Nonlinear Control and Stability of Approximately Symmetric Systems of Coordinating Robotic Agents

Bill Goodwine

***Abstract*— Stability and convergence properties of control and other algorithms in robotics are essential aspects of the development of truly autonomous and deployable real-world robotic systems. Because of the desire for flexible, adaptive and reactive robotic systems, i.e., *global* operation, nonlinear analyses and tools are especially important in robotics. Hence, Lyapunov methods are at the core of many critical robotics methodologies. This paper considers Lyapunov stability for approximately symmetric robotic systems. Many robotic systems, such as swarms and fleets of mobile robots, distributed sensor networks, etc., are comprised many identical interacting agents. Our prior work has considered stability aspects of control and dynamics of such systems. This paper extends those results to the important case where all the agents are not identical, which is important for real-world applications where it is not possible to have exactly identical agents. Importantly, these results do not require the components to have small differences. However, the bounds on the nature of the solutions will obviously depend on how different the agents are.**

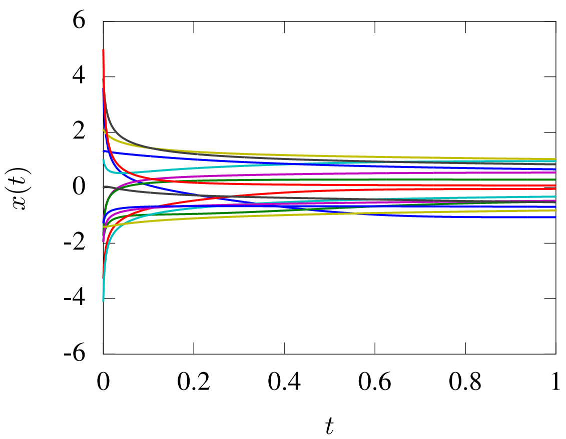

Fig. 1. Nonsymmetric consensus stability for 15-agent system.

I. INTRODUCTION

The use of multi-agent systems has a transformative potential for the ubiquitous use of robotics in real-world applications. This partially is due to: 1) robustness, because if a small subset of agents fail, the overall performance of a large-scale multi-agent system is not likely to be significantly reduced, 2) adaptability and agility, because, in addition to any adaptability built into an individual agent's control, significant adaptability and agility in behavior may emerge from the *coordinated* efforts of the agents, significantly beyond what is possible for a single agent, and 3) economics, because in many applications it may be simultaneously cheaper and more effective to accomplish a task using multiple, cheaper, components than a single more sophisticated and more expensive, one.

Despite these advantages, other than sensor networks, actually deployed cooperating agent robotic systems are not yet ubiquitous. This is partly because their promise depends on a critical control aspect, which is that in order to bring about the desired system behavior, coordination among the agents is necessary, which is a difficult theoretical and technological control problem. One aspect of this difficulty is that if the agents are coordinating their behavior the system has a very high dimensional and coupled state space, making analysis and design of the control problem difficult.

Many recent research efforts have attempted to address this problem. Some efforts have focused on *compositionality* and *composability* wherein the relationship between system properties and the properties of individual components is determined [1], [2]. While the results in this paper are general and apply to a broad class of problems, the most natural focus in robotics would be on problems such as formation control and consensus. Formation control has a natural relationship with Lyapunov methods in that potential functions are an appealing means to define a formation. Of course there is a *vast* literature on formation control, and a representative set of references include [3]–[8] and the book [9]. Similarly consensus has received much attention and is an important subject in robotics because of the need for distributed agents to reach a consensus on many things ranging from collected sensor data to headings and velocities for flocking behavior. While it is emphasized that our results are much more applicable than just to the consensus problem, the main example in this paper is a nonlinear consensus one. In contrast to most work in consensus which focuses on the interconnectedness of the system (see for example, [10], [11]), we focus on system properties which are invariant with a pre-defined regularity in the system structure with an emphasis in this paper on robustness when the system is not exactly symmetric.

Prior work of the author has focused on control of *symmetric systems* [12]–[18]. The precise meaning of a symmetric system will be defined subsequently, but the simplest example is a system that is composed of identical agents interacting with each other in a highly-structured manner. In such a case, it makes sense that the equations of motion for the system will exhibit significant structure which may be exploited for analysis and design in control.

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Bill Goodwine is with the Department of Aerospace & Mechanical Engineering at the University of Notre Dame, Notre Dame, IN 46556 USA `billcontrols.ame.nd.edu`.

One shortcoming of that prior work is that it requires the system to be exactly symmetric in that all the agents must either be identical (or at least diffeomorphically related). This work is part of a series of efforts to extend those results to the case where the agents in the system are only approximately symmetric. This will encompass the realities that it is not possible in the real world to have agents that are exactly identical as well as the fact that it may be desirable for the agents to differ. This extension currently is in four directions: the combinations of 1) point-based *vs.* set-based stability and 2) model variations *vs.* persistent non-symmetric and non-autonomous inputs to the system. These four areas grow out of the fact that Lyapunov analyses for stability are delineated along similar lines: Lyapunov stability of a point *vs.* set-based LaSalle's invariance principle and stability *vs.* boundedness. Closely related current work by the author includes two other conference submission, [19], [20]. The first of these deals with results for boundedness of solutions where a stable symmetric system is subjected to persistent nonautonomous inputs and the second deals with robust stability of approximately symmetric systems to *sets*. The latter submission has much more restrictive conditions on the symmetry breaking due to using invariance principle results instead of Lyapunov stability, which is the focus of the present paper.

The rest of this paper is organized as follows. Section II is background information summarizing some of our prior work and defines a symmetric system. Section III presents the main result for stability of approximately symmetric systems. Section IV presents a consensus example. Finally, Section V presents conclusions and outlines our future work.

II. SYMMETRIC AND APPROXIMATELY SYMMETRIC SYSTEMS

This section is a summary of our results from [12] and gives an overview of symmetric systems and the relationship among symmetric systems with different numbers of components. It also extends these to allow for additive symmetry-breaking terms to each component.

Consider the "building block" for symmetric systems illustrated in Figure 2. The $w^-(t)$ are the outputs from the component and u, $v^-(t)$ and $v^+(t)$ are the inputs. The signals $v^\pm$ represent coupling with the other components and u are the control inputs. We wish to consider fully nonlinear symmetric systems of the form

$$\begin{aligned} \dot{x}_i(t) &= f_i\big(x_i(t)\big) + \sum_{j=1}^{m_i} g_{i,j}\big(x_i(t)\big) u_{i,j}\,(t) \\ w_i^-(t) &= w_i^-\,(x_i(t)) \\ w_i^+(t) &= w_i^+\,(x_i(t))\,. \end{aligned} \tag{1}$$

If the system is controlled via feedback, the outputs from the neighbors appear in the control input for component i in Equation 1, which can be expressed by

$$u_{i,j}(t) = u_{i,j}\left(x_i(t), w_{i-1}^+(x_{i-1}(t)), w_{i+1}^-(x_{i+1}(t))\right). \tag{2}$$

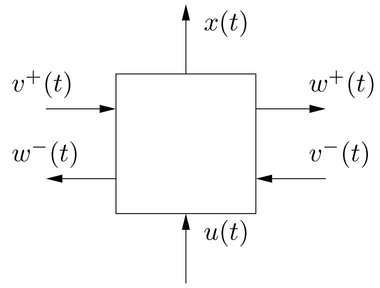

Fig. 2. System building block in one spatial dimension.

The block in Figure 2 has only $+$ and $-$ inputs and outputs, so we consider systems defined on *groups* in order to allow systems with a more general interconnection structure. Recall, that a *group* is a set, G with

1) a binary associative operation, $G \times G \to G$,
2) an identity element e such that $eg = ge = g$ for all $g \in G$, and
3) for every $g \in G$ there exists an element $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$.

Let $|G|$ denote the number of elements in a set G.

Following the development in [21], interconnections in systems we consider will be represented by a set of *generators*, denoted by X. If X is a subset of a group G, then the smallest subgroup of G containing X is called the *subgroup generated by* X. For the rest of this paper assume that G itself is the group generated by X and that if $s \in X$, then $s^{-1} \in X$ as well. *Relations* define constraints among the generators, and are of the form $s_1 s_2 \ldots s_m = e$ for $s_1, \ldots, s_m \in X$. Finally, we represent systems by a *Cayley graph*, which is a graph with vertices that are the elements of a group, G, generated by the subset X, with a directed edge from g_1 to g_2 if $g_2 = sg_1$ for some $s \in X$ (see [22]). A directed edge from node g_1 to g_2 represents that a coupling input to g_2 is equal to an output from g_1.

The following example will be developed throughout this paper. It has a similar interconnection structure to the formation control problem we have published in [12], but the dynamics are for a consensus problem rather than formation control. It is also similar to that in [19], but focuses on boundedness under structural perturbations to the system dynamics rather than persistent nonautonomous input terms.

Example 1: Consider the system with the Cayley graph illustrated in Figure 3. Each vertex has edges connecting to four other vertices and hence the system is generated by four elements. Let g denote a vertex, *i.e.*, $g \in \{-2, -1, 0, 1, \ldots, N-3\} = G$. Consider the generators $X = \{-2, -1, 1, 2\}$, the group operation to be addition and the relation $s^N = e = 0$. This relation makes the group operation of addition to be $\operatorname{mod} N$, and hence the group is the quotient of the set of integers $\mathbb{Z}$ where elements of $\mathbb{Z}$ that differ by an integer multiple of N are equivalent. $\diamond$

Next, we develop some notation. For a system on the group G with generators $X = \{s_1, s_2, \ldots, s_{|X|}\}$, let x_g denote the states corresponding to $g \in G$, $Xg = \{s_1 g, s_2 g, \ldots, s_{|X|} g\}$

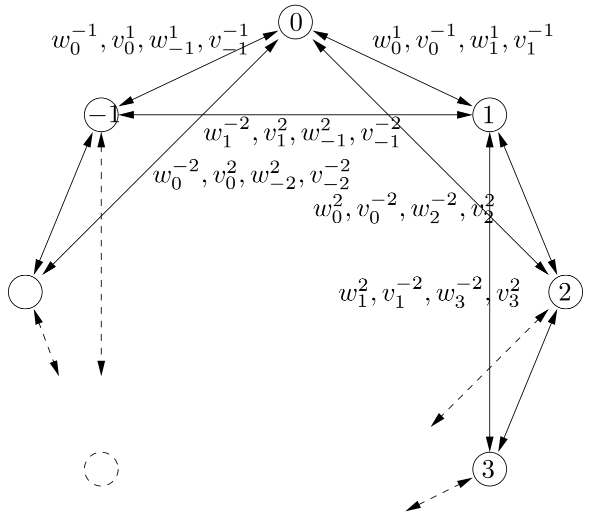

Fig. 3. System topology for Example 1.

denote the neighbors of component $g \in G$, x_{Xg} denote the states of the neighbors of $g \in G$, and x_{XXg} denote the states of the neighbors of the neighbors, etc. For a component g, let $\left\{w_g^{s_1}, w_g^{s_2}, \dots, w_g^{s_{|X|}}\right\}$ denote of outputs, and correspondingly let $\left\{v_g^{s_1}, v_g^{s_2}, \dots, v_g^{s_{|X|}}\right\}$ denote the inputs. In this more general setting, the dynamics of a component, $g \in G$ are represented by

$$\begin{aligned} \dot{x}_g(t) &= f_g\left(x_g(t)\right) \\ &\quad + \sum_{j=1}^{m_g} g_{g,j}\left(x_g(t)\right) u_{g,j}\left(x_g(t), v_g^{s_1}(t), , \dots, v_g^{s_{|X|}}(t)\right) \\ w_g^s(t) &= w_g^s\left(x_g(t)\right), \end{aligned} \tag{3}$$

for all $s \in X$.

Definition 1: A system with with dynamics given by Equation 3 has *periodic interconnections* if

$$v_g^s\left(t\right) = w_{s^{-1}g}^s\left(x_{s^{-1}g}(t)\right), \tag{4}$$

for all $g \in G$ and $s \in X$. Furthermore, if

$$\begin{aligned} f_{g_1}(x) &= f_{g_2}(x), & g_{g_1,j}(x) &= g_{g_2,j}(x), \\ w_{g_1}^s(x) &= w_{g_2}^s(x), & m_{g_1} &= m_{g_2} = m \end{aligned} \tag{5}$$

for all $s \in X$, $g_1, g_2 \in G$, $x \in \mathbb{R}^n$ and $j \in \{1, \dots, m\}$, then G has *symmetric components*. Finally, if the control laws also satisfy

$$\begin{aligned} &u_{g_1,j}\left(x_1, w_{s_1^{-1}g_1}^{s_1}(x_2), \dots, w_{s_{|X|}^{-1}g_1}^{s_{|X|}}(x_{|X|+1})\right) = \\ &\qquad u_{g_2,j}\left(x_1, w_{s_1^{-1}g_2}^{s_1}(x_2), \dots, w_{s_{|X|}^{-1}g_2}^{s_{|X|}}(x_{|X|+1})\right) \end{aligned} \tag{6}$$

for all $g_1, g_2 \in G$, $j \in \{1, \dots, m\}$, $s \in X$ and $(x_1, x_2, \dots, x_{|X|+1}) \in \mathbb{R}^n \times \mathbb{R}^n \times \dots \times \mathbb{R}^n$ then the system is called a *symmetric system on G*. $\diamond$

Example 2: We will return to Example 1 and add dynamics to each component. Let the equation of motion for the ith robot be

$$\dot{x}_i = u_i = k \sum_{j \in \mathcal{N}} \left(x_j - x_i\right)^3 \tag{7}$$

where $\mathcal{N} = \{i-2, i-1, i+1, i+2\} \pmod N$. We will show this is a symmetric system.

In Example 1 we showed that this be represented by the graph illustrated in Figure 3 with $G = \{-2, -1, 0, 1, 2, \dots, N-3\}$, the group operation to be addition and $X = \{-2, -1, 1, 2\}$ with the relation $s^N = 0$, $N \geq 5$. Also observe from Equation 7, the control for robot i depends on its own state as well as the states for robots $i-2$, $i-1$, $i+1$ and $i+2$, which are equivalent to the four generators. Hence, define all four outputs for robot i to be the vector of the robot's position, *i.e.*, $w_i^s = x_i$ where $s \in X = \{-2, -1, 1, 2, \}$. Define the inputs to $i \in \{-2, -1, \dots, N-3\}$ to be $v_i^s = x_{i-s}$, $s \in \{-2, -1, 1, 2\}$ which satisfies Equation 4. The dynamics, as given in Equation 7 satisfy Equation 5. Finally, the feedback law given in Equation 7 satisfies Equation 6. Because these hold for all $i \in \{-2, -1, 0, \dots, N-3\}$ the system is a symmetric system. $\diamond$

Now, we will define two systems to be *equivalent* if they are symmetric with identical components which are interconnected in the same manner, but possibly with a different number of components.

Definition 2: Two symmetric systems on the finite groups G_1 and G_2 are *equivalent* if G_1 and G_2 are generated by the same set of generators, X,

$$\begin{aligned} f_{g_1}(x) &= f_{g_2}(x), \qquad g_{g_1,j}(x) = g_{g_2,j}(x), \\ w_{s^{-1}g_1}^s(x) &= w_{s^{-1}g_2}^s(x) \end{aligned} \tag{8}$$

and the feedback part of the control laws satisfy

$$\begin{aligned} &u_{g_1,j}\left(x_1(t), w_{s_1^{-1}g_1}^{s_1}(x_2(t)), \dots, w_{s_{|X|}^{-1}g_1}^{s_{|X|}}(x_{|X|+1}(t))\right) = \\ &u_{g_2,j}\left(x_1(t), w_{s_1^{-1}g_2}^{s_1}(x_2(t)), \dots, w_{s_{|X|}^{-1}g_2}^{s_{|X|}}(x_{|X|+1}(t))\right) \end{aligned} \tag{9}$$

for all $g_1 \in G_1$, $g_2 \in G_2$, $s \in X$, $x \in \mathbb{R}^n$, $\left(x_1, x_2, \dots, x_{|X|+1}\right) \in \mathbb{R}^n \times \mathbb{R}^n \times \dots \times \mathbb{R}^n$ and $j \in \{1, \dots, m\}$ where $m = m_{g_1} = m_{g_2}$. $\triangleright$

Example 3: Continuing Example 2 consider two systems with components that satisfy Equation 7 and components belonging to

$$\begin{aligned} G_1 &= \{-2, -1, 0, 1, 2, \dots, N-3\} \\ G_2 &= \{-2, -1, 0, 1, 2, \dots, M-3\} \end{aligned}$$

where $M > N$. Because the dynamics of all the components are identical and the feedback definitions are identical, these systems are equivalent. Both have generating sets $X = \{-2, -1, 1, 2\}$ with the only difference being the relation for G_1 is $s^N = 0$ and the relation for G_2 is $s^M = 0$. $\diamond$

Finally, we define an approximately symmetric system.

Definition 3: Consider

$$\begin{aligned}\dot{x}_g(t) &= f_g\left(x_g(t)\right) + p_g\left(x_g(t), v_g^{s_1}(t), , \ldots, v_g^{s_{|X|}}(t)\right)\\ &\quad + \sum_{j=1}^{m_g} g_{g,j}\left(x_g(t)\right) u_{g,j}\left(x_g(t), v_g^{s_1}(t), , \ldots, v_g^{s_{|X|}}(t)\right)\\ w_g^s(t) &= w_g^s\left(x_g(t)\right),\end{aligned} \tag{10}$$

for all $s \in X$. If, in the absence of the p_g vector field, the system satisfies all the requirements of a symmetric system, then it is an *approximately symmetric system*. The system obtained by setting $p_g = 0$ for all $g \in G$ is called *the corresponding symmetric system*. If two different approximately symmetric have equivalent corresponding symmetric systems, then they are called *equivalent approximately symmetric systems*.

Note the form of p_g is very general and hence can encompass additive perturbations on the drift term, f_g, on the control vector fields g_g or on the control inputs.

Example 4: Returning to Example 3, adding a perturbation to each agent's dynamics of the form

$$\dot{x}_i = k \sum_{j \in \mathcal{N}} \left(x_j - x_i\right)^3 + p_i\left(x_{-2}, \ldots, x_N\right). \tag{11}$$

This is an approximately symmetric system regardless of the specific form of p_i.

For notational convenience, we will concatenate all the states and vector fields from each component into one system description of the form, $\dot{x}_G = f_G(x_G) p_G(x_G) + g_G(x_G) u(t)$ where

$$f_G(x_G) = \begin{bmatrix} f_{g_1}(x_{g_1}) \\ f_{g_2}(x_{g_2}) \\ \vdots \\ f_{g_{|G|}}(x_{g_{|G|}}) \end{bmatrix}, \quad p_G(x_G) = \begin{bmatrix} p_{g_1}(x_{g_1}) \\ p_{g_2}(x_{g_2}) \\ \vdots \\ p_{g_{|G|}}(x_{g_{|G|}}) \end{bmatrix}$$

(and similarly for g_G) with $x_G = \begin{bmatrix} x_{g_1} & \cdots & x_{g_{|G|}} \end{bmatrix}^T$. The $x_{g_i} \in \mathbb{R}^n$ are the states of the g_ith component of the system.

III. STABILITY OF APPROXIMATELY SYMMETRIC SYSTEMS

The main result in this paper is based on incorporating the following result into the symmetric context. Theorem 1 is a standard result from the literature.

Theorem 1: Consider

$$\dot{x} = f(x, t) \tag{12}$$

and

$$\dot{x} = f(x, t) + g(x, t). \tag{13}$$

If $f(x,t)$ is continuous in x and t, $f(0,t) = 0$, $f(x,t)$ is Lipschitz in x and the zero solution of Equation 12 is uniform asymptotically stable, then if $\|g(x,t)\| < \delta$, the solution to Equation 13 satisfies $\|x(t)\| < \delta$ for any initial condition where $\|x(t_0)\| < \delta$ for $t \geq t_0 \geq 0$. $\lhd$

For a proof, see [23]. In words, if an equilibrium of a system is uniformly asymptotically stable, then if bounded additive perturbations are added to the system, the solutions are still *bounded*. Also, in this paper we limit our attention to autonomous systems, so any asymptotically stable equilibrium is also uniformly asymptotically stable.

The idea of the main result in this paper is that if an exactly symmetric system is uniformly asymptotically stable, then the bounded perturbed terms result in, at most, solutions which deviate from asymptotically stable behavior at most by the amount of the bound on the perturbation terms. Furthermore, this result holds for the entire equivalence class of systems, so a robotics control engineer only needs to check *one* system in the entire class to ensure stable operation of the entire class of systems. We first state one of our prior results, which we will shortly extend to the nonsymmetric case.

Proposition 1: Given a symmetric system on a finite group G with generators X, assume there is a function $V_G : \mathcal{D}_G \rightarrow \mathbb{R}$ that is smooth on some open domain $\mathcal{D}_G \subset \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ ($|G|$ times) such that

1) V_G may be expressed as the sum of terms corresponding to each component where

$$V_g : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{1+|X|\text{times}} \rightarrow \mathbb{R}$$

$$\begin{aligned} V_G(x_G) &= \sum_{g \in G} V_g\left(x_g, x_{Xg}\right)\\ &= \sum_{g \in G} V_g\left(x_g, w_{s_1^{-1}g}^{s_1}(x_{s_1^{-1}g}), \ldots, w_{s_{|X|}^{-1}g}^{s_{|X|}}(x_{s_{|X|}^{-1}g})\right), \end{aligned} \tag{14}$$

for all $x \in \mathcal{D}_G$,

2) the individual functions corresponding to each component in G are equal as functions, *i.e.*,

$$V_{g_1} = V_{g_2} = V \tag{15}$$

for all $g_1, g_2 \in G$, and

3) for any one of the $g \in G$,

$$\frac{\partial V_G}{\partial x_g}(x_G)\left(f_g(x_g) + \sum_{j=1}^{m} g_{g,j}(x_g) u_{g,j}\left(x_g, x_{Xg}\right)\right) < 0 \tag{16}$$

for all $x_G \in \mathcal{D}_G$.

Then

1) $\dot{V}_G(x) < 0$ for all $x \in \mathcal{D}_G$ and
2) for any equivalent symmetric system on $\hat{G}$, there is a $V_{\hat{G}}$ such that $\dot{V}_{\hat{G}} < 0$ on some open domain, $\mathcal{D}_{\hat{G}}$. $\lhd$

Proof: (SKETCH) Note that for $h \in G$, because only V_h and its neighbors depend on x_h,

$$\begin{aligned} \frac{\partial V_G}{\partial x_h}(x_G) &= \frac{\partial}{\partial x_h}\left(\sum_{g \in G} V_g\left(x_g, x_{Xg}\right)\right)\\ &= \frac{\partial}{\partial x_h}\left(\sum_{s=e, s \in X} V\left(x_{sh}, x_{Xsh}\right)\right) \end{aligned}$$

where e is the identity element in G. Hence,

$$\dot{V}_G(x_G) = \sum_{g\in G}\left[\frac{\partial}{\partial x_g}\left(\sum_{s=e,s\in X} V\left(x_{sg}, x_{Xsg}\right)\right)\right.$$
$$\left.\left(f_g(x_g) + \sum_{j=1}^{m} g_{g,j}(x_g)u_{g,j}\left(x_g, x_{Xg}\right)\right)\right]. \quad (17)$$

By hypothesis, one of the terms in the sum is negative definite, and for a given g, the term in square brackets is a function with a domain that is the Cartesian product among the states of g, the states of the neighbors of g and the states of the neighbors of the neighbors of g, which is a set of the form $\mathcal{D} \times \cdots \times \mathcal{D}$. Because the Lyapunov functions corresponding to each component are identical, we may take

$$\mathcal{D}_G = \underbrace{\mathcal{D} \times \cdots \times \mathcal{D}}_{|G|\text{times}} \quad (18)$$

for some subset $\mathcal{D} \subset \mathbb{R}^n$. Hence, because the domains of each function is restricted to the same range of values, then negative definiteness of one of them implies the same for all of them.

The fact that the result then holds for any equivalent symmetric system follows a similar argument. For a detailed proof, see [12]. ∎

The main result follows from combining Theorem 1 and Proposition 1.

Proposition 2: Given an approximately symmetric system of the form of Equation 10, assume the corresponding symmetric system satisfies the hypotheses of Proposition 1 and that

$$\|p_G(x_G)\| < \delta.$$

Then for any initial condition satisfying $\|x_G(t_0)\| < \delta$, the solutions of the approximately symmetric system satisfy

$$\|x_G(t)\| < \delta \quad (19)$$

for all $t \geq t_0$. Furthermore, solutions to any equivalent approximately symmetric system also satisfies Equation 19. ◁

Proof: If the corresponding symmetric system satisfies the hypotheses of Proposition 1, then it has a Lyapunov function with a negative definite derivative. Hence, it is asymptotically stable. Then by Theorem 1, if the perturbation terms satisfy $\|p_G(x_G)\| < \delta$, the solutions are such that $\|x_G(t)\| < \delta$.

Similarly, if the corresponding symmetric system satisfies Proposition 1, then any symmetric system equivalent to it has a Lyapunov function, $V_{\hat{G}}$ satisfying $\dot{V}_{\hat{G}} < 0$, and hence is also asymptotically stable. Hence, by Theorem 1, if the perturbation terms satisfy $\left\|p_{\hat{G}}\left(x_{\hat{G}}\right)\right\| < \delta$, the solutions satisfy $\left\|x_{\hat{G}}\right\| < \delta$. ∎

In words, what Proposition 2 provides is that if the corresponding symmetric system for an approximately symmetric system has the right properties to be asymptotically stable, then

- the perturbed system has bounded solutions with a known bound, and,
- *any* equivalent approximately symmetric system also has bounded solutions if the perturbation terms are bounded.

IV. EXAMPLE

We will illustrate the application of the results to the approximately symmetric system from Example 4. We will consider two equivalent symmetric systems one with five agents and one with fifteen agents. In each case, the equations of motion are given by Equation 11 with $k = 1$ and we take the perturbation terms to be of the form

$$p_i(x) = k_i \tan^{-1}(x_2) \quad (20)$$

with $k_1 = 3$, $k_2 = 6$ and $k_4 = -9$ and the rest of the $k_i = 0$. With these perturbation terms, $\delta = \sqrt{126}\pi/2 \approx 17.6$, and any solution with initial conditions satisfying $\|x(0)\| < \delta$ will have a norm less than δ for all time.

We need to show the symmetric system is uniformly asymptotically stable and the perturbation terms are bounded. If that is true, then by Proposition 2, the approximately symmetric system will have solutions satisfying the bound given in Equation 19 as will the solutions for any equivalent approximately symmetric system. Because the system is autonomous, if it is asymptotically stable, it is uniformly asymptotically stable. To show asymptotic stability, following the logic and using some of the results in in [24], define a disagreement vector, x^δ by

$$x = \text{avg}(x)\mathbf{1} + x^\delta$$

in which case, if the average of the state values is invariant, then x^δ has the same equations of motion as x, with the important difference that the origin is the equilibrium point for x^δ instead of the consensus value for x.

If we choose

$$V = \frac{1}{2}\left\|x^\delta\right\|$$

then

$$\begin{aligned}
\dot{V} = x^\delta \dot{x}^\delta &= \sum_{i=1}^{N} x_i^\delta \dot{x}_i^\delta = \sum_{i=1}^{N} x_i^\delta \left(\sum_{j\in\mathcal{N}} \left(x_j^\delta - x_i^\delta\right)^3\right) \\
&= \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}}\left(x_i^\delta\left(x_j^\delta - x_i^\delta\right)^3 + x_j^\delta\left(x_i^\delta - x_j^\delta\right)^3\right) \\
&= \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}}\left(x_i^\delta\left(x_j^\delta - x_i^\delta\right)^3 - x_j^\delta\left(x_j^\delta - x_i^\delta\right)^3\right) \\
&= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}}\left(x_j^\delta - x_i^\delta\right)^4.
\end{aligned}$$

This is zero when $x_i^\delta = x_j^\delta$. However, by the definition of the disagreement vector, this is only possible when they are zero. Hence, the origin is uniformly asymptotically stable and by Proposition 2, the system given by Equations 11 and

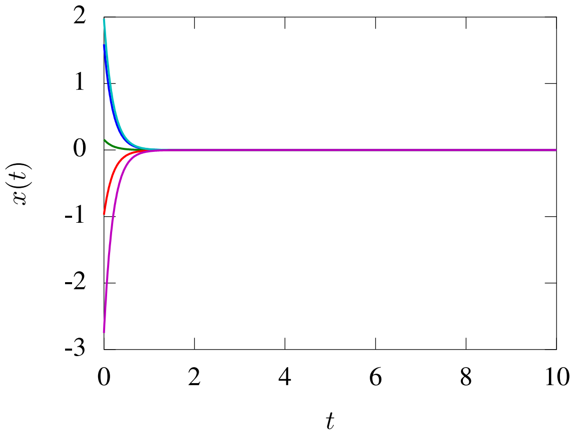

Fig. 4. Asymptotic stability for corresponding symmetric five-agent system.

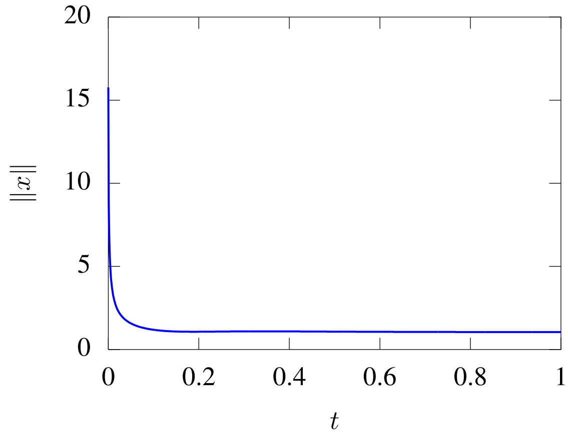

Fig. 6. Norm of solution for approximately symmetric five-agent system.

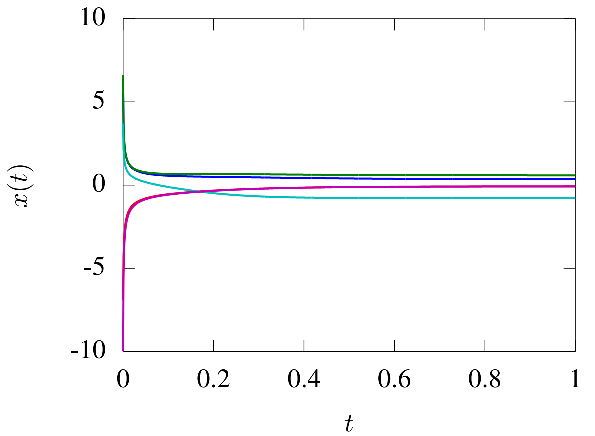

Fig. 5. Stability bound for approximately symmetric five-agent system.

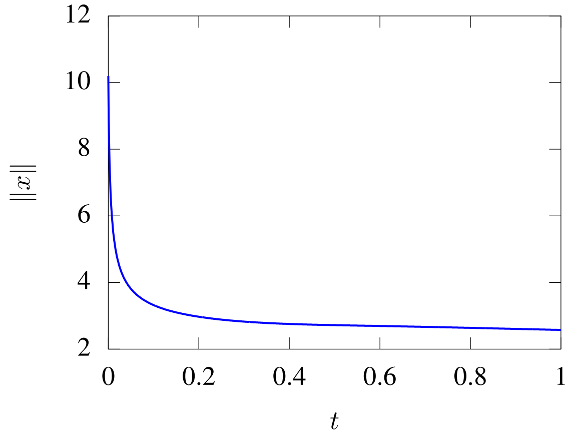

Fig. 7. Norm of solution for approximately symmetric fifteen-agent system.

20 has solutions that satisfy $\left\|x^{\delta}(t)\right\| < \delta$ as long as the initial conditions are less than δ.

Note that in this specific example, because the analysis is on the dynamics of the disagreement vector, the symmetry-breaking perturbations must be of the form that does not alter the term $\operatorname{avg}(x)$, which is the case with the given $p_i(x)$. Asymptotic stability for the five-agent symmetric system is illustrated in Figure 4. With the perturbation terms, bounded deviations from the asymptotic symmetric solution is guaranteed by Proposition 2, as is illustrated in Figure 5. The norm is plotted in Figure 6 illustrating the fact that the norm of the solution satisfies the bound given by $\|x\| < \delta$.

By Proposition 2, these results must hold for any equivalent system as well. For a fifteen agent system with the same perturbation terms on agents one, two and four, the solution is illustrated in Figure 1 and the norm of the solution is illustrated in Figure 7, showing it also satisfies the bound given by Proposition 2.

V. CONCLUSIONS AND FUTURE WORK

This paper presented results which provide a guaranteed bound on solutions for *approximately* symmetric systems. It is an important extension of existing work in the literature on symmetric systems in that it allows for a much broader application of the results because it does not require an exactly symmetric system. As long as the system possesses an underlying symmetric structure which has asymptotically stable associated dynamics, then perturbations (not necessarily small) may prevent *asymptotic* stability, but are guaranteed to not produce solutions which grow unbounded. Importantly, because of the underlying symmetric structure, these results hold for an *entire equivalence class* of approximately symmetric systems. This is valuable for robotics engineers because it 1) it only requires checking one member of a whole class of systems and 2) the check can be on any member of the class, so the one that is simplest to analyze may be used.

Future work is focused on several related issues. First, the results in this paper provide a bound with a very simple restriction on the perturbation, which is simply that the perturbation be bounded. More structural requirements on the perturbation will allow for stronger results, such as stability and asymptotic stability, as opposed to boundedness, in the presence of symmetry-breaking perturbations. Second, the work in this paper focused on systems with a stable equilibrium. Set-based results, based on LaSalle's theorem or related invariance principles, are an important focus for investigation because many robotic formation control problems do not have a single, stable equilibrium, but rather a set of an infinite number acceptable formations.

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