Dissipativity of Hybrid Systems: Feedback Interconnections and Networks

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Abstract

We study the notions of QSR dissipativity and passivity in hybrid systems. The work presented in this report can mainly be divided into two parts- (i) feedback and parallel interconnections and, (ii) networks. First, it is shown that the negative feedback interconnection of two QSR dissipative hybrid systems is QSR dissipative. Also, passivity is preserved when passive hybrid systems are connected in parallel. It is also shown that QSR dissipativity and hence passivity are closely related to the stability of hybrid systems. In the second part of this work, passivity of networked hybrid input-output automata, in presence of network delays is studied. The issues of both the constant and time varying delays is addressed using a wave variable transform based approach.

1 INTRODUCTION

Hybrid systems are a class of dynamical systems that can have both continuous as well as discrete dynamics [1], [2]. The continuous dynamics of hybrid system are typically represented using linear or non-linear differential or difference equations. Discrete dynamics can be represented using Automata, Finite State Machine or Petri Nets. Stability analysis of hybrid systems poses several technical challenges due to the complex interaction of continuous and discrete dynamics. A brief overview of stability in hybrid systems can be found in [3] and [4].

Dissipativity and passivity based approaches can be used to analyze system behavior. The concept of passivity originated from circuit theory where circuit elements that dissipate energy and do not generate any of their own were termed passive. [5] A more generalized notion of energy was used to extend the concept of dissipativity and passivity to other systems. In general, dissipative systems with appropriate choice of supply rate functions (e.g. QSR dissipativity, passivity) can be shown to be stable [6]. Additionally, passivity is preserved over a class of interconnection structures (e.g. negative feedback), allowing for compositional analysis of large scale systems [7].

In this work, a notion of dissipativity and passivity of hybrid systems modeled as hybrid input output automata is introduced. Important results on passivity and stability of interconnected systems are discussed. A wave variable transform (WVT) based technique is proposed to ensure passive interconnection of networked hybrid systems.

As discussed earlier, dissipativity and passivity based approaches have been used to analyze a wide class of systems [8]. However, there has been very little work done on the passivity and dissipativity of hybrid systems. Dissipativity of hybrid systems modeled as impulsive dynamical systems is discussed in [9]. However, the work did not discuss the results for feedback and parallel interconnection of dissipative hybrid systems. [10] introduced dissipativity of a class of hybrid automata where system states are continuous during discrete transitions. The work done in this paper differs from [10] in that the notion of dissipativity introduced here holds for a larger class of hybrid systems where system states are allowed to have discontinuities at discrete transitions. Moreover, in contrast to [9], hybrid input output automata are considered and results on passivity and stability of interconnected systems are presented. We show that passivity for hybrid systems is preserved over a class of interconnection structures (e.g. negative feedback), allowing for compositional analysis of large scale systems. However, in case of networked interconnection, this property may not hold due to network effects such as delays. One of the approaches to deal with network delays uses WVT [11], [12], [13]. WVT was earlier used to preserve the passivity of networked interconnection of switched systems [14]. We extend this result to hybrid systems which allows for the analysis of a larger class of networked systems.

This report is organized as follows. Section 2 describes the hybrid system model and defines dissipativity and a few special cases for the same. Section 3 presents results on the stability of interconnections of dissipative hybrid systems. In section 4, feedback interconnection of hybrid input output automata over a network with delays is considered. Both the cases of constant as well as time varying delays are addressed. A wave variable transform based approach is presented to guarantee the passivity of this interconnection. Finally, section 5 contains concluding remarks.

2 Dissipativity for Hybrid Systems

Dissipativity is an energy based approach to characterize system behaviour. A dissipative system is one that stores and dissipates energy without generating any of its own. Thus the internally stored energy is upper bounded by the externally supplied energy. The notion of stored energy is defined typically in terms of a non-negative storage function and that of supplied energy by a supply rate function [5].

Hybrid systems have a tight interaction of continuous and discrete dynamics. Dissipativity of just the continuous modes of hybrid system is not sufficient for the dissipativity of hybid system as a whole [10]. Therefore, the definition of dissipativity for hybrid systems should take into account not only the continuous state flow but also the energy added at discrete mode transitions. Moreover, in case of no transition, the definition should reduce to general notion of dissipativity.

In this section, we introduce a definition for the dissipativity and passivity of hybrid input-output automaton which takes into account the points discussed above.

Consider a hybrid system represented by a hybrid input-output (I/O) automaton

$$\Sigma = \{Q, Z, F, H, Init, Inv, E, G, R\} \tag{1}$$

where

- Q is the set of discrete states.
- $Z = \{U, Y, X\}$ is the set of continuous input, output and state variables.
- $F = \{f_i\}$ is the set of vector fields $f_i(\cdot, \cdot) : X \times U \longrightarrow \mathbb{R}^n$. In other words f_i is the equation governing state dynamics in the i^{th} discrete state.
- $H = \{h_i\}$ is the set of output equations $h_i(\cdot, \cdot) : X \times U \longrightarrow \mathbb{R}^r$.
- $Init \subset Q \times X$ denotes the valid set of initial conditions.
- $Inv : Q \longrightarrow 2^X$ denotes the portion of X where each $q \in Q$ may be active.
- $E \subset Q \times Q$ is the set of all possible discrete transitions called as edges.
- $G : E \longrightarrow 2^X$ is the guard set which assigns the conditions under which a transition in E may take place; and

- $R : E \times X \longrightarrow 2^X$ is the reset map for continuous state $x \in X$. It indicates the value of state immediately after a transition $e \in E$. Thus, it helps identify if there is any discontinuous change in state at discrete transition.

Hybrid systems can have different state variables in different modes in which case the states can be concatenated together to form a single state vector x. Same approach can be used with input and output variables as well.

A hybrid I/O automaton combines the differential equation representing continuous state dynamics, and automaton representing discrete dynamics into a single framework. The above system description is motivated by the hybrid I/O automaton discussed in [15]. The only difference is that here, invariant and guard sets, and reset maps are specified explicitly.

Definition 1. *[16] A hybrid input-output automaton (1) is said to be dissipative if for each mode $q_i \in Q$, there exists a storage function $V_i(x)$ such that*

$$\underline{\alpha}(\|x\|) \leq V_i(x) \leq \overline{\alpha}(\|x\|)$$

for class-$\mathcal{K}$ functions $\underline{\alpha}$ and $\overline{\alpha}$ to satisfy the following:

1. *For all $q_i \in Q$, there exist continuous energy supply rate functions $\omega_c^i : U \times Y \longrightarrow \mathbb{R}$ such that when q_i is active between switching instants t_{k-1} and t_k, $\forall t_{k-1} \leq t_1 \leq t_2 \leq t_k$*

$$V_i(x(t_2)) \leq V_i(x(t_1)) + \int_{t_1}^{t_2} \omega_c^i(u, y)dt \tag{2}$$

2. *There exists a discrete energy supply rate $\omega_d : X \times E \longrightarrow \mathbb{R}$ such that for each switching instant t_k, where the transition can be denoted $e = (q_i, q_j)$,*

$$V_j(x(t_k^+)) \leq V_i(x(t_k^-)) + \omega_d(x, e), \tag{3}$$

and the rate ω_d is bounded by a class-$\mathcal{K}$ function $W(\|x\|)$, $\omega_d(x, e) \leq W(\|x\|)$.

There are several possible choices for the continuous supply rate functions. Different choices of supply rate functions dictate different important properties for dissipative systems. One popular choice of supply rate function is QSR dissipativity [17]. The notion of QSR dissipativity is closely related to that of stability. With some additional constraints, it is possible to define QSR dissipativity for hybrid I/O automaton as well.

Definition 2. *A hybrid input-output automaton (1) is said to be QSR dissipative if it is dissipative with respect to*

$$\omega_c^i = \begin{bmatrix} y^T & u^T \end{bmatrix} \begin{bmatrix} Q_i & S_i \\ S_i^T & R_i \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}, \quad \forall q_i \in Q \tag{4}$$

and if for all discrete mode switching times t_k the discrete supply rate satisfies,

$$\sum_{k=1}^{T_n} \omega_d(x(t_k^-), e) \leq \phi_d(t) \tag{5}$$

where T_n is the number of switches till time t and $\phi_d(t)$ is absolutely integrable.

Definition 3. *A hybrid system (1) is said to be passive if it is QSR dissipative with $Q_i = 0$, $R_i = 0$ and $S_i = \frac{1}{2}I$, $\forall q_i \in Q$.*

Additionally, system (1) is called output strictly passive (OSP) if it is QSR dissipative with $Q_i = \epsilon_i I$, $R_i = 0$ and $S_i = \frac{1}{2}I$, $\forall q_i \in Q$.

This notion of passivity indicates that for a hybrid system to be passive all the modes must be passive when active and the energy added to the system at discrete transitions must be bounded as well. Clearly, in case of no discrete transitions, this definition reduces to classical notion of passivity.

In general, passive systems are stable. Additionally, passivity is preserved over a class of interconnection structures. In the next section, it is shown that these properties hold for passive hybrid I/O automata as well.

3 Stability of Interconnected Hybrid Systems

One of the major advantages of using passivity is its scalability property [7] which allows for the compositional analysis and design of large scale systems. In this section, we present results to show that QSR dissipativity, and thus passivity is preserved over negative feedback and parallel interconnections for hybrid I/O automata. We further show the relation between QSR dissipativity and stability.

Consider two hybrid I/O automata

$$\Sigma_1 = \{Q^{(1)}, Z^{(1)}, F^{(1)}, H^{(1)}, Init^{(1)}, Inv^{(1)}, E^{(1)}, G^{(1)}, R^{(1)}\} \tag{6}$$

$$\Sigma_2 = \{Q^{(2)}, Z^{(2)}, F^{(2)}, H^{(2)}, Init^{(2)}, Inv^{(2)}, E^{(2)}, G^{(2)}, R^{(2)}\} \tag{7}$$

connected in feedback as shown in Fig. 1. This feedback interconnection forms a new hybrid system which is a mapping from $w \to y$ where

$$w = \begin{bmatrix} w^{(1)} \\ w^{(2)} \end{bmatrix} \text{ and } y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \end{bmatrix}.$$

The resulting system has modes which not only depend on the modes of Σ_1 and Σ_2 but also on the interaction between them. The issue of meaningful interconnection of hybrid systems is discussed in [18]. Let the number of modes in Σ_1 be M_1 and that in Σ_2 be M_2. Then the maximum number of modes for the feedback system will be $M = M_1 \times M_2$. Note that the possible modes for actual feedback system might be less than this because some of the modes might not be possible due to edge and guard sets of the two systems.

Let the set of switching instants of Σ_1 be $\mathcal{T}^{(1)} = \{t_k^{(1)}\}$ and that for Σ_2 be $\mathcal{T}^{(2)} = \{t_k^{(2)}\}$. The set of switching instants of interconnected systems can be represented as $\mathcal{T} = \{t_k\}$.

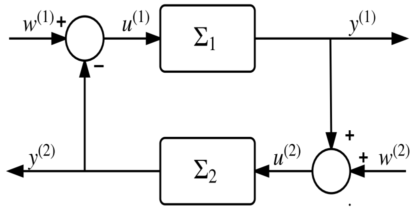

Figure 1: Feedback interconnection of two passive hybrid systems.

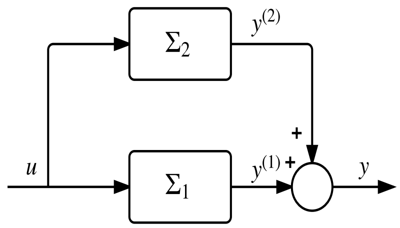

Figure 2: Parallel interconnection of two passive hybrid systems.

Theorem 1. *The feedback interconnection of two QSR dissipative hybrid I/O automaton (6) forms a new hybrid I/O automata that is QSR dissipative with respect to the continuous supply rate*

$$\omega_c^{ij}(t) = y^T Q_{ij} y + 2y^T S_{ij} w + w^T R_{ij} w$$

where

$$Q_{ij} = \begin{bmatrix} Q_i + R_j & -S_i + S_j^T \\ -S_i^T + S_j & Q_j + R_i \end{bmatrix}$$
$$S_{ij} = \begin{bmatrix} S_i & R_j \\ -R_i & S_j \end{bmatrix}, R_{ij} = \begin{bmatrix} R_i & 0 \\ 0 & R_j \end{bmatrix}$$

Proof. The feedback interconnection of two hybrid I/O automata (6), as shown in Fig. 1, can also be represented in the form of hybrid I/O automata Σ given as,

$$\Sigma = \{Q, Z, F, H, Init, Inv, E, G, R\}$$

where

- $Q : q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in Q$. This assumes elements of $Q^{(1)}$ and $Q^{(2)}$ have unique labels.
- $Z = U, Y, X$ where $X : x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \in X, \forall x^{(1)} \in X^{(1)}$ and $x^{(2)} \in X^{(2)}$. U and Y can be defined

in a similar manner as well.

- $F : f_{ij}(x,u) = \begin{bmatrix} f_i^{(1)}(x^{(1)}, u^{(1)}) \\ f_j^{(2)}(x^{(2)}, u^{(2)}) \end{bmatrix} \in F$. In a similar way we can form H as well.
- $Init : \left(\begin{bmatrix} x_0^{(1)} \\ x_0^{(2)} \end{bmatrix}, \begin{bmatrix} q_0^{(1)} \\ q_0^{(2)} \end{bmatrix} \right) \in Init, \forall (x_0^{(1)}, q_0^{(1)}) \in Init^{(1)}$ and $(x_0^{(2)}, q_0^{(2)}) \in Init^{(2)}$.
- $Inv : Inv^{(1)}(q^{(1)}) \times Inv^{(2)}(q^{(2)}) \subset Inv$.
- $E = E^{(1)} \bigcup E^{(2)}$. G and R are formed in a similar manner.

Suppose at any time t, $t_{k-1} < t < t_k$, active modes of Σ_1 and Σ_2 be i and j respectively. Since both these systems are QSR dissipative, there exist storage functions $V_i^{(1)}$ and $V_j^{(2)}$ such that,

$$\int_{t_1}^{t_2} \begin{bmatrix} y^{(1)T} & u^{(1)T} \end{bmatrix} \begin{bmatrix} Q_i & S_i \\ S_i^T & R_i \end{bmatrix} \begin{bmatrix} y^{(1)} \\ u^{(1)} \end{bmatrix} d\tau \geq V_i^{(1)}(x^{(1)}(t_2)) - V_i^{(1)}(x^{(1)}(t_1))$$

$$\int_{t_1}^{t_2} \begin{bmatrix} y^{(2)T} & u^{(2)T} \end{bmatrix} \begin{bmatrix} Q_j & S_j \\ S_j^T & R_j \end{bmatrix} \begin{bmatrix} y^{(2)} \\ u^{(2)} \end{bmatrix} d\tau \geq V_j^{(1)}(x^{(2)}(t_2)) - V_j^{(1)}(x^{(2)}(t_1))$$

$\forall t_{k-1} \leq t_1 \leq t_2 \leq t_k$. Adding the above two equations and substituting the loop relations,

$$u^{(1)} = w^{(1)} - y^{(2)},$$
$$u^{(2)} = w^{(2)} + y^{(1)}$$

we get,

$$\int_{t_1}^{t_2} \begin{bmatrix} y^T & w^T \end{bmatrix} \begin{bmatrix} Q_{ij} & S_{ij} \\ S_{ij}^T & R_{ij} \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} d\tau \geq V_{ij}(x(t_2)) - V_{ij}(x(t_1)) \tag{8}$$

where,

$$y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \end{bmatrix}, w = \begin{bmatrix} w^{(1)} \\ w^{(2)} \end{bmatrix} \tag{9}$$

$$Q_{ij} = \begin{bmatrix} Q_i + R_j & -S_i + S_j^T \\ -S_i^T + S_j & Q_j + R_i \end{bmatrix}$$
$$S_{ij} = \begin{bmatrix} S_i & R_j \\ -R_i & S_j \end{bmatrix}, \text{ and } R_{ij} = \begin{bmatrix} R_i & 0 \\ 0 & R_j \end{bmatrix} \tag{10}$$

Equation (8) shows that the closed loop system is dissipative when active. Next we need to consider the energy supplied at switching instants.

Suppose, at time t_k, Σ_1 switches from mode i to i' and Σ_2 from mode j to j'. Then, there exist discrete energy supply rates $\omega_d^{(1)}$ and $\omega_d^{(2)}$ such that

$$
\begin{aligned}
&V_{i'}^{(1)}(x^{(1)}(t_k^+)) \leq V_i^{(1)}(x^{(1)}(t_k^-)) + \omega_d^{(1)}(x^{(1)}, e^{(1)}) \\
&\sum_{k=1}^{T_n} \omega_d^{(1)}(x^{(1)}, e^{(1)}) \leq \phi_d^{(1)}(t)
\end{aligned} \tag{11}
$$

and

$$
\begin{aligned}
&V_{j'}^{(2)}(x^{(2)}(t_k^+)) \leq V_j^{(2)}(x^{(2)}(t_k^-)) + \omega_d^{(2)}(x^{(2)}, e^{(2)}) \\
&\sum_{k=1}^{T_n} \omega_d^{(2)}(x^{(2)}, e^{(2)}) \leq \phi_d^{(2)}(t)
\end{aligned} \tag{12}
$$

where $\phi_d^{(1)}$ and $\phi_d^{(2)}$ are absolutely integrable.

If $E^{(1)}$ and $E^{(2)}$ are the edge sets of Σ_1 and Σ_2 respectively then any transition represented by edge e of the feedback system will either be contained in $E^{(1)}$ or $E^{(2)}$ or both i.e. $e \in E^{(1)}$ or $e \in E^{(2)}$ or $e \in E^{(1)} \bigcap E^{(2)}$.

For each switching instant t_k of the feedback system shown in Fig. 1 define,

$$
\omega_d(x(t_k^-, e)) = \begin{cases} \omega_d^{(1)}(x(t_k^-, e)), & e \in E^{(1)} \text{ and } e \notin E^{(2)} \\ \omega_d^{(2)}(x(t_k^-, e)), & e \in E^{(2)} \text{ and } e \notin E^{(1)} \\ \omega_d^{(1)}(x(t_k^-, e)) + \omega_d^{(2)}(x(t_k^-, e)), & e \in E^{(1)} \bigcap E^{(2)} \end{cases}
$$

$$
V_{ij}(x(t_k^-)) = V_i^{(1)}(x(t_k^-)) + V_j^{(2)}(x(t_k^-))
$$

$$
V_{i'j'}(x(t_k^+)) = V_{i'}^{(1)}(x(t_k^+)) + V_{j'}^{(2)}(x(t_k^+))
$$

Using (11) and (12), at each discrete transition,

$$
V_{i'j'}(x(t_k^+)) - V_{ij}(x(t_k^-)) \leq \omega_d(x(t_k^-), e) \tag{13}
$$

and,

$$
\sum_{k=1}^{T_n} \omega_d(x(t_k^-), e) \leq \phi_d(t) \tag{14}
$$

where $\phi_d(t)$ is absolutely integrable.

From (8), (13) and (14) it is clear that the negative feedback interconnection of Σ_1 and Σ_2 is QSR dissipative. □

Remark: Note that similar result for passive systems was shown in [16].

Corollary 1. *The feedback interconnection of two passive hybrid I/O automata (6) forms a new hybrid I/O automata that is passive.*

Corollary 2. *The feedback interconnection of OSP hybrid I/O automata (6) forms a new hybrid I/O automata that is OSP.*

The proof for above two corollaries follows directly from Theorem 1.

Theorem 2. *[16] The parallel interconnection of two passive hybrid I/O automata is passive.*

Proof. Consider the parallel interconnection of two hybrid I/O automata as shown in Fig. 2. Since both Σ_1 and Σ_2 are passive, $\forall t_k < t_1 < t_2 < t_{k+1}$

$$\int_{t_1}^{t_2} {y^{(1)}}^T u \; d\tau \geq V_i^{(1)}(x^{(1)}(t_2)) - V_i^{(1)}(x^{(1)}(t_1))$$

$$\int_{t_1}^{t_2} {y^{(2)}}^T u \; d\tau \geq V_j^{(2)}(x^{(2)}(t_2)) - V_j^{(2)}(x^{(2)}(t_1))$$

Let the storage function V_{ij} for interconnection be

$$V_{ij}(x(t)) = V_i^{(1)}(x^{(1)}(t)) + V_j^{(2)}(x^{(2)}(t))$$

Adding the above two equations,

$$\int_{t_1}^{t_2} (y^{(1)} + y^{(2)})^T u \; d\tau \geq V_i^{(1)}(x^{(1)}(t_1)) + V_j^{(2)}(x^{(2)}(t_1)) - V_i^{(1)}(x^{(1)}(t_2)) - V_j^{(2)}(x^{(2)}(t_2))$$

$$\text{or,} \qquad \int_{t_1}^{t_2} y^T u \; d\tau \geq V_{ij}(x(t_2)) - V_{ij}(x(t_1))$$

.

This proves that all the modes of parallel interconnection are passive when active.

Similar to the proof for Theorem 1, we can also prove the remaining conditions required to demonstrate the passivity of parallel interconnection of passive hybrid systems.

□

We are often interested in analyzing the stability of systems under consideration. The following theorem relates QSR dissipativity to stability of hybrid systems. The traditional notion of Lyapunov stability might not always be applicable to hybrid systems as different modes might not have common equilibria. Also, the state vectors across different modes might be different. However, by placing suitable restrictions on state space to define equilibria of hybrid automata, Lyapunov stability can be talked about [19], [20]. Here, we make the following assumptions.

Assumption 1. *A hybrid I/O automaton has a single state space when*

- $x^{(i)} = x^{(j)}$ $\forall q_i,\ q_j \in Q$, *i.e., the continuous state vector represents same quantities in each discrete state, and*
- $\exists\ x_e = 0$ *such that* $f_i(x_e, 0) = 0$, $\forall i$.

Theorem 3. *Consider a QSR dissipative hybrid I/O automaton. If the system satisfies Assumption 1 and $Q_i \leq 0$ $\forall q_i \in Q$, then the unforced ($u(t) = 0$) system is Lyapunov stable.*

Proof. Lyapunov stability of a hybrid I/O automata can be ensured if, $\forall \epsilon > 0, \exists \delta > 0$ such that

$$\|x(t_0)\| \leq \delta \Rightarrow \|x(t)\| \leq \epsilon, \forall t \geq 0$$

Since $\phi_d(t)$ is absolutely integrable, given $\epsilon > 0$, it is possible to pick $T > t$ such that

$$\int_T^\infty \phi_d(t)dt \leq \overline{\alpha}(\epsilon/2)$$

where $\overline{\alpha}$ is defined in Definition 1.

The final discrete mode of the system can be denoted q_j. To guarantee $\|x(T)\| \leq \epsilon/2$, $V_j(x(T)) \leq \overline{\alpha}(\|x(T)\|) \leq \overline{\alpha}(\epsilon/2) = \epsilon_v$. The final mode q_j is active for time interval $I_{T_n} = [t_{T_n}, T]$. Dissipativity implies the storage function over this final time interval is bounded,

$$V_j(x(T)) \leq V_j(t_{T_n}) + \int_{t_{T_n}}^T \omega_c^j(0, y)dt. \tag{15}$$

The previous automaton mode from which it switched to q_j can be denoted as q_i. Dissipativity provides a bound on $V_j(\cdot)$ and $V_i(\cdot)$ at this final switch:

$$V_j(x(t_{T_n}^+)) \leq V_i(x(t_{T_n}^-)) + \omega_d(x(t_{T_n}^-), e_{T_n}) \tag{16}$$

where $e_{T_n} = (q_i, q_j)$. This process can be repeated until the first time arrival is reached where mode q_1 is active,

$$V_1(x(t_1)) \leq V_1(x(t_0)) + \int_{t_0}^{t_1} \omega_c^1(0, y)dt. \tag{17}$$

Recall that the hybrid system under consideration is QSR dissipative with $Q_i \leq 0$, $\forall q_i \in Q$. Therefore, the integrals involving ω_c^i are all less than zero and can be removed. Above equations can be summarized as,

$$V_j(x(T)) \leq V_1(t_0) + \sum_{k=1}^{T_n} \omega_d(x(t_k^-), e_k). \tag{18}$$

Since each $\omega_d(x(t_k^-), e_k)$ term is bounded by a class$-\kappa$ function, there always exist successive bounds on $\|x(t_k)\|$ to bound the storage function. Following this procedure guarantees that $\sum_{k=1}^{T_n} \omega_d(x(t_k^-), e_k) < \epsilon/2$ which guarantees that there exists δ such that:

$$0 < \delta \leq \overline{\alpha}^{-1}\big(\epsilon_v - \sum_{k=1}^{T_n} \omega_d(x(t_k^-), e_k)\big). \tag{19}$$

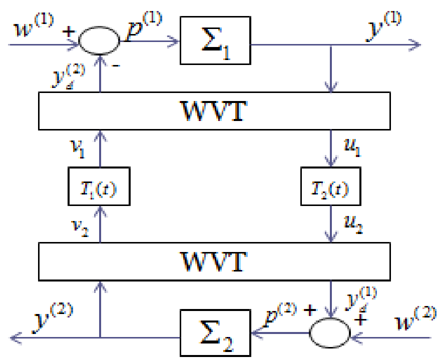

Figure 3: Two hybrid systems connected over a network with delays using wave variable transform.

This choice guarantees $V_j(x(T)) \leq \epsilon_v$ which implies $\|x(t)\| \leq \epsilon$, $\forall t$ which shows Lyapunov stability.

□

Remark: Note that a related result for dissipative systems (not necessarily QSR) can be found in [16].

The above theorems, when used together, provide a means to analyze and design stable large scale systems by breaking them into smaller systems.

4 Stability of Networked Hybrid Systems

4.1 Network structure

In the previous section we talked about the stability of feedback and parallel interconnections of hybrid I/O automata. It was shown that the feedback interconnection of two passive hybrid I/O automata forms a new hybrid I/O automaton that is passive. However, in large scale systems, this interconnection is often over a network. The presence of communication networks to transmit data among the two systems may introduce delays. This might lead to loss of passivity.

Consider two passive hybrid I/O automata (6) in feedback, communicating over a network with time delays $T_1(t)$ and $T_2(t)$. One of the approaches to handle network delays uses WVT [11] as shown in Fig. 3. WVT has earlier been used to ensure passivity in case of non-switched and switched systems [14]. The key idea behind using this approach is to treat the network with delays as a 2-port network. If this two port network is passive, i.e., if the power is either stored or dissipated, then it ensures the passivity of feedback interconnection under consideration.

We can analyze the power flow in the network by subtracting the product of input-output at one port of the network and the product of input-output at the other port of network. Note that the notion of power flow considered here does not represent the actual physical power flow but power in the sense of passivity. It was shown in [13] that the standard form of network with delays is not passive. Therefore, nothing can be said about the passivity of feedback interconnection under consideration.

In the subsequent sections, we show that WVT based approach can be used to guarantee the passivity of feedback interconnection of hybrid I/O automata in the presence of network delays.

4.2 Network with constant delays

As explained earlier, the presence of delays in the network can compromise the passivity of feedback interconnection of hybrid systems. In this section we consider the case when network delays are constant.

Consider the networked interconnection of two passive hybrid I/O automata as shown in Fig. 3. First we consider the case where the network delays T_1 and T_2 are constant. We can use the WVT discussed in [11] which is an input output transform given as

$$\begin{aligned} \begin{bmatrix} u_1 \\ \hat{v}_1 \end{bmatrix} &= \frac{1}{\sqrt{2b}} \begin{bmatrix} I & bI \\ -I & bI \end{bmatrix} \begin{bmatrix} y_d^{(2)} \\ y^{(1)} \end{bmatrix} \\ \begin{bmatrix} \hat{u}_2 \\ v_2 \end{bmatrix} &= \frac{1}{\sqrt{2b}} \begin{bmatrix} I & bI \\ -I & bI \end{bmatrix} \begin{bmatrix} y^{(2)} \\ y_d^{(1)} \end{bmatrix} \end{aligned} \tag{20}$$

where

$$\begin{aligned} \hat{u}_2 &= u_1(t - T_1) \\ \hat{v}_1 &= v_2(t - T_2) \end{aligned} \tag{21}$$

It can be shown that the introduction of WVT preserves the passivity of interconnection.

Theorem 4. *The feedback interconnection of two passive hybrid systems (6), connected over a network with constant delays, using the wave variable transform (20), as shown in Fig. 3 is passive.*

Proof. Let i and j be the active modes for Σ_1 and Σ_2 respectively between the two consecutive switching instants t_k and t_{k+1}. The amount of energy stored in the network between any two discrete transitions can be computed as,

$$V_N = \int_{t_1}^{t_2} (u_1^T(\tau)u_1(\tau) + v_2^T(\tau)v_2(\tau) - \hat{u}_2^T(\tau)\hat{u}_2(\tau) - \hat{v}_1^T(\tau)\hat{v}_1(\tau))d\tau \tag{22}$$

where

$$V_N = \int_{t_1}^{t_2} (u_1^T(\tau)u_1(\tau) + v_2^T(\tau)v_2(\tau) - u_1^T(\tau - T_1)u_1(\tau - T_1) - v_2^T(\tau - T_2)v_2(\tau - T_2))d\tau \tag{23}$$

But,

$$\int_{t_1}^{t_2} u_2^T(\tau - T_1)u_2(\tau - T_1)d\tau = \int_{t_1 - T_1}^{t_2 - T_1} u_1^T(s)u_1(s)ds$$

Similarly,

$$\int_{t_1}^{t_2} v_1^T(\tau - T_1)v_1(\tau - T_1)d\tau = \int_{t_1-T_2}^{t_2-T_2} v_2^T(s)v_2(s)ds \tag{24}$$

Using these in (23),

$$V_N = S(t_2) - S(t_1)$$

where $S(t) = \int_{t-T_1}^{t} u_1^T(s)u_1(s)ds + \int_{t-T_2}^{t} v_2^T(s)v_2(s)ds$. Also, using WVT (20)

$$\begin{aligned} V_N &= \int_{t_1}^{t_2} {y^{(1)}}^T y_d^{(2)} - {y^{(2)}}^T y_d^{(1)} d\tau \\ \implies \int_{t_1}^{t_2} &(y^T w - y^T p)d\tau = S(t_2) - S(t_1) \end{aligned} \tag{25}$$

where

$$y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \end{bmatrix}, w = \begin{bmatrix} w^{(1)} \\ w^{(2)} \end{bmatrix}, p = \begin{bmatrix} p^{(1)} \\ p^{(2)} \end{bmatrix}$$

Since mapping $p^{(1)} \longrightarrow y^{(1)}$ and $p^{(2)} \longrightarrow y^{(2)}$ is passive, $V_i^{(1)}$ and $V_j^{(2)}$ exist for Σ_1 and Σ_2 respectively that satisfy,

$$\int_{t_1}^{t_2} {p^{(1)}}^T y^{(1)} d\tau \geq V_i^{(1)}(x^{(1)}(t_2)) - V_i^{(1)}(x^{(1)}(t_1))$$

$$\int_{t_1}^{t_2} {p^{(2)}}^T y^{(2)} d\tau \geq V_i^{(2)}(x^{(2)}(t_2)) - V_i^{(2)}(x^{(2)}(t_1))$$

$\forall t_k \leq t_1 \leq t_2 \leq t_{k+1}$.

Adding the above two equations and using it in (24),

$$\int_{t_1}^{t_2} p^T y \; d\tau \geq V_{ij}(x(t_2)) - V_{ij}(x(t_1))$$

where $V_{ij}(x(t)) = V_i^{(1)}(x^{(1)}(t)) + V_j^{(2)}(x^{(2)}(t)) + S(t)$. Therefore,

$$\int_{t_1}^{t_2} y^T w \; d\tau \geq V_{ij}(x(t_2)) - V_{ij}(x(t_1)) \tag{26}$$

$t_k \leq t_1 \leq t_2 \leq t_{k+1}$.

This proves the first condition for passivity of the interconnection, i.e., any mode is passive when active.

In order to analyze the system behavior at discrete transitions, we can proceed in a manner similar to that in proof of Theorem 1. Thus, we can say that the system shown in Fig. 3 is passive. □

This result helps us ensure the passivity of networked passive hybrid systems connected in feedback in presence of constant network delays. However, as shown in [13], this version of WVT can not be used in the case where delays are time varying. In the next section we present the use of modified WVT for this purpose.

4.3 Network with time varying delays

In this section, we discuss the use of modified WVT to preserve the passivity of feedback interconnection of two hybrid I/O automata in presence of time varying network delays. Modified WVT was introduced in [12] to handle time varying delays in networks. It was later used to ensure the passivity of feedback interconnection of both non-switched and switched systems over a network with time varying delays [14]. Here, we extend this approach to networked passive hybrid systems. The idea is to use the same input output transformation matrix (20) as in the case of constant delays. However, before feeding the output of WVT to actual system, it is multiplied with functions $g_1(t)$ and $g_2(t)$ to take care of varying time delay. This is known as modified wave variable transform.

$$\begin{aligned} \hat{u}_2 &= g_1(t)u_1(t - T_1(t)) \\ \hat{v}_1 &= g_2(t)v_2(t - T_2(t)) \end{aligned} \tag{27}$$

$$g_1^2(t) \le 1 - \frac{dT_1}{dt}, g_2^2(t) \le 1 - \frac{dT_2}{dt}$$
$$\frac{dT_1}{dt} \le 1, \frac{dT_2}{dt} \le 1$$

and b and g_1, g_2 are design parameters of WVT.

Theorem 5. *The feedback interconnection of two passive hybrid systems (6), connected over a network with measurable time-varying delays using the modified wave variable transform, is passive.*

Proof. There are two parts to the proof, first we prove that any mode is passive when active and secondly we deal with passivity at switching instants. First part of the proof follows from the result in [14]. However, for the sake of completeness we mention it again with respect to hybrid systems (6).

We can prove the above theorem by an energy based approach. The amount of energy stored in the network is obtained by taking the difference of energy going into and out of the network i.e. for all $t_k \le t_1 \le t_2 \le t_{k+1}$

$$V_N = \int_{t_1}^{t_2} (u_1^T(\tau)u_1(\tau) + v_2^T(\tau)v_2(\tau) - \hat{u}_2^T(\tau)\hat{u}_2(\tau) - \hat{v}_1^T(\tau)\hat{v}_1(\tau))d\tau \tag{28}$$

$$\begin{aligned} V_N = \int_{t_1}^{t_2} (u_1^T(\tau)u_1(\tau) + v_2^T(\tau)v_2(\tau) - g_1^2(\tau)u_1^T(\tau - T_1(\tau))u_1(\tau - T_1(\tau)) \\ - g_2^2(\tau)v_1^T(\tau - T_2(\tau))v_1(\tau - T_2(\tau)))d\tau \end{aligned} \tag{29}$$

But,

$$\int_{t_1}^{t_2} g_1^2(\tau)u_1^T(\tau - T_1(\tau))u_1(\tau - T_1(\tau))d\tau \le \int_{t_1}^{t_2} \left(1 - \frac{dT_1}{d\tau}\right) u_1^T(\tau - T_1(\tau))u_1(\tau - T_1(\tau))d\tau$$
$$= \int_{t_1 - T_1(t_1)}^{t_2 - T_1(t_2)} u_1^T(s)u_1(s)ds$$

Similarly,

$$\int_{t_1}^{t_2} g_2^2(\tau)v_2^T(\tau - T_2(\tau))v_2(\tau - T_2(\tau))d\tau \le \int_{t_1 - T_2(t_1)}^{t_2 - T_2(t_2)} v_2^T(s)v_2(s)ds \tag{30}$$

$$V_N \ge \int_{t_2 - T_1(t_2)}^{t_2} u_1^T(s)u_1(s)ds + \int_{t_2 - T_2(t_2)}^{t_2} v_2^T(s)v_2(s)ds$$
$$- \int_{t_1 - T_1(t_1)}^{t_1} u_1^T(s)u_1(s)ds - \int_{t_1 - T_2(t_1)}^{t_1} v_2^T(s)v_2(s)ds$$
$$\text{or, } V_N \ge S(t_2) - S(t_1) \tag{31}$$

where

$$S(t) = \int_{t - T_1(t)}^{t} u_1^T(s)u_1(s)ds + \int_{t - T_2(t)}^{t} v_2^T(s)v_2(s)ds.$$

Using this and (20) in (29),

$$\int_{t_1}^{t_2} y^T w d\tau \ge \int_{t_1}^{t_2} y^T p d\tau + S(t_2) - S(t_1) \tag{32}$$

where y, w and p have the same meaning as in proof of Theorem 4. The remaining part of proof follows from that of Theorem 4 where it can be shown that the individual modes of the feedback interconnected system are passive when active and the energy supplied to the system at discrete transitions is bounded.

This proves that use of modified WVT preserves the passivity of negative feedback interconnection of two passive hybrid I/O automata in presence of network with time varying delays. □

Corollary 3. *Consider the negative feedback interconnection of two passive hybrid I/O automata (6) connected over a network with time varying delays using modified WVT. If the interconnected system satisfies Assumption 1, then the resulting system is stable.*

Proof. This corollary is a direct result of Theorem 3 and Theorem 5 used together. □

One of the major advantages of using WVT based approach is that it does not put a bound on the time delays. The only assumption is that the rate of change of network delays does not exceed the rate at which the actual time is changing.

The results presented in this work are applicable to a wide variety of systems. For example, if there are no discontinuities allowed at discrete transition, i.e., the reset map is of the form

$R(e, x) = x$, then the results presented here automatically hold for switched systems which is in agreement with [14]. The two interacting systems can be different in nature as well such as a hybrid plant and switched controller or a logic controller. The only point is to make sure that interconnection is meaningful [18]. Thus the results presented here help analyze passivity and thus the stability of a large class of networked systems in a compositional manner.

5 Conclusion

In this work a notion of dissipativity for hybrid input output automaton is introduced. The close relationship between QSR dissipativity and Lyapunov stability of hybrid system is highlighted. It is shown that when two QSR dissipative hybrid automata are connected in feedback then the resulting system is QSR dissipative. Similarly, passivity is preserved over parallel interconnection as well. This scalability property along with the relationship with stability provides a way for compositional analysis of large scale hybrid systems. The interconnection of hybrid systems over network with time delays was also addressed. A wave variable transform based approach is presented to preserve the passivity of feedback interconnection of passive hybrid systems over a network with delays.

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