F. Hsu, and Y.-T. Juang, " Robust control design for linear systems with uncertain parameters, " *Int. J. Contr.*, vol. 46, no. 5, pp. 1557-1567, 1987.

[28] K. Zhou, P. P. Khargonekar, J. Stoustrup, and H. H. Niemann, " Robust Stability and performance of uncertain systems in state space, " in *Proc. 25th IEEE Conf. Decision Contrl.*, Tucson, AZ, pp. 662-667, 1992.