# ISIS Technical Report Draft: Robust Self-Triggered Real-Time Scheduling For Stabilization Of Passive/Dissipative Systems

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# ISIS Technical Report Draft: Robust Self-Triggered Real-Time Scheduling For Stabilization Of Passive/Dissipative Systems

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## I. INTRODUCTION

The majority of feedback control laws are nowadays implemented on digital platforms since microprocessors offer many advantages of running real-time operating systems. This creates the possibility of sharing the computational resources among control and other kinds of applications thus reducing the deployment costs of complex control systems[9]. Moreover, in distributed environment, control systems are often implemented through a shared communication media where controllers, sensors and actuators exchange data. However, since the executions of the controllers are traditionally implemented in a periodic fashion where the inter-execution time $T$ is a constant units of time which is selected based on a worst-case scenario to guarantee the performance of the actuator for all possible operation points, thus the control task is executed at the same rate regardless of the states of the plant and leads to inefficient implementations in terms of processor usage or available communication bandwidth.

To overcome this drawback of the periodic paradigm, several researchers suggested the idea of event-triggered control, see [3]-[4] and [6]. In event-triggered real-time scheduling algorithm, the control tasks are executed whenever a certain error becomes large when compared with the states' norm of the plant (so the triggering condition is based on the full-state of the plant). The event-triggered technique reduces resource usage and provides a high degree of robustness (since the plant is measured continuously). Unfortunately, in many case it requires dedicated hardware to monitor the plant permanently otherwise one might run the risk of consuming the processor time.

Self-triggered real-time scheduling strategy is studied in [5],[7],[8],[9] and it takes the advantage of the event-triggered technique without resorting to extra hardware. The key idea of self-triggered control is to compute the next instants of time at which the control action is to be recomputed based on the current or the last states' measurements of the plant. A first attempt to explore self-triggered paradigm for linear systems was developed in [5], by discretizing the plant, and in [2] for linear $\mathcal{H}_\infty$ controllers. A study on self-triggered scheduling for nonlinear dynamic systems is shown in [7] and [9], where a simple self-trigger condition based on the norm of the current states is proposed by exploiting the properties of the trajectories of homogeneous control systems. However, for the self-triggered scheduling strategy, the intervals of time in which no attention is devoted to the plant pose a concern regarding the robustness of self-triggered implementations.

In this report, we propose a robust self-triggered real-time scheduling strategy for stabilization of passive/dissipative systems. We assume that the model of the plant is a passive or a disspative system, and we assume that the structure uncertainty is a $\mathcal{L}_2$ stable dynamic system in a feedback/feedforward interconnection with the model of the plant. We derived the self-triggered real-time scheduling strategies for both cases and we have also shown that the inter-execution time under the proposed scheduling strategy is non-trivial.

The rest of this report is organized as follows. We first set the notions and introduce some background on passive/dissipative systems in section II; the problem statement is made in section III; the self-triggered scheduling strategy when the uncertainty is in a feedback interconnection with the model of the plant is proposed in section IV and the corresponding self-triggered scheduling strategy when the uncertainty is in a feed-forward interconnection with the model of the plant is proposed in section V; several examples are provide several examples in section VI; finally, conclusion is made in section VII.

## II. Notations and Background Material

To set the background and notation for what follows, we need to introduce some basic concepts on passive and dissipative systems.

Consider the following dynamic system which can be used to describe both linear and nonlinear system:

$$H : \begin{cases} \dot{x} = f(x, u) \\ y = h(x, u) \end{cases} \tag{1}$$

where $x \in X \subset \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$ and $y \in Y \subset \mathbb{R}^m$ are the state, input and output variables, respectively, and $X$, $U$ and $Y$ are the state, input and output spaces, respectively. The representation $x(t) = \phi(t, t_0, x_0, u)$ is used to denote the state at time $t$ reached from the initial state $x_0$ at $t_0$.

**Definition 1(Supply Rate [1]).** *The supply rate $\omega(t) = \omega(u(t), y(t))$ is a real valued function defined* *on $U \times Y$, such that for any $u(t) \in U$ and $x_0 \in X$ and $y(t) = h(\phi(t, t_0, x_0, u))$, $\omega(t)$ satisfies*

$$\int_{t_0}^{t_1} |\omega(\tau)| d\tau < \infty \tag{2}$$

**Definition 2(Dissipative System [1]).** *System H with supply rate $\omega(t)$ is said to be dissipative if there exists a nonnegative real function $V(x) : X \to \mathbb{R}^+$, called the storage function, such that, for all $t_1 \geq t_0 \geq 0$, $x_0 \in X$ and $u \in U$,*

$$V(x_1) - V(x_0) \leq \int_{t_0}^{t_1} \omega(\tau) d\tau \tag{3}$$

*where $x_1 = \phi(t_1, t_0, x_0, u)$ and $\mathbb{R}^+$ is a set of nonnegative real numbers.*

**Definition 3(Passive System [1])**. *The dynamic system given in (1) is said to be* ***passive*** *if there exists a $\mathcal{C}^1$ storage function $V(x) \geq 0$ such that*

$$\dot{V} = \frac{\partial V(x)}{\partial x} f(x(t), u(t)) \leq -S(x) + u(t)^T y(t) \ \forall t \tag{4}$$

*for some positive semi-definite function $S(x)$. We say it is* ***strictly passive*** *if $S(x) > 0$.*

**Definition 4(Excess/Shortage of Passivity [2]).** *System $H$ is said to be:*

- *Input Feed-forward Passive (IFP) if it is dissipative with respect to supply rate $\omega(u, y) = u^T y - \nu u^T u$ for some $\nu \in \mathbb{R}$, denoted as IFP($\nu$).*
- *Output Feedback Passive (OFP) if it is dissipative with respect to the supply rate $\omega(u, y) = u^T y - \rho y^T y$ for some $\rho \in \mathbb{R}$, denoted as OFP($\rho$).*
- *Input Feed-forward Output Feedback Passive (IF-OFP) if it is dissipative with respect to the supply rate $\omega(u, y) = u^T y - \rho y^T y - \nu u^T u$ for some $\rho \in \mathbb{R}$ and $\nu \in \mathbb{R}$, denoted as IF-OFP($\nu, \rho$).*

A positive $\nu$ or $\rho$ means that the system has an excess of passivity; otherwise, the system is lack of passivity.

**Definition 5(Zero-State Observability and Detectability [2]).** *Consider the system $H$ with zero input, that is $\dot{x} = f(x, 0)$, $y = h(x, 0)$, and let $Z \subset \mathbb{R}^n$ be its largest positively invariant set contained in $\{x \in \mathbb{R}^n | y = h(x, 0) = 0\}$. We say $H$ is zero-state detectable(ZSD) if $x = 0$ is asymptotically stable conditionally to $Z$. if $Z = \{0\}$, we say that $H$ is zero-state observable (ZSO).*

## III. Problem Statement

We consider a nonlinear control system $H$ given by

$$H : \begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases} \tag{5}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input and $y \in \mathbb{R}^m$ is the output. We assume that $H$ is a passive system, which means that there exists a nonnegative storage function $V(x) : \mathbb{R}^n \to \mathbb{R}^+$, such that

$$\dot{V}(x) \leq u^T y. \tag{6}$$

We know that if $H$ is zero-state detectable(ZSD), then under the feedback control law

$$u = -Ky \tag{7}$$

where $K > 0$ could be a scalar or an $m \times m$ positive definite matrix, the origin of $H$ is asymptotically stable.

In real time, the implementation of the feedback control law (7) on an embedded processor is typically done by sampling the output $y$ at time instants

$$t_0,\ t_1,\ t_2,\ t_3,\ t_4,\ \ldots,$$

computing the control action $u(t_i) = -Ky(t_i)$ and updating the actuator at time instants

$$t_0 + \triangle_0,\ t_1 + \triangle_1,\ t_2 + \triangle_2,\ t_3 + \triangle_3,\ t_4 + \triangle_4,\ \ldots,$$

where $\triangle_i \geq 0$ represents the time required to read the output from the sensor, compute the control action and update the actuators. This means that a sequence of measurements

$$y(t_0),\ y(t_1),\ y(t_2),\ y(t_3),\ y(t_4),\ \ldots,$$

corresponds to a sequence of actuation updates

$$u(t_0 + \triangle_0),\ u(t_1 + \triangle_1),\ u(t_2 + \triangle_2),\ u(t_3 + \triangle_3),\ u(t_4 + \triangle_4),\ \ldots,$$

as shown in Figure 1.

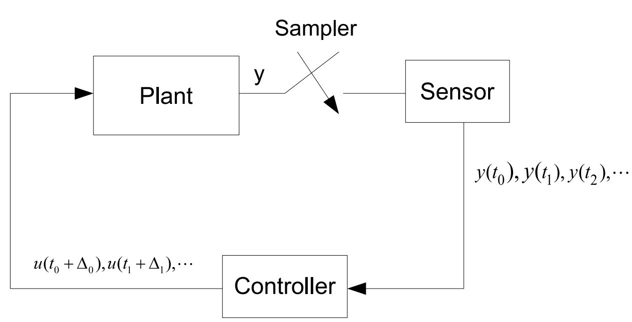


Fig. 1: Implementation of the feedback law in real time

Between actuator updates, the control law $u$ is held constant according to

$$t \in [t_i + \triangle_i, t_{i+1} + \triangle_{i+1}] \Rightarrow u(t) = u(t_i + \triangle_i) = -Ky(t_i). \tag{8}$$

Furthermore, the sequence of times $t_0, t_1, t_2, t_3, \ldots$ is typically periodic, which means that $t_{i+1} - t_i = T$, where $T > 0$ is the sampling period; in this case, we can regard the execution of the output feedback control law (7) as being "time-triggered". If we define the measurement error at the actuator to be

$$t \in [t_i + \triangle_i, t_{i+1} + \triangle_{i+1}] \Rightarrow e(t) = y(t) - y(t_i), \tag{9}$$

then

$$\dot{x} = f(x, -Ky(t_i)) = f(x, -K(y(t) - e(t))),\ t \in [t_i + \triangle_i, t_{i+1} + \triangle_{i+1}]. \tag{10}$$

Since system $H$ is passive, with (6),(7) and (9) we can obtain

$$\begin{aligned} \dot{V}(x) &\leq u^T y = -K(y-e)^T y = -Ky^T y + Ke^T y \\ &\leq K\|e\|_2\|y\|_2 - K\|y\|_2^2 \end{aligned} \tag{11}$$

so if

$$\|e\|_2 \leq \|y\|_2,\ for\ t \in [t_i + \triangle_i, t_{i+1} + \triangle_{i+1}],\ \forall i, \tag{12}$$

we will have $\dot{V}(x) \leq 0, \forall t$, then the closed-loop system is asymptotically stable since the plant is passive and ZSD. One can easily show that a sufficient condition for (12) to hold is given by

$$\|e\|_2 \leq \frac{1}{2}\|y(t_i)\|_2,\ for\ t \in [t_i + \triangle_i, t_{i+1} + \triangle_{i+1}],\ \forall i, \tag{13}$$

and based on this sufficient condition, we have proposed a self-triggered scheduling strategy for stabilization of passive/disspative plant in [10]. Now the problem is: when the given plant model which is passive or dissipative, is subject to model uncertainty, can we derive a "robust"scheduling strategy so that we are still able to stabilize the plant while taking advantage of the self-triggered scheduling strategy as we have discussed in [10]?

The rest of this reports proposes possible ways to solve this problem: we assume that the model of the plant is a passive or a dissipative system, and we assume that the structure uncertainty is a $\mathcal{L}_2$ stable dynamic system; we first study the case when the model uncertainty is in a feedback interconnection with the model of the plant in section IV; another case when the model uncertainty is in a feed-forward interconnection with the model of the plant is discussed in section V. For both cases, we have also shown that the inter-execution time under the proposed scheduling strategies is non-trivial and we can decide the next time at which the control action needs to be re-computed based on the past measurements of the plant's output.

## IV. Self-Triggered Real-Time Scheduling In Presence Of Output Feedback Uncertainty

**Theorem 1.**Consider the control system as shown in Fig.2, where the plant model is given by

$$\Sigma_o : \begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases} \tag{14}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^n$ is the control input and $y \in \mathbb{R}^n$ is the output; $\Sigma_o$ is a ZSD passive system satisfying the passive inequality given by

$$\dot{V}(x) \leq u^T y \tag{15}$$

with $V(x) \geq 0$. We assume that the plant model is subject to feedback model uncertainty, where the

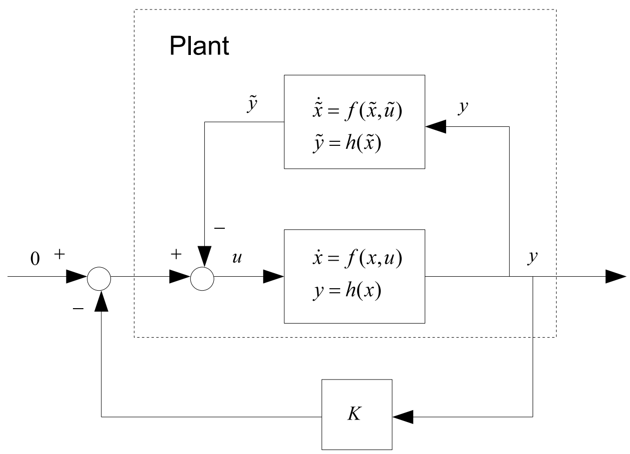


Fig. 2: Output Feedback Uncertainty

model uncertainty is a $\mathcal{L}_2$ stable dynamic system with finite $\mathcal{L}_2$ gain $\Gamma$, and the system's dynamic is given by

$$\Sigma_\Delta : \begin{cases} \dot{\tilde{x}} = \tilde{f}(\tilde{x}, \tilde{u}) \\ \tilde{y} = \tilde{h}(\tilde{x}). \end{cases} \tag{16}$$

If the following conditions are satisfied

1) $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous on compacts;
2) $h : \mathbb{R}^n \to \mathbb{R}^n$ is a static nonlinear function of $x$ which belongs to a sector $[\beta, \alpha]$ such that $\beta x^T x \leq x^T h(x) \leq \alpha x^T x$, where $\alpha\beta > 0$;
3) $\|\frac{\partial h(x)}{\partial x}\|_2 \leq \gamma$, where $0 < \gamma < \infty$;

4) $\tilde{\beta}\tilde{u}^T\tilde{u} \le \tilde{u}^T\tilde{y} \le \tilde{\alpha}\tilde{u}^T\tilde{u}$, where $-\infty < \tilde{\beta} \le 0 \le \tilde{\alpha} < \infty$(notice that if we know the $\mathcal{L}_2$ gain $\Gamma$ of $\Sigma_\Delta$, then we could simply choose $\tilde{\beta} = -\Gamma$ and $\tilde{\alpha} = \Gamma$);

then under the following scheduling strategy, the passive system under the control action $u(t) = -Ky(t_i)$ for $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}]$, where $K > max\{-\tilde{\beta}, 0\}$, is asymptotically stable:

- $t_0 = t_0 + \triangle_0$;
- $t_1 = t_0 + \tau_0$, $\tau_0 = \frac{1}{\gamma L(2+\zeta+\Gamma)} \ln\left(1 + \frac{2+\zeta+\Gamma}{1+\zeta+\Gamma}\hat{\sigma}\right)$;
- $t_{i+1} = t_i + \triangle_i + \tau$, $i = 1, 2, \ldots$;

where $t_i$ is the time at which the sensor gets the measurement of the output $y(t_i)$, and $\zeta = \max\{\frac{1}{|\alpha|}, \frac{1}{|\beta|}\}$; $L$ is the Lipschitz constant of $f(x, u)$; choose $\tilde{\sigma}, \hat{\sigma}, \hat{\sigma}'$ as constants such that $0 < \tilde{\sigma} < \hat{\sigma} < \hat{\sigma}' < \frac{K+\tilde{\beta}}{2K+\tilde{\beta}}$ and

$$\tau = \frac{1}{\gamma L(2+\zeta+\Gamma)} \ln\left(\frac{\frac{1+\zeta+\Gamma}{2+\zeta+\Gamma} + \hat{\sigma}}{\frac{1+\zeta+\Gamma}{2+\zeta+\Gamma} + \tilde{\sigma}}\right). \tag{17}$$

$\triangle_i$ is the estimated admissible execution delay of the actuator after the sensor gets the $i$th measurement of the system's output, and

$$\triangle_i = \min\left[\varepsilon_i^-, \varepsilon_i^+\right], \tag{18}$$

where

$$\varepsilon_i^- = \frac{1}{\gamma L(2+\zeta+\Gamma)} \ln\left(\frac{(2+\zeta+\Gamma)\tilde{\sigma}\|y(t_i)\|_2}{(1+\zeta+\Gamma)\|y(t_i)\|_2 + \|y(t_i) - y(t_{i-1})\|_2} + 1\right), \tag{19}$$

and

$$\varepsilon_i^+ = \frac{1}{\gamma L(2+\zeta+\Gamma)} \ln\left(\frac{\frac{1+\zeta+\Gamma}{2+\zeta+\Gamma} + \hat{\sigma}'}{\frac{1+\zeta+\Gamma}{2+\zeta+\Gamma} + \hat{\sigma}}\right). \tag{20}$$

*Proof.* Let $e(t) = y(t) - y(t_i)$ for $t \in [t_i + \triangle_i, t_{i+1} + \triangle_{i+1}]$ denote the error of the measurement at the actuator, and let $\tilde{e}(t) = y(t) - y(t_i)$ for $t \in [t_i, t_{i+1}]$ denote the error of the measurement at the sensor. Since the model of the plant $\Sigma_o$ is passive, we have $\dot{V}(x) \le u^T y$. For $t \in [t_i + \triangle_i, t_{i+1} + \triangle_{i+1}]$, we have $u(t) = -Ky(t_i) - \tilde{y}(t)$, and we can obtain

$$\begin{aligned}\dot{V}(x) &\le [-Ky(t_i) - \tilde{y}(t)]^T y(t) = -Ky(t_i)^T y(t) - \tilde{y}(t)^T y(t) \\ &= -K[y(t) - e(t)]^T y(t) - \tilde{y}(t)^T y(t) \\ &= -Ky(t)^T y(t) + Ke(t)^T y(t) - \tilde{y}(t)^T y(t),\end{aligned} \tag{21}$$

and thus

$$\begin{aligned}\dot{V}(x) &\leq -K\|y\|_2^2 + K\|e\|_2\|y\|_2 - \tilde{y}^T y \\ &\leq -K\|y\|_2^2 + K\|e\|_2\|y\|_2 - \tilde{\beta}\|y\|_2^2 \\ &= -(K+\tilde{\beta})\|y\|_2^2 + K\|e\|_2\|y\|_2.\end{aligned} \tag{22}$$

So, if $K > max\{-\tilde{\beta}, 0\}$, and $\|e\|_2 \leq \frac{K+\tilde{\beta}}{K}\|y\|_2$ for $t \in [t_i + \triangle_i, t_{i+1} + \triangle_{i+1}]$, $\forall i$, then we will have $\dot{V}(x) \leq 0$, $\forall t$, and the closed-loop system is asymptotically stable since we assume that $\Sigma_o$ is ZSD. Moreover, since $e(t) = y(t) - y(t_i)$ for $t \in [t_i + \triangle_i, t_{i+1} + \triangle_{i+1}]$, we have $\|e\|_2 \geq \|y(t_i)\|_2 - \|y\|_2$, and we can obtain a sufficient condition for $\|e\|_2 \leq \frac{K+\tilde{\beta}}{K}\|y\|_2$ to hold which is given by

$$\|e\|_2 \leq \frac{K+\tilde{\beta}}{2K+\tilde{\beta}}\|y(t_i)\|_2,\ \ for\ t \in [t_i + \triangle_i, t_{i+1} + \triangle_{i+1}]. \tag{23}$$

If we consider non-zero execution delay of the actuator, the evolution of $\frac{\|e(t)\|_2}{\|y(t_i)\|_2}$ should appear as shown in Fig.3. For $t \in [t_i, t_i + \Delta_i]$, we have $e(t) = y(t) - y(t_{i-1})$ and $\tilde{e}(t) = y(t) - y(t_i)$, and we can

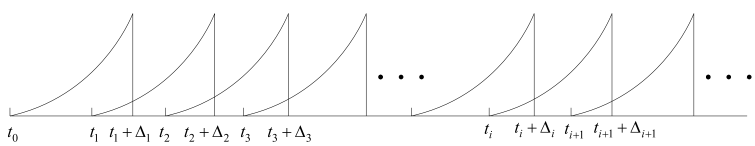


Fig. 3: evolution of $\frac{\|e(t)\|_2}{\|y(t_i)\|_2}$

obtain

$$\begin{aligned}\frac{d}{dt}\|\tilde{e}(t)\|_2 &\leq \|\dot{\tilde{e}}(t)\|_2 = \|\dot{y}(t)\|_2 = \|\frac{\partial y}{\partial x}\dot{x}\|_2 \leq \|\frac{\partial y}{\partial x}\|_2\|\dot{x}\|_2 \\ &\leq \gamma\|f(x,u)\|_2 = \gamma\|f(x, -Ky(t_{i-1}) - \tilde{y})\|_2 \\ &= \gamma\|f(x, -K(y-e) - \tilde{y})\|_2 \\ &\leq \gamma L\big[\|x\|_2 + \|y\|_2 + \|e\|_2 + \|\tilde{y}\|_2\big] \\ &\leq \gamma L\big[\zeta\|y\|_2 + \|y\|_2 + \|\tilde{e} + y(t_i) - y(t_{i-1})\|_2 + \Gamma\|y\|_2\big] \\ &= \gamma L\big[(1+\zeta+\Gamma)\|y\|_2 + \|\tilde{e} + y(t_i) - y(t_{i-1})\|_2\big] \\ &= \gamma L\big[(1+\zeta+\Gamma)\|\tilde{e} + y(t_i)\|_2 + \|\tilde{e} + y(t_i) - y(t_{i-1})\|_2\big] \\ &\leq \gamma L(2+\zeta+\Gamma)\|\tilde{e}\|_2 + \gamma L(1+\zeta+\Gamma)\|y(t_i)\|_2 + \gamma L\|y(t_i) - y(t_{i-1})\|_2,\end{aligned} \tag{24}$$

where $\zeta = \max\{\frac{1}{|\alpha|}, \frac{1}{|\beta|}\}$. So the evolution of $\|\tilde{e}(t)\|_2$ during the time $[t_i, t_i + \Delta_i]$ is bounded by the solution of

$$\dot{\phi}(t) = \gamma L(2+\zeta+\Gamma)\phi(t) + \gamma L(1+\zeta+\Gamma)\|y(t_i)\|_2 + \gamma L\|y(t_i) - y(t_{i-1})\|_2, \tag{25}$$

with $\phi(t_i) = y(t_i) - y(t_i) = 0$, the solution to (25) is given by

$$\phi(t) = \frac{(1+\zeta+\Gamma)\|y(t_i)\|_2 + \|y(t_i) - y(t_{i-1})\|_2}{2+\zeta+\Gamma}\big[e^{\gamma L(2+\zeta+\Gamma)(t-t_i)} - 1\big]. \tag{26}$$

So if we choose $0 < \tilde{\sigma} < \frac{K+\tilde{\beta}}{2K+\tilde{\beta}}$, and let $\phi(t_i + \Delta_i) = \tilde{\sigma}\|y(t_i)\|_2$, we can get an estimate of $\Delta_i$, if we denote it by $\varepsilon_i^-$, then $\varepsilon_i^-$ is given by

$$\varepsilon_i^- = \frac{1}{\gamma L(2+\zeta+\Gamma)}\ln\big[1 + \frac{(2+\zeta+\Gamma)\tilde{\sigma}\|y(t_i)\|_2}{(1+\zeta+\Gamma)\|y(t_i)\|_2 + \|y(t_i) - y(t_{i-1})\|_2}\big], \tag{27}$$

and notice that $\varepsilon_i^- > 0$ for any $\tilde{\sigma} > 0$ and $\|y(t_i)\|_2$ cannot goes to zero in a short time because by applying output feedback to the passive system model, the output $y(t)$ goes to zero asymptotically.

Assume that the actuator updates the control action at $t = t_i + \Delta_i$, and choose a $\hat{\sigma}$ such that

$$0 < \tilde{\sigma} < \hat{\sigma} < \frac{K+\tilde{\beta}}{2K+\tilde{\beta}},$$

then for $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}]$, we have

$$e(t) = \tilde{e}(t) = y(t) - y(t_i), \tag{28}$$

and we can obtain

$$\begin{aligned}
\frac{d}{dt}\|e(t)\|_2 &\le \|\dot{e}(t)\|_2 = \|\dot{y}(t)\|_2 = \|\frac{\partial y}{\partial x}\dot{x}\|_2 \\
&\le \|\frac{\partial y}{\partial x}\|_2\|\dot{x}\|_2 \le \gamma\|f(x, -K(y-e) - \tilde{y})\|_2 \\
&\le \gamma L\big[\|x\|_2 + \|y\|_2 + \|e\|_2 + \|\tilde{y}\|_2\big] \\
&\le \gamma L\big[\zeta\|y\|_2 + \|y\|_2 + \|e\|_2 + \Gamma\|y\|_2\big] \\
&= \gamma L\big[(1+\zeta+\Gamma)\|y\|_2 + \|e\|_2\big] \\
&= \gamma L\big[(1+\zeta+\Gamma)\|e + y(t_i)\|_2 + \|e\|_2\big] \\
&\le \gamma L(2+\zeta+\Gamma)\|e\|_2 + \gamma L(1+\zeta+\Gamma)\|y(t_i)\|_2,
\end{aligned} \tag{29}$$

so the evolution of $\|e(t)\|_2$ during $[t_i + \Delta_i, t_{i+1} + \Delta_{i+1}]$ is bounded by the solution of

$$\dot{\phi}(t) = \gamma L(2+\zeta+\Gamma)\phi(t) + \gamma L(1+\zeta+\Gamma)\|y(t_i)\|_2, \tag{30}$$

with $\phi(t_i + \Delta_i) = \|y(t_i + \Delta_i) - y(t_i)\|_2 = \tilde{\sigma}\|y(t_i)\|_2$, we can get the solution to (30) which is given by

$$\phi(t) = \frac{(1+\zeta+\Gamma)\|y(t_i)\|_2 + (2+\zeta+\Gamma)\tilde{\sigma}\|y(t_i)\|_2}{2+\zeta+\Gamma}e^{\gamma L(2+\zeta+\Gamma)(t-t_i-\Delta_i)} - \frac{1+\zeta+\Gamma}{2+\zeta+\Gamma}\|y(t_i)\|_2. \tag{31}$$

Assume that at $t = t_{i+1}$, we have $\phi(t_{i+1}) = \hat{\sigma}\|y(t_i)\|_2$, then an estimate of the time it takes for $\frac{\|e(t)\|_2}{\|y(t_i)\|_2}$ to evolve from $\tilde{\sigma}$ to $\hat{\sigma}$ is given by

$$\tau = \frac{1}{\gamma L(2+\zeta+\Gamma)} \ln \big(\frac{\hat{\sigma} + \frac{1+\zeta+\Gamma}{2+\zeta+\Gamma}}{\tilde{\sigma} + \frac{1+\zeta+\Gamma}{2+\zeta+\Gamma}}\big) \tag{32}$$

and notice that for any $\hat{\sigma} > \tilde{\sigma} > 0$, we have $\tau > 0$. Next, assume that $t = t_{i+1} + \Delta_{i+1}$, we have $\phi(t_{i+1} + \Delta_{i+1}) = \hat{\sigma}^{'}\|y(t_i)\|_2$, where we have

$$0 < \tilde{\sigma} < \hat{\sigma} < \hat{\sigma}^{'} < \frac{K + \tilde{\beta}}{2K + \tilde{\beta}}.$$

Since $e(t) = y(t) - y(t_i)$ for $t \in [t_{i+1}, t_{i+1} + \Delta_{i+1}]$, we can still get an estimate of the time it takes for $\frac{\|e(t)\|_2}{\|y(t_i)\|_2}$ to evolve from $\hat{\sigma}$ to $\hat{\sigma}^{'}$ based on (30). If we denote it by $\varepsilon_i^{+}$, then we can obtain

$$\tau = \frac{1}{\gamma L(2+\zeta+\Gamma)} \ln \big(\frac{\hat{\sigma}^{'} + \frac{1+\zeta+\Gamma}{2+\zeta+\Gamma}}{\hat{\sigma} + \frac{1+\zeta+\Gamma}{2+\zeta+\Gamma}}\big). \tag{33}$$

Now we could give a lower bound of $\Delta_i$ which is given by $\big($notice that $\Delta_i$ is the execution delay of the actuator after the sensor gets the measurement $y(t_i)\big)$:

$$\Delta_i = \min[\varepsilon_i^{-}, \varepsilon_i^{+}], \tag{34}$$

and the corresponding estimate of the time for the sensor to get the next new measurement is given by

$$t_{i+1} = t_i + \Delta_i + \tau. \tag{35}$$

Since we choose$0 < \tilde{\sigma} < \hat{\sigma} < \hat{\sigma}^{'} < \frac{K+\tilde{\beta}}{2K+\tilde{\beta}}$, for $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}], \forall i$, we can guarantee that $\dot{V}(x) \leq 0, \forall t$. Thus, if the model of the plant is ZSD, then the closed-loop system under the proposed self-triggered scheduling strategy is asymptotically stable. ■

**Remark 1**: For the linear system case, consider the model of the plant given by

$$\Sigma_o : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx, \end{cases} \tag{36}$$

and the feedback uncertainty given by

$$\Sigma_\Delta : \begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u} \\ \tilde{y} = \tilde{C}\tilde{x}, \end{cases} \tag{37}$$

the closed-loop system is shown in Fig.4. We assume that $\Sigma_o$ is passive, and $\Sigma_\Delta$ is a $\mathcal{L}_2$ stable system with finite $\mathcal{L}_2$ gain $\Gamma$. For $t \in [t_i, t_i + \Delta_i]$, we have $e(t) = y(t) - y(t_{i-1})$ and $\tilde{e}(t) = y(t) - y(t_i)$, so we

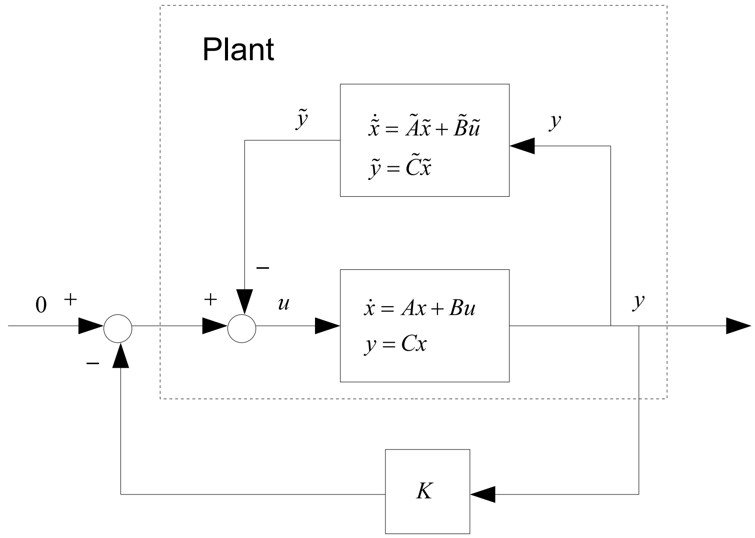


Fig. 4: Linear Feedback Uncertainty

can obtain

$$\begin{aligned} \frac{d}{dt}\|\tilde{e}(t)\|_2 \leq \|\dot{\tilde{e}}(t)\|_2 = \|\dot{y}(t)\|_2 = \|\frac{\partial y}{\partial x}\dot{x}\|_2 = \|C\dot{x}\|_2 \\ = \|CAx + CBu\|_2 \leq \|CAx\|_2 + \|CBu\|_2, \end{aligned} \tag{38}$$

if $\frac{\|CAx\|_2}{\|y\|_2} = \frac{(x^T A^T C^T CAx)^{\frac{1}{2}}}{(x^T C^T Cx)^{\frac{1}{2}}} \leq \zeta$, where $0 \leq \zeta < \infty$, then we can get

$$\begin{aligned} \frac{d}{dt}\|\tilde{e}(t)\|_2 &\leq \|CAx + CBu\|_2 \leq \|CAx\|_2 + \|CBu\|_2 \\ &\leq \zeta\|y\|_2 + \|CBu\|_2 \\ &= \zeta\|\tilde{e} + y(t_i)\|_2 + \| - CB(Ky(t_{i-1}) + \tilde{y})\|_2 \\ &\leq \zeta\|\tilde{e}\|_2 + \zeta\|y(t_i)\|_2 + \|CBKy(t_{i-1})\|_2 + \|CB\|_2\|\tilde{y}\|_2 \\ &\leq \zeta\|\tilde{e}\|_2 + \zeta\|y(t_i)\|_2 + \|CBKy(t_{i-1})\|_2 + \|CB\|_2\Gamma\|y\|_2 \\ &= \zeta\|\tilde{e}\|_2 + \zeta\|y(t_i)\|_2 + \|CBKy(t_{i-1})\|_2 + \|CB\|_2\Gamma\|\tilde{e} + y(t_i)\|_2 \\ &\leq (\zeta + \|CB\|_2\Gamma)\|\tilde{e}\|_2 + (\zeta + \|CB\|_2\Gamma)\|y(t_i)\|_2 + \|CBKy(t_{i-1})\|_2 \end{aligned} \tag{39}$$

so the evolution of $\|\tilde{e}(t)\|_2$ during the time $[t_i, t_i + \Delta_i]$ is bounded by the solution of

$$\dot{\phi}(t) = (\zeta + \|CB\|_2\Gamma)\phi(t) + \|CBKy(t_{i-1})\|_2 + (\zeta + \|CB\|_2\Gamma)\|y(t_i)\|_2 \tag{40}$$

with $\phi(t) = \|y(t_i) - y(t_i)\|_2 = 0$, the solution to (40) is given by

$$\phi(t) = \frac{\|CBKy(t_{i-1})\|_2 + (\zeta + \|CB\|_2\Gamma)\|y(t_i)\|_2}{\zeta + \|CB\|_2\Gamma}\left[e^{(\zeta + \|CB\|_2\Gamma)(t - t_i)} - 1\right] \tag{41}$$

so in this case, $\varepsilon_i^-$ is given by

$$\varepsilon_i^- = \frac{1}{\zeta + \|CB\|_2\Gamma} \ln \big[1 + \frac{(\zeta + \|CB\|_2\Gamma)\tilde{\sigma}\|y(t_i)\|_2}{\|CBKy(t_{i-1})\|_2 + (\zeta + \|CB\|_2\Gamma)\|y(t_i)\|_2}\big]. \tag{42}$$

For $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}]$, we have $e(t) = y(t) - y(t_i)$, and we can obtain

$$\begin{aligned} \frac{d}{dt}\|e\|_2 &\leq \|\dot{e}\|_2 = \|\dot{y}\|_2 = \|C\dot{x}\|_2 = \|C(Ax + Bu)\|_2 \leq \|CAx\|_2 + \|CBu\|_2 \\ &\leq \zeta\|y\|_2 + \| - CB(Ky(t_i) + \tilde{y})\|_2 \\ &\leq \zeta\|y\|_2 + \|CBKy(t_i)\|_2 + \|CB\tilde{y}\|_2 \\ &\leq (\zeta + \|CB\|_2\Gamma)\|e\|_2 + (\zeta + \|CB\|_2\Gamma + \|CBK\|_2)\|y(t_i)\|_2, \end{aligned} \tag{43}$$

so the evolution of $\|e(t)\|_2$ during $[t_i + \Delta_i, t_{i+1} + \Delta_{i+1}]$ is bounded by the solution of

$$\dot{\phi}(t) = (\zeta + \|CB\|_2\Gamma)\phi(t) + (\zeta + \|CB\|_2\Gamma + \|CBK\|_2)\|y(t_i)\|_2. \tag{44}$$

Based on this, we can get

$$\varepsilon_i^+ = \frac{1}{\zeta + \|CB\|_2\Gamma} \ln \big[\frac{\zeta + \|CB\|_2\Gamma + \|CBK\|_2 + (\zeta + \|CB\|_2\Gamma)\hat{\sigma}'}{\zeta + \|CB\|_2\Gamma + \|CBK\|_2 + (\zeta + \|CB\|_2\Gamma)\hat{\sigma}}\big], \tag{45}$$

and

$$\tau = \frac{1}{\zeta + \|CB\|_2\Gamma} \ln \big[\frac{\zeta + \|CB\|_2\Gamma + \|CBK\|_2 + (\zeta + \|CB\|_2\Gamma)\hat{\sigma}}{\zeta + \|CB\|_2\Gamma + \|CBK\|_2 + (\zeta + \|CB\|_2\Gamma)\tilde{\sigma}}\big]. \tag{46}$$

**Remark 2**: If the model of the plant is not passive, but it satisfies the following dissipative inequality given by

$$\dot{V}(x) \leq u^T y + \rho y^T y \tag{47}$$

where $\rho > 0$ is the smallest value such that the above dissipative inequality holds, then we need choose $K$ carefully. Since for $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}]$, we have $u = -Ky(t_i) - \tilde{y}$, we can obtain

$$\begin{aligned} \dot{V}(x) &\leq u^T y + \rho y^T y = [-Ky(t_i) - \tilde{y}]^T y + \rho y^T y \\ &\leq -Ky^T(t_i)y - \tilde{y}^T y + \rho y^T y = -K(y - e)^T y - \tilde{y}^T y + \rho y^T y \\ &= -Ky^T y + Ke^T y - \tilde{y}^T y + \rho y^T y \\ &= -(K - \rho)y^T y + Ke^T y - \tilde{y}^T y \\ &\leq -(K - \rho + \tilde{\beta})\|y\|_2^2 + K\|e\|_2\|y\|_2, \end{aligned} \tag{48}$$

so if we choose $K > 0$ and $K > \rho - \tilde{\beta}$, and if $\|e\|_2 \leq \frac{K-\rho+\tilde{\beta}}{K}\|y\|_2$ for $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}]$, $\forall i$, then $\dot{V}(x) \leq 0, \forall t$. One could show that a sufficient condition for $\|e\|_2 \leq \frac{K-\rho+\tilde{\beta}}{K}\|y\|_2, \forall t$ is given by

$$\|e\|_2 \leq \frac{K-\rho+\tilde{\beta}}{2K-\rho+\tilde{\beta}}\|y(t_i)\|_2,\ for\ t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}], \forall i. \quad (49)$$

So in this case, the $\tilde{\sigma}, \hat{\sigma}, \hat{\sigma}^{'}$ in the proposed self-triggered scheduling strategy in Theorem 1 should be properly chosen such that

$$0 < \tilde{\sigma} < \hat{\sigma} < \hat{\sigma}^{'} < \frac{K-\rho+\tilde{\beta}}{2K-\rho+\tilde{\beta}}. \quad (50)$$

**Remark 3**: Assumption 2) in Theorem 1 is rather conservative, since we restrict output $y = h(x)$ of the plant to belong to a bounded sector. But most of the time, this assumption can be relaxed as long as we can get

$$\|\dot{y}\|_2 \leq p_1\|y\|_2 + p_2\|e\|_2 \quad (51)$$

for some constant $p_1 \geq 0$ and $p_2 > 0$, and similar self-triggered scheduling strategy can be obtained. Also in this case, the output $y$ does not to have the same dimension as the state $x$.

## V. Self-Triggered Real-Time Scheduling In Presence Of Feed-forward Uncertainty

**Theorem 2.**Consider the control system as shown in Fig.5, where the plant model is given by

$$\Sigma_o : \begin{cases} \dot{x} = f(x, u) \\ y_0 = h(x) \end{cases} \tag{52}$$

$x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^n$ is the control input and $y_0 \in \mathbb{R}^n$ is the output; $\Sigma_o$ is a ZSD passive system satisfying the passive inequality given by

$$\dot{V}(x) \leq u^T y_0 \tag{53}$$

with $V(x) \geq 0$. We assume that the plant model is subject to feed-forward model uncertainty, where the

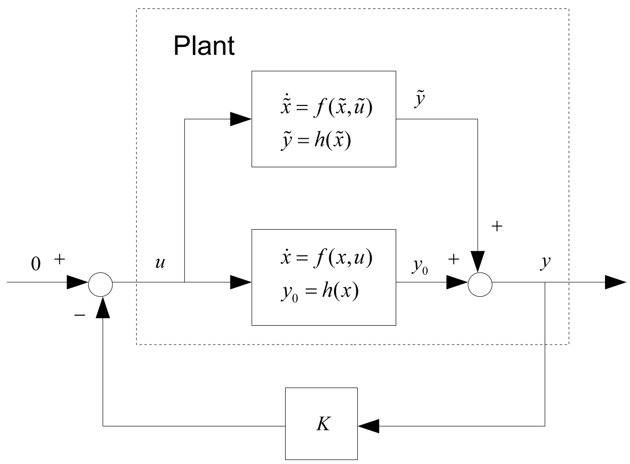


Fig. 5: Feed-forward Uncertainty

model uncertainty is a $\mathcal{L}_2$ stable dynamic system with finite $\mathcal{L}_2$ gain $\Gamma$ which is given by

$$\Sigma_\Delta : \begin{cases} \dot{\tilde{x}} = \tilde{f}(\tilde{x}, \tilde{u}) \\ \tilde{y} = \tilde{h}(\tilde{x}), \end{cases} \tag{54}$$

where $\tilde{x} \in \mathbb{R}^n$ is the state, $\tilde{u} \in \mathbb{R}^n$ is the control input and $\tilde{y} \in \mathbb{R}^n$ is the output. If the following conditions are satisfied

1) $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz continuous on compacts;
2) $h : \mathbb{R}^n \to \mathbb{R}^n$ is a static nonlinear function of $x$ which belongs to a sector $[\beta_o, \alpha_o]$ such that $\beta_o x^T x \leq x^T h(x) \leq \alpha_o x^T x$, where $\alpha_o \beta_o > 0$;
3) $\tilde{h} : \mathbb{R}^n \to \mathbb{R}^n$ is a static nonlinear function of $\tilde{x}$ which belongs to a sector $[\beta_\Delta, \alpha_\Delta]$ such that $\beta_\Delta \tilde{x}^T \tilde{x} \leq \tilde{x}^T \tilde{h}(\tilde{x}) \leq \alpha_\Delta \tilde{x}^T \tilde{x}$, where $\alpha_\Delta \beta_\Delta > 0$;

4) $\|\frac{\partial h(x)}{\partial x}\|_2 \leq \gamma_1$, where $0 < \gamma_1 < \infty$, and $\|\frac{\partial \tilde{h}(\tilde{x})}{\partial \tilde{x}}\|_2 \leq \gamma_2$, where $0 < \gamma_2 < \infty$;
5) $\tilde{\beta}\tilde{u}^T\tilde{u} \leq \tilde{u}^T\tilde{y} \leq \tilde{\alpha}\tilde{u}^T\tilde{u}$, where $-\infty < \tilde{\beta} \leq 0 \leq \tilde{\alpha} < \infty$(notice that if we know the $\mathcal{L}_2$ gain $\Gamma$ of $\Sigma_\Delta$, then we could simply choose $\tilde{\beta} = -\Gamma$ and $\tilde{\alpha} = \Gamma$);

then under the following scheduling strategy, the passive system under the control action $u(t) = -Ky(t_i)$ for $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}]$, where $K$ satisfies $K > 0$ and $K + \tilde{\beta}K^2 - \delta > 0(\delta > 0$ is a constant), is asymptotically stable:

- $t_0 = t_0 + \triangle_0$;
- $t_1 = t_0 + \tau_0$, $\tau_0 = \frac{1}{\gamma_1 L_1 \zeta_1} \ln\big(1 + \frac{\gamma_1 L_1 \zeta_1 \hat{\sigma}}{\gamma_1 L_1 \zeta_1 (1+\Gamma K) + \gamma_2 L_2 \zeta_2 \Gamma K}\big)$;
- $t_{i+1} = t_i + \triangle_i + \tau$, $i = 1, 2, \ldots$;

where $t_i$ is the time at which the sensor gets the measurement of the output $y(t_i)$, and $\zeta_1 = \max\{\frac{1}{|\alpha_o|}, \frac{1}{|\beta_o|}\}$, $\zeta_2 = \max\{\frac{1}{|\alpha_\Delta|}, \frac{1}{|\beta_\Delta|}\}$; $L_1$ is the Lipschitz constant of $f(x,u)$ and $L_2$ is the Lipschitz constant of $\tilde{f}(\tilde{x},\tilde{u})$; choose $\tilde{\sigma}, \hat{\sigma}, \hat{\sigma}'$ as constants such that $0 < \tilde{\sigma} < \hat{\sigma} < \hat{\sigma}' < \sigma$, where

$$\sigma = \frac{\sqrt{\frac{K+\tilde{\beta}K^2-\delta}{\frac{1}{4\delta}(K+2\tilde{\beta}K^2)^2-\tilde{\beta}K^2}}}{1+\sqrt{\frac{K+\tilde{\beta}K^2-\delta}{\frac{1}{4\delta}(K+2\tilde{\beta}K^2)^2-\tilde{\beta}K^2}}}, \tag{55}$$

and

$$\tau = \frac{1}{\gamma_1 L_1 \zeta_1} \ln\big(\frac{\gamma_1 L_1 \zeta_1 (1+\Gamma K) + \gamma_2 L_2 \zeta_2 \Gamma K + \gamma_1 L_1 \zeta_1 \hat{\sigma}}{\gamma_1 L_1 \zeta_1 (1+\Gamma K) + \gamma_2 L_2 \zeta_2 \Gamma K + \gamma_1 L_1 \zeta_1 \tilde{\sigma}}\big); \tag{56}$$

$\triangle_i$ is the estimated admissible execution delay of the actuator after the sensor gets the $i$th measurement of the system's output $y(t)$, and

$$\triangle_i = \min\big[\varepsilon_i^-, \varepsilon_i^+\big], \tag{57}$$

where

$$\varepsilon_i^- = \frac{1}{\gamma_1 L_1 \zeta_1} \ln\big(1 + \frac{\tilde{\sigma}\|y(t_i)\|_2}{\|y(t_i)\|_2 + \frac{K\Gamma(\gamma_1 L_1 \zeta_1 + \gamma_2 L_2 \zeta_2)\|y(t_{i-1})\|_2}{\gamma_1 L_1 \zeta_1}}\big) \tag{58}$$

and

$$\varepsilon_i^+ = \frac{1}{\gamma_1 L_1 \zeta_1} \ln\big(\frac{\gamma_1 L_1 \zeta_1 (1+\Gamma K) + \gamma_2 L_2 \zeta_2 \Gamma K + \gamma_1 L_1 \zeta_1 \hat{\sigma}'}{\gamma_1 L_1 \zeta_1 (1+\Gamma K) + \gamma_2 L_2 \zeta_2 \Gamma K + \gamma_1 L_1 \zeta_1 \hat{\sigma}}\big). \tag{59}$$

*Proof.* Since the model of the plant is passive, we have

$$\dot{V}(x) \leq u^T y_0, \tag{60}$$

since $y = y_0 + \tilde{y}$, we have $\dot{V}(x) \leq u^T(y - \tilde{y})$; since $u = -Ky(t_i) = -K(y - e)$ for $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}]$, we have

$$\begin{aligned}\dot{V} &\leq u^T(y-\tilde{y}) = u^T y - u^T\tilde{y} \leq u^T y - \tilde{\beta} u^T u \\ &= -K(y-e)^T y - \tilde{\beta}K^2(y-e)^T(y-e) \\ &= -Ky^T y + Ke^T y - \tilde{\beta}K^2(y-e)^T(y-e) \\ &= (-K - \tilde{\beta}K^2)y^T y + (K + 2\tilde{\beta}K^2)e^T y - \tilde{\beta}K^2 e^T e \\ &= -[\delta y^T y - (K+2\tilde{\beta}K^2)e^T y + \frac{1}{4\delta}(K+2\tilde{\beta}K^2)^2 e^T e] \\ &+ (-K - \tilde{\beta}K^2 + \delta)y^T y + [-\tilde{\beta}K^2 + \frac{1}{4\delta}(K+2\tilde{\beta}K^2)^2]e^T e\end{aligned} \tag{61}$$

where $\delta > 0$ is a constant. So if

$$\|e\|_2 \leq \sqrt{\frac{K + \tilde{\beta}K^2 - \delta}{\frac{1}{4\delta}(K + 2\tilde{\beta}K^2)^2 - \tilde{\beta}K^2}}\|y\|_2 \tag{62}$$

and if we choose $K > 0$ such that $K + \tilde{\beta}K^2 - \delta > 0$, then we can get $\dot{V}(x) \leq 0$. It can be shown that a sufficient condition for (61) to hold for any $t$ is given by

$$\|e\|_2 \leq \sigma\|y(t_i)\|_2, \;\; for\ t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}], \forall i \tag{63}$$

where

$$\sigma = \frac{\sqrt{\frac{K+\tilde{\beta}K^2-\delta}{\frac{1}{4\delta}(K+2\tilde{\beta}K^2)^2-\tilde{\beta}K^2}}}{1 + \sqrt{\frac{K+\tilde{\beta}K^2-\delta}{\frac{1}{4\delta}(K+2\tilde{\beta}K^2)^2-\tilde{\beta}K^2}}}. \tag{64}$$

If we consider non-zero execution delay of the actuator, the evolution of $\frac{\|e(t)\|_2}{\|y(t_i)\|_2}$ appears as the same as shown in Fig.3, where $e(t) = y(t) - y(t_i)$ for $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}]$ denotes the error of the measurement at the actuator. Again, let $\tilde{e}(t)$ denote the error of measurement at the sensor. For $t \in [t_i, t_i + \Delta_i]$, we have $\tilde{e}(t) = y(t) - y(t_i)$ and $e(t) = y(t) - y(t_{i-1})$, based on this we can obtain

$$\begin{aligned}\frac{d}{dt}\|\tilde{e}(t)\|_2 &\leq \|\dot{\tilde{e}}(t)\|_2 = \|\dot{y}(t)\|_2 = \|\dot{y}_0(t) + \dot{\tilde{y}}(t)\|_2 \leq \|\dot{y}_0(t)\|_2 + \|\dot{\tilde{y}}(t)\|_2 \\ &= \|\frac{\partial y_0}{\partial x}\dot{x}\|_2 + \|\frac{\partial \tilde{y}}{\partial \tilde{x}}\dot{\tilde{x}}\|_2 \leq \gamma_1\|\dot{x}\|_2 + \gamma_2\|\dot{\tilde{x}}\|_2 \\ &= \gamma_1\|f(x,u)\|_2 + \gamma_2\|\tilde{f}(\tilde{x},\tilde{u})\|_2 \\ &\leq \gamma_1 L_1\|x\|_2 + \gamma_2 L_2\|\tilde{x}\|_2,\end{aligned} \tag{65}$$

if assmption 2) and 3) are satisfied, then we have $\|x\|_2 \leq \zeta_1\|y_0\|_2$ and $\|\tilde{x}\|_2 \leq \zeta_2\|\tilde{y}\|_2$, where $\zeta_1 = \max\{\frac{1}{|\alpha_o|}, \frac{1}{|\beta_0|}\}$ and $\zeta_2 = \max\{\frac{1}{|\alpha_\Delta|}, \frac{1}{|\beta_\Delta|}\}$. Then we can get

$$\begin{aligned}
\frac{d}{dt}\|\tilde{e}(t)\|_2 &\leq \gamma_1 L_1\zeta_1\|y_0\|_2 + \gamma_2 L_2\zeta_2\|\tilde{y}\|_2 \\
&= \gamma_1 L_1\zeta_1\|y-\tilde{y}\|_2 + \gamma_2 L_2\zeta_2\|\tilde{y}\|_2 \\
&\leq \gamma_1 L_1\zeta_1\|y\|_2 + (\gamma_1 L_1\zeta_1 + \gamma_2 L_2\zeta_2)\|\tilde{y}\|_2 \\
&\leq \gamma_1 L_1\zeta_1\|\tilde{e}+y(t_i)\|_2 + (\gamma_1 L_1\zeta_1 + \gamma_2 L_2\zeta_2)\Gamma K\|y(t_{i-1})\|_2 \\
&\leq \gamma_1 L_1\zeta_1\|\tilde{e}\|_2 + \gamma_1 L_1\zeta_1\|y(t_i)\|_2 + (\gamma_1 L_1\zeta_1 + \gamma_2 L_2\zeta_2)\Gamma K\|y(t_{i-1})\|_2.
\end{aligned} \tag{66}$$

So the evolution of $\|\tilde{e}\|_2$ during the time $[t_i, t_i+\Delta_i]$ is bounded by the solution of

$$\dot{\phi}(t) = \gamma_1 L_1\zeta_1\phi(t) + \gamma_1 L_1\zeta_1\|y(t_i)\|_2 + (\gamma_1 L_1\zeta_1 + \gamma_2 L_2\zeta_2)\Gamma K\|y(t_{i-1})\|_2, \tag{67}$$

with $\phi(t_i) = \|y(t_i) - y(t_i)\|_2 = 0$, the solution to (66) is given by

$$\phi(t) = \big(\|y(t_i)\|_2 + \frac{(\gamma_1 L_1\zeta_1 + \gamma_2 L_2\zeta_2)K\|y(t_{i-1})\|_2}{\gamma_1 L_1\zeta_1}\big)\big(e^{\gamma_1 L_1\zeta_1(t-t_i)} - 1\big). \tag{68}$$

Assume that at $t = t_i + \Delta_i$, we have $\phi(t_i+\Delta_i) = \tilde{\sigma}\|y(t_i)\|_2$, where $0 < \tilde{\sigma} < \sigma$, then we could get an estimate of $\Delta_i$ based on (67), if we denote it by $\varepsilon_i^-$, then $\varepsilon_i^-$ is given by

$$\varepsilon_i^- = \frac{1}{\gamma L_1\zeta_1}\ln\big(1 + \frac{\tilde{\sigma}\|y(t_i)\|_2}{\|y(t_i)\|_2 + \frac{(\gamma_1 L_1\zeta_1+\gamma_2 L_2\zeta_2)\Gamma K\|y(t_{i-1})\|_2}{\gamma_1 L_1\zeta_1}}\big). \tag{69}$$

Notice that $\varepsilon_i^- > 0$ for any $\tilde{\sigma} > 0$ and $\|y(t_i)\|_2$ cannot goes to zero in a short time because by applying output feedback to the passive system model, the output $y(t)$ goes to zero asymptotically.

Assume that the actuator updates the control action at $t = t_i + \Delta_i$, for $t \in [t_i+\Delta_i, t_{i+1}+\Delta_{i+1}]$, we have $e(t) = y(t) - y(t_i)$, and

$$\begin{aligned}
\frac{d}{dt}\|e(t)\|_2 &\leq \|\dot{e}(t)\|_2 = \|\dot{y}(t)\|_2 = \|\dot{y}_0(t) + \dot{\tilde{y}}(t)\|_2 \leq \|\dot{y}_0(t)\|_2 + \|\dot{\tilde{y}}(t)\|_2 \\
&\leq \gamma_1\|\dot{x}\|_2 + \gamma_2\|\dot{\tilde{x}}\|_2 \leq \gamma_1 L_1\|x\|_2 + \gamma_2 L_2\|\tilde{x}\|_2 \\
&\leq \gamma_1 L_1\zeta_1\|y_0\|_2 + \gamma_2 L_2\zeta_2\|\tilde{y}\|_2 = \gamma_1 L_1\zeta_1\|y-\tilde{y}\|_2 + \gamma_2 L_2\zeta_2\|\tilde{y}\|_2 \\
&\leq \gamma_1 L_1\zeta_1\|y\|_2 + (\gamma_1 L_1\zeta_1 + \gamma_2 L_2\zeta_2)\|\tilde{y}\|_2 \\
&\leq \gamma_1 L_1\zeta_1\|e+y(t_i)\|_2 + \gamma_2 L_2\zeta_2)\Gamma K\|y(t_i)\|_2 \\
&\leq \gamma_1 L_1\zeta_1\|e\|_2 + [\gamma_1 L_1\zeta_1(1+\Gamma K) + \gamma_2 L_2\zeta_2\Gamma K]\|y(t_i)\|_2,
\end{aligned} \tag{70}$$

so the evolution of $\|e(t)\|_2$ for $t \in [t_i+\Delta_i, t_{i+1}+\Delta_{i+1}]$ is bounded by the solution of

$$\dot{\phi}(t) = \gamma_1 L_1\zeta_1\phi(t) + [\gamma_1 L_1\zeta_1(1+\Gamma K) + \gamma_2 L_2\zeta_2\Gamma, K]\|y(t_i)\|_2 \tag{71}$$

with $\phi(t_i+\Delta_i)=\tilde{\sigma}\|y(t_i)\|_2$, the solution to (70) is given by

$$\phi(t)=\frac{\gamma_1 L_1\zeta_1(1+\Gamma K)+\gamma_2 L_2\zeta_2\Gamma K}{\gamma_1 L_1\zeta_1}\|y(t_i)\|_2\big(e^{\gamma_1 L_1\zeta_1 t}-1\big)+\tilde{\sigma}\|y(t_i)\|_2 e^{\gamma_1 L_1\zeta_1 t}. \tag{72}$$

Choose $0<\tilde{\sigma}<\hat{\sigma}<\hat{\sigma}'<\sigma$, then we can get an estimate of the time for $\frac{\|e\|_2}{\|y(t_i)\|_2}$ to evolve from $\tilde{\sigma}$ to $\hat{\sigma}$ and from $\hat{\sigma}$ to $\hat{\sigma}'$, if we denote them by $\tau$ and $\varepsilon_i^+$, then we can obtain

$$\tau=\frac{1}{\gamma_1 L_1\zeta_1}\ln\big(\frac{\gamma_1 L_1\zeta_1(1+\Gamma K)+\gamma_2 L_2\zeta_2\Gamma K+\gamma_1 L_1\zeta_1\hat{\sigma}}{\gamma_1 L_1\zeta_1(1+\Gamma K)+\gamma_2 L_2\zeta_2\Gamma K+\gamma_1 L_1\zeta_1\tilde{\sigma}}\big), \tag{73}$$

and

$$\varepsilon_i^+=\frac{1}{\gamma_1 L_1\zeta_1}\ln\big(\frac{\gamma_1 L_1\zeta_1(1+\Gamma K)+\gamma_2 L_2\zeta_2\Gamma K+\gamma_1 L_1\zeta_1\hat{\sigma}'}{\gamma_1 L_1\zeta_1(1+\Gamma K)+\gamma_2 L_2\zeta_2\Gamma K+\gamma_1 L_1\zeta_1\hat{\sigma}}\big), \tag{74}$$

and notice that for any $0<\tilde{\sigma}<\hat{\sigma}<\hat{\sigma}'$, we have $\tau>0$ and $\varepsilon_i^+>0$.

Now, we could give a lower bound of $\Delta_i$, where $\Delta_i$ is the execution delay of the actuator after the sensor gets the measurement $y(t_i)$:

$$\Delta_i=\min[\varepsilon_i^-,\varepsilon_i^+], \tag{75}$$

and the corresponding estimate of the time for the sensor to get the next new measurement is given by

$$t_{i+1}=t_i+\Delta_i+\tau. \tag{76}$$

Since we choose $0<\tilde{\sigma}<\hat{\sigma}<\hat{\sigma}'<\sigma$, then by applying the proposed scheduling strategy in Theorem 2, we can guarantee that $\dot{V}(x)\le 0,\forall t$. Then one can prove that the closed-loop system is asymptotically stable since we assume that $\Sigma_o$ is passive and ZSD, $\Sigma_\Delta$ is $\mathcal{L}_2$ stable. ■

**Remark 4**: For the linear system case, consider the model of the plant given by

$$\Sigma_o:\begin{cases}\dot{x}=Ax+Bu\\ y=Cx,\end{cases} \tag{77}$$

and the feed-forward uncertainty given by

$$\Sigma_\Delta:\begin{cases}\dot{\tilde{x}}=\tilde{A}\tilde{x}+\tilde{B}\tilde{u}\\ \tilde{y}=\tilde{C}\tilde{x},\end{cases} \tag{78}$$

the closed-loop system is shown in Fig.6. We assume that $\Sigma_o$ is passive, and $\Sigma_\Delta$ is a $\mathcal{L}_2$ stable system with finite $\mathcal{L}_2$ gain $\Gamma$. For $t\in[t_i,t_i+\Delta_i]$, we have $e(t)=y(t)-y(t_{i-1})$ and $\tilde{e}(t)=y(t)-y(t_i)$, and we can obtain

$$\begin{aligned}\frac{d}{dt}\|\tilde{e}(t)\|_2&\le\|\dot{\tilde{e}}(t)\|_2=\|\dot{y}(t)\|_2\le\|\dot{\tilde{y}}(t)\|_2+\|\dot{y}_0(t)\|_2\\&=\|\tilde{C}\tilde{A}\tilde{x}+\tilde{C}\tilde{B}u\|_2+\|CAx+CBu\|_2,\end{aligned} \tag{79}$$

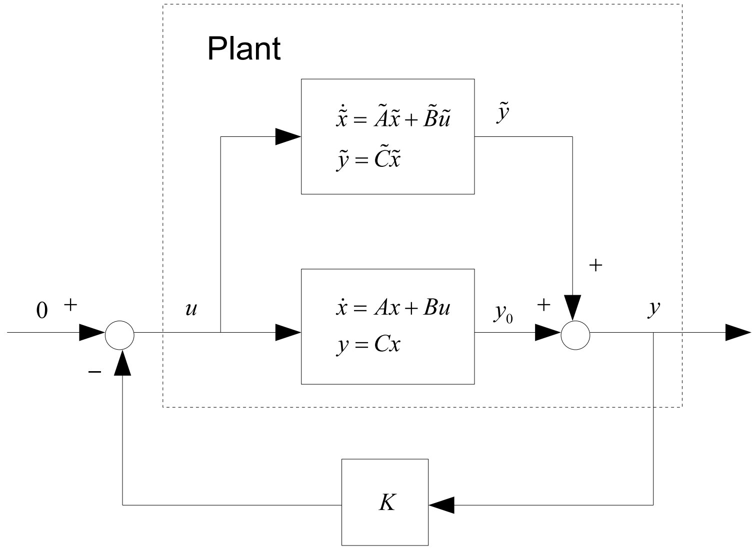


Fig. 6: Feed-forward Uncertainty

if $\frac{\|CAx\|_2}{\|y_0\|_2} = \frac{(x^T A^T C^T CAx)^{\frac{1}{2}}}{(x^T C^T Cx)^{\frac{1}{2}}} \leq \zeta_1$ and $\frac{\|\tilde{C}\tilde{A}\tilde{x}\|_2}{\|\tilde{y}\|_2} = \frac{(\tilde{x}^T \tilde{A}^T \tilde{C}^T \tilde{C}\tilde{A}\tilde{x})^{\frac{1}{2}}}{(\tilde{x}^T \tilde{C}^T \tilde{C}\tilde{x})^{\frac{1}{2}}} \leq \zeta_2$, then we have

$$\begin{aligned}
\frac{d}{dt}\|\tilde{e}(t)\|_2 &\leq \zeta_2\|\tilde{y}\|_2 + \|\tilde{C}\tilde{B}u\|_2 + \zeta_1\|y_0\|_2 + \|CBu\|_2 \\
&\leq \zeta_2\Gamma\|u\|_2 + \|\tilde{C}\tilde{B}u\|_2 + \zeta_1\|y - \tilde{y}\|_2 + \|CBu\|_2 \qquad (80)\\
&\leq \zeta_1\|\tilde{e}\|_2 + K\big(\zeta_1\Gamma + \zeta_2\Gamma + \|\tilde{C}\tilde{B}\|_2 + \|CB\|_2\big)\|y(t_{i-1})\|_2 + \zeta_1\|y(t_i)\|_2.
\end{aligned}$$

For $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}]$, we have $e(t) = y(t) - y(t_i)$, and we can verify that

$$\frac{d}{dt}\|e(t)\|_2 \leq \zeta_1\|\tilde{e}\|_2 + \big[K(\zeta_1\Gamma + \zeta_2\Gamma + \|\tilde{C}\tilde{B}\|_2 + \|CB\|_2) + \zeta_1\big]\|y(t_i)\|_2. \qquad (81)$$

So in this case, we have

$$\varepsilon_i^- = \frac{1}{\zeta_1}\ln\big[1 + \frac{\zeta_1\tilde{\sigma}\|y(t_i)\|_2}{K\big(\zeta_1\Gamma + \zeta_2\Gamma + \|\tilde{C}\tilde{B}\|_2 + \|CB\|_2\big)\|y(t_{i-1})\|_2 + \zeta_1\|y(t_i)\|_2}\big], \qquad (82)$$

$$\varepsilon_i^+ = \frac{1}{\zeta_1}\ln\big[\frac{K(\zeta_1\Gamma + \zeta_2\Gamma + \|\tilde{C}\tilde{B}\|_2 + \|CB\|_2) + \zeta_1 + \zeta_1\hat{\sigma}'}{K(\zeta_1\Gamma + \zeta_2\Gamma + \|\tilde{C}\tilde{B}\|_2 + \|CB\|_2) + \zeta_1 + \zeta_1\hat{\sigma}}\big], \qquad (83)$$

and

$$\tau = \frac{1}{\zeta_1}\ln\big[\frac{K(\zeta_1\Gamma + \zeta_2\Gamma + \|\tilde{C}\tilde{B}\|_2 + \|CB\|_2) + \zeta_1 + \zeta_1\hat{\sigma}}{K(\zeta_1\Gamma + \zeta_2\Gamma + \|\tilde{C}\tilde{B}\|_2 + \|CB\|_2) + \zeta_1 + \zeta_1\tilde{\sigma}}\big]. \qquad (84)$$

(85)

**Remark 5**: Consider the case when the model of the plant is not passive, but it satisfies the following dissipative inequality given by

$$\dot{V}(x) \leq u^T y_0 + \rho y_0^T y_0 \tag{86}$$

where $\rho > 0$ is the smallest value such that the above dissipative inequality holds. Then one can show that for $t \in [t_i + \Delta_i, t_{i+1} + \Delta_{i+1}]$, we have

$$\begin{aligned}
\dot{V}(x) &\leq u^T y_0 + \rho y_0^T y_0 \\
&= -K(y-e)^T y_0 + \rho(y-\tilde{y})^T(y-\tilde{y}) \\
&= -(K-\rho)y^T y + (K-2\rho)y^T\tilde{y} + \rho\tilde{y}^T\tilde{y} + Ke^T y - Ke^T\tilde{y} \\
&\leq -(K-\rho)\|y\|_2^2 + |K-2\rho|\|y\|_2\|\tilde{y}\|_2 + \rho\|\tilde{y}\|_2^2 + K\|e\|_2\|y\|_2 + K\|e\|_2\|\tilde{y}\|_2 \\
&\leq (-K+\rho+K|K-2\rho|\Gamma+\rho K^2\Gamma^2)\|y\|_2^2 \\
&+ (K^2\Gamma+\rho\Gamma^2K^2)\|e\|_2^2 + (K|K-2\rho|\Gamma+K+\Gamma K^2)\|e\|_2\|y\|_2 \\
&= -\big[\delta\|y\|_2^2 - (K|K-2\rho|\Gamma+K+\Gamma K^2)\|e\|_2\|y\|_2 + \frac{1}{4\delta}(K|K-2\rho|\Gamma+K+\Gamma K^2)^2\|e\|_2^2\big] \\
&+ (-K+\rho+K|K-2\rho|\Gamma+\rho K^2\Gamma^2+\delta)\|y\|_2^2 \\
&+ [K^2\Gamma+\rho\Gamma^2K^2+\frac{1}{4\delta}(K|K-2\rho|\Gamma+K+\Gamma K^2)^2]\|e\|_2^2
\end{aligned} \tag{87}$$

where $\delta > 0$ is a constant, and $\Gamma$ is the finite $\mathcal{L}_2$ gain of the uncertainty. One can see that not for any finite gain $\Gamma$ we can find a $K > 0$ and $\delta > 0$ such that

$$(K-\rho-K|K-2\rho|\Gamma-\rho K^2\Gamma^2-\delta)\|y\|_2^2 \geq [K^2\Gamma+\rho\Gamma^2K^2+\frac{1}{4\delta}(K|K-2\rho|\Gamma+K+\Gamma K^2)^2]\|e\|_2^2 \tag{88}$$

to guarantee that $\dot{V}(x) \leq 0$. But for the case when we can find such $K$ and $\delta$ which satisfy (87), then let

$$C_1 = K-\rho-K|K-2\rho|\Gamma-\rho K^2\Gamma^2-\delta, \tag{89}$$

and

$$C_2 = \frac{1}{4\delta}(K|K-2\rho|\Gamma+K+\Gamma K^2)^2, \tag{90}$$

the corresponding triggering signal $\sigma$ in Theorem 2 is then given by $\sigma = \frac{\sqrt{\frac{C_1}{C_2}}}{1+\sqrt{\frac{C_1}{C_2}}}$, and we can develop similar self-triggered scheduling strategy as discussed in the proof of Theorem 2 to stabilize the closed-loop system.

**Remark 6**: Assumption 2) and 3) in Theorem 2 are also very conservative. Most of the time, these two assumptions can be relaxed as long as we can get $\|\dot{y}_0\|_2 \leq p_0\|y_0\|_2$ and $\|\dot{\tilde{y}}\|_2 \leq \tilde{p}\|\tilde{y}\|_2$, for some constant $p_0 > 0$ and $\tilde{p} > 0$, then similar self-triggered scheduling strategy can be obtained. Also in this case, $y_0$ and $\tilde{y}$ do not have to have the same dimension as $x$ and $\tilde{x}$.

## VI. Example

**Example 1.** Consider the model of the plant which is a linear passive system given by

$$\Sigma_o : \begin{cases} \dot{x}_1(t) = -5x_1(t) - x_2(t) \\ \dot{x}_2(t) = -x_2(t) + u(t) \\ y(t) = x_2(t), \end{cases} \tag{91}$$

assume its feedback uncertainty is given by

$$\Sigma_\Delta : \begin{cases} \dot{\tilde{x}}_1(t) = \tilde{x}_2(t) \\ \dot{\tilde{x}}_2(t) = -a\tilde{x}_1^3(t) - b\tilde{x}_2(t) + \tilde{u}(t), \ (a > 0, b > 0) \\ \tilde{y}(t) = \tilde{x}_2(t). \end{cases} \tag{92}$$

If we choose $V(x) = \frac{1}{2}x_2^2$ for the system $\Sigma_o$, then we have

$$\dot{V}(x) = \dot{x}_2 x_2 = (-x_2 + u)x_2 = uy - y^2, \tag{93}$$

so $\Sigma_o$ is passive, also notice that it is ZSD. If we choose $\tilde{V}(\tilde{x}) = \frac{1}{4}a\tilde{x}_1^4 + \frac{1}{2}\tilde{x}_2^2$, we have

$$\dot{\tilde{V}}(\tilde{x}) = -b\tilde{y}^2 + \tilde{u}\tilde{y}, \tag{94}$$

One could verify that the finite $\mathcal{L}_2$ gain of $\Sigma_\Delta$ is $\frac{1}{b}$. In this case, we can get

$$\varepsilon_i^- = \frac{1}{1+\frac{1}{b}} \ln\Big[\frac{(1+\frac{1}{b})\tilde{\sigma}\|y(t_i)\|_2}{(1+\frac{1}{b})\|y(t_i)\|_2 + K\|y(t_{i-1})\|_2} + 1\Big], \tag{95}$$

$$\varepsilon_i^+ = \frac{1}{1+\frac{1}{b}} \ln\Big[\frac{(1+\frac{1}{b})\hat{\sigma}' + 1 + \frac{1}{b} + K}{(1+\frac{1}{b})\hat{\sigma} + 1 + \frac{1}{b} + K}\Big], \tag{96}$$

and

$$\tau = \frac{1}{1+\frac{1}{b}} \ln\Big[\frac{(1+\frac{1}{b})\hat{\sigma} + 1 + \frac{1}{b} + K}{(1+\frac{1}{b})\tilde{\sigma} + 1 + \frac{1}{b} + K}\Big]. \tag{97}$$

According to Theorem 1, we need to choose $K$ such that $K > \frac{1}{b} - 1$ and $K > 0$. The simulation result when $a = 3$, $b = 2$, $K =$, $\tilde{\sigma} = 0.05$, $\hat{\sigma} = 0.4714$ and $\hat{\sigma}' = 0.5214$ are shown in Fig.7-Fig.10, where the solid line shows the simulation results when we measure the output of the plant every $0.001s$ and update the control action without delay, the dashed line shows the simulation results of the self-triggered scheduling strategy. Minimum admissible delay $\triangle$ obtained from the self-triggered scheduling simulation is $0.0134s$, the corresponding inter-execution time $\tau + \triangle$ is $0.1246 + 0.0134 = 0.1380s$. The sensor gets 31 measurements of system's output in $4s$ and the actuator updates the control action 30 times in $4s$.

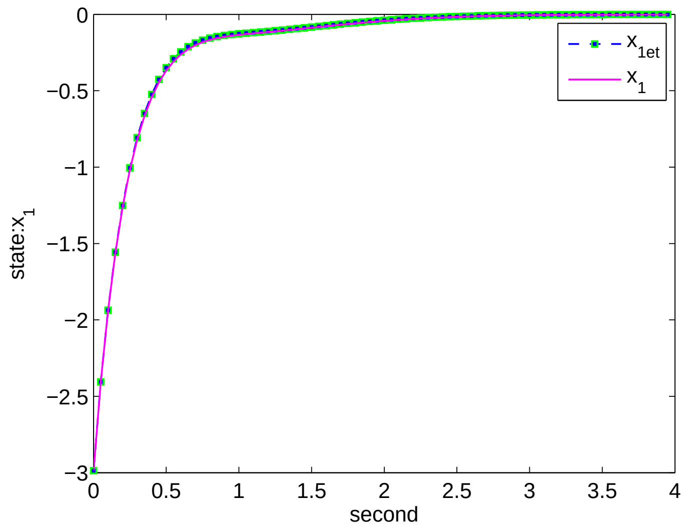


Fig. 7: evolution of the states

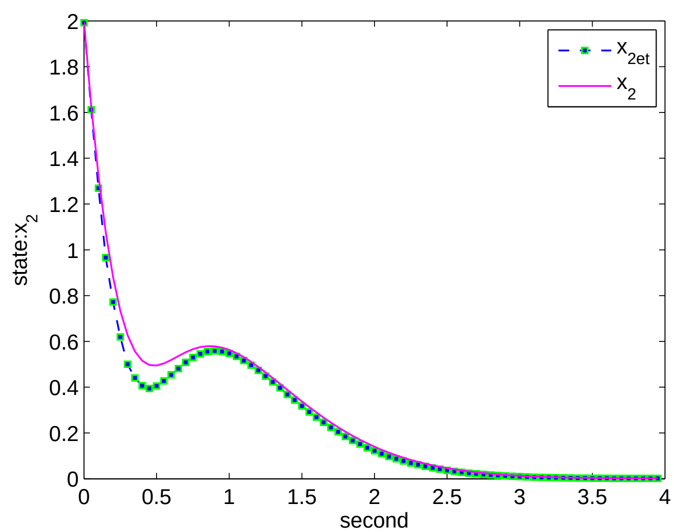


Fig. 8: evolution of the states

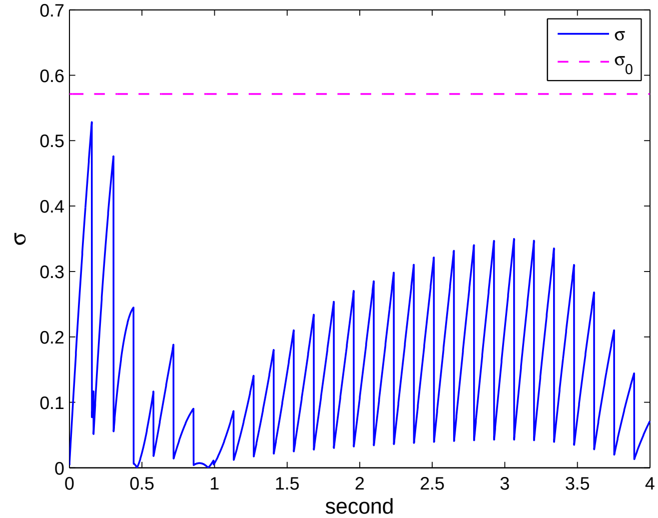


Fig. 9: evolution of $\frac{\|e(t)\|_2}{\|y(t_i)\|_2}$

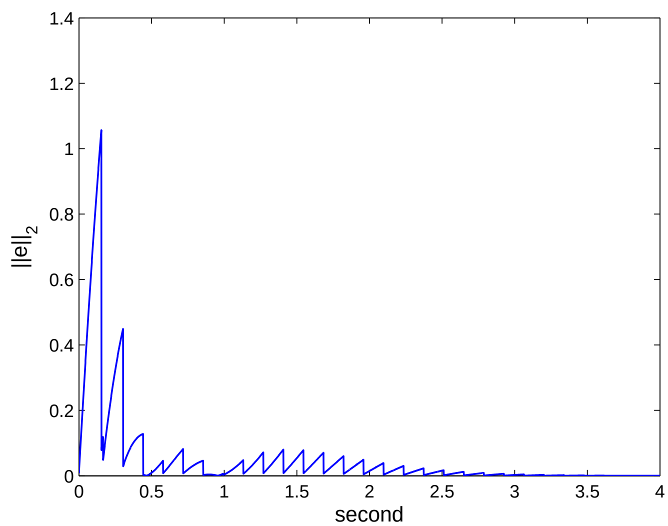


Fig. 10: evolution of $\|e(t)\|_2$

**Example 2.** Consider the model of the plant which is given by

$$\Sigma_o : \begin{cases} \dot{x}_1(t) = -3x_1^3(t) + x_1(t)x_2(t) \\ \dot{x}_2(t) = 3x_2(t) + 2u(t) \\ \ y(t) = x_2(t), \end{cases} \tag{98}$$

assume its feedback uncertainty is given by

$$\Sigma_\Delta : \begin{cases} \dot{\tilde{x}}_1(t) = \tilde{x}_2(t) \\ \dot{\tilde{x}}_2(t) = -a\tilde{x}_1^3(t) - b\tilde{x}_2(t) + \tilde{u}(t), \ (a > 0, b > 0) \\ \ \tilde{y}(t) = \tilde{x}_2(t). \end{cases} \tag{99}$$

We can see that $\Sigma_o$ is ZSD but unstable. If we choose the storage function $V(x) = \frac{1}{4}x_2^2$ for $\Sigma_o$, we can get

$$\dot{V}(x) = uy + 1.5y^2. \tag{100}$$

From example 1, we know that if we choose the storage function $\tilde{V}(\tilde{x}) = \frac{1}{4}a\tilde{x}_1^4 + \frac{1}{2}\tilde{x}_2^2$ for $\Sigma_\Delta$, then we could verify that $\Sigma_\Delta$ has finite $\mathcal{L}_2$ gain $\frac{1}{b}$. Based on Remark 2 of Theorem 1, we need to choose $K > 1.5 + \frac{1}{b}$, and $0 < \tilde{\sigma} < \hat{\sigma} < \hat{\sigma}' < \frac{K-1.5-\frac{1}{b}}{2K-1.5-\frac{1}{b}}$. In this case, we can obtain

$$\varepsilon_i^- = \frac{1}{3+\frac{2}{b}} \ln\big[\frac{(3+\frac{2}{b})\tilde{\sigma}\|y(t_i)\|_2}{(3+\frac{2}{b})\|y(t_i)\|_2 + 2K\|y(t_{i-1})\|_2} + 1\big], \tag{101}$$

$$\varepsilon_i^+ = \frac{1}{3+\frac{2}{b}} \ln\big[\frac{(3+\frac{2}{b})\hat{\sigma}' + 3 + \frac{2}{b} + 2K}{(3+\frac{2}{b})\hat{\sigma} + 3 + \frac{2}{b} + 2K}\big], \tag{102}$$

and

$$\tau = \frac{1}{3+\frac{2}{b}} \ln\big[\frac{(3+\frac{2}{b})\hat{\sigma} + 3 + \frac{2}{b} + 2K}{(3+\frac{2}{b})\tilde{\sigma} + 3 + \frac{2}{b} + 2K}\big]. \tag{103}$$

The simulation result when $a = 3$, $b = 2$, $K = 7$, $\tilde{\sigma} = 0.05$, $\hat{\sigma} = 0.3167$ and $\hat{\sigma}' = 0.3667$ are shown in Fig.11-Fig.14, where the solid line shows the simulation results when we measure the output of the plant every $0.001s$ and update the control action without delay, the dashed line shows the simulation results of the self-triggered scheduling strategy. Minimum admissible delay $\triangle$ obtained from the self-triggered scheduling simulation is $0.0026s$, the corresponding inter-execution time $\tau + \triangle$ is $0.0142 + 0.0026 = 0.0168s$. The sensor gets 62 measurements of system's output in $1s$ and the actuator updates the control action 61 times in $1s$.

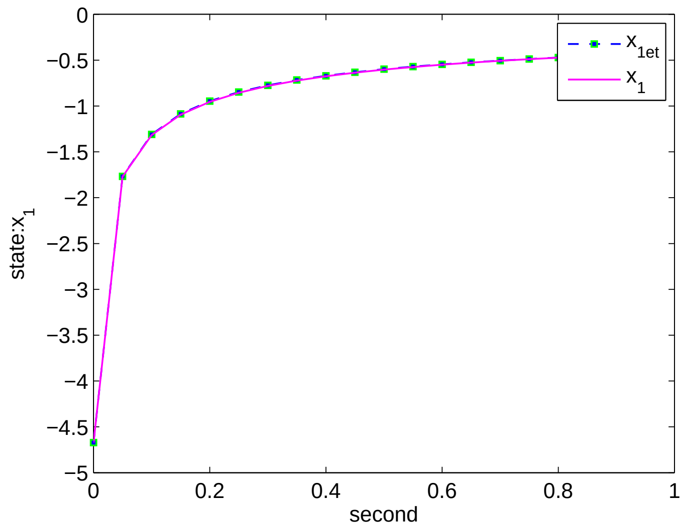


Fig. 11: evolution of the states

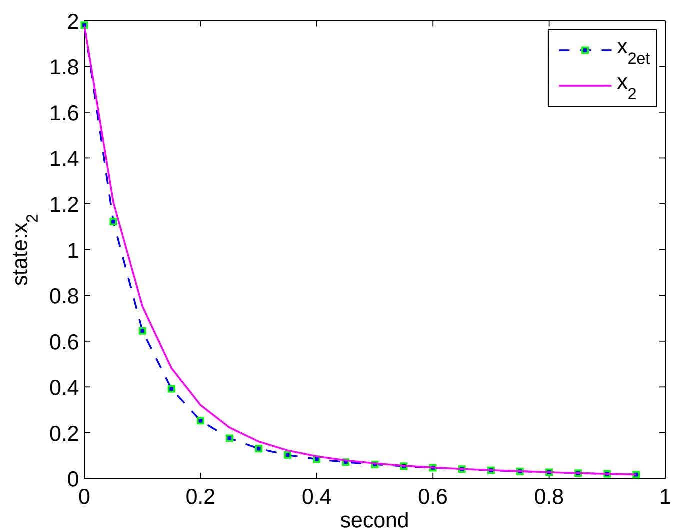


Fig. 12: evolution of the states

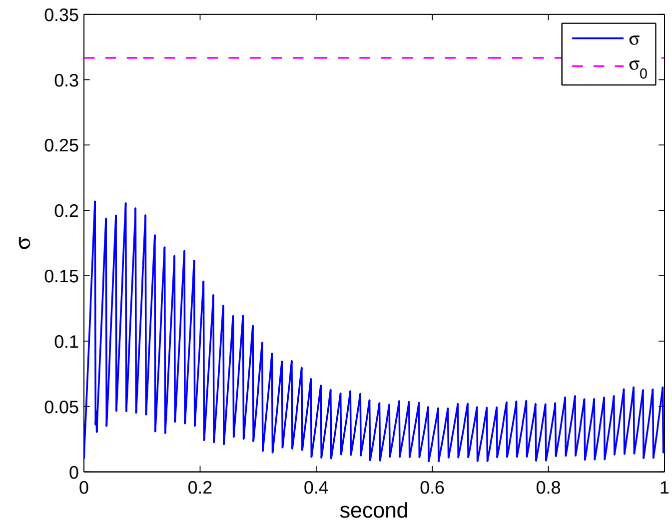


Fig. 13: evolution of $\frac{\|e(t)\|_2}{\|y(t_i)\|_2}$

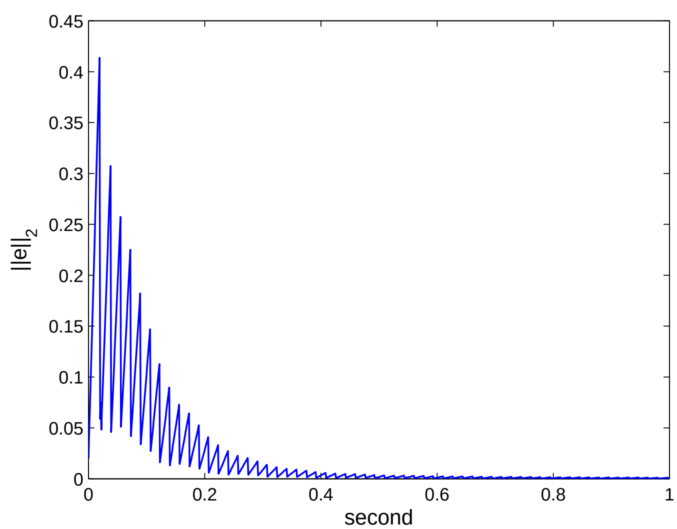


Fig. 14: evolution of $\|e(t)\|_2$

**Example 3.** Consider the model of the plant which is given by

$$\Sigma_o : \begin{cases} \dot{x}_1(t) = -3x_1^3(t) + x_1(t)x_2(t) \\ \dot{x}_2(t) = 3x_2(t) + 2u(t) \\ \ y(t) = x_2(t), \end{cases} \tag{104}$$

assume its feed-forward uncertainty is given by

$$\Sigma_\Delta : \begin{cases} \dot{\tilde{x}}_1(t) = -3\tilde{x}_1(t) - \tilde{x}_2(t) \\ \dot{\tilde{x}}_2(t) = -b\tilde{x}_2(t) + u(t), \ (a > 0, b > 0) \\ \ \tilde{y}(t) = \tilde{x}_2(t). \end{cases} \tag{105}$$

If we choose the storage function $V(x) = \frac{1}{4}x_2^2$ for $\Sigma_o$, we can get

$$\dot{V}(x) = uy + 1.5y^2. \tag{106}$$

also, we can see that $\Sigma_o$ is ZSD. By choosing $\tilde{V}(\tilde{x}) = \frac{1}{2}\tilde{x}^2$, we can verify that $\Sigma_\Delta$ has finite $\mathcal{L}_2$ gain $\frac{1}{b}$. According to Remark 5 of Theorem 3, if $b = 10$, then we could choose $K = 3$, $\delta = 0.5$ such that (88) is satisfied and the corresponding $\sigma$ is 0.2404. In this case, we can obtain

$$\varepsilon_i^- = \frac{1}{8K + \frac{6K}{b} + 3} \ln\big[\frac{(3 + \frac{6K}{b} + 8K)\tilde{\sigma}\|y(t_i)\|_2}{(4K + 3 + \frac{3K}{b})\|y(t_i)\|_2 + (4K + \frac{3K}{b})\|y(t_i) - y(t_{i-1})\|_2} + 1\big], \tag{107}$$

$$\varepsilon_i^+ = \frac{1}{8K + \frac{6K}{b} + 3} \ln\big[\frac{4K + 3 + \frac{3K}{b} + (8K + \frac{6K}{b} + 3)\hat{\sigma}'}{4K + 3 + \frac{3K}{b} + (8K + \frac{6K}{b} + 3)\hat{\sigma}}\big], \tag{108}$$

and

$$\tau = \frac{1}{8K + \frac{6K}{b} + 3} \ln\big[\frac{4K + 3 + \frac{3K}{b} + (8K + \frac{6K}{b} + 3)\hat{\sigma}}{4K + 3 + \frac{3K}{b} + (8K + \frac{6K}{b} + 3)\tilde{\sigma}}\big]. \tag{109}$$

The simulation result when $b = 10$, $K = 3$, $\tilde{\sigma} = 0.05$, $\hat{\sigma} = 0.2324$ and $\hat{\sigma}' = 0.2824$ are shown in Fig.15-Fig.18, where the solid line shows the simulation results when we measure the output of the plant every $0.001s$ and update the control action without delay, the dashed line shows the simulation results of the self-triggered scheduling strategy. Minimum admissible delay $\triangle$ obtained from the self-triggered scheduling simulation is $0.0021s$, the corresponding inter-execution time $\tau + \triangle$ is $0.0092 + 0.0021 = 0.0113s$. The sensor gets 267 measurements of system's output in $3s$ and the actuator updates the control action 266 times in $3s$.

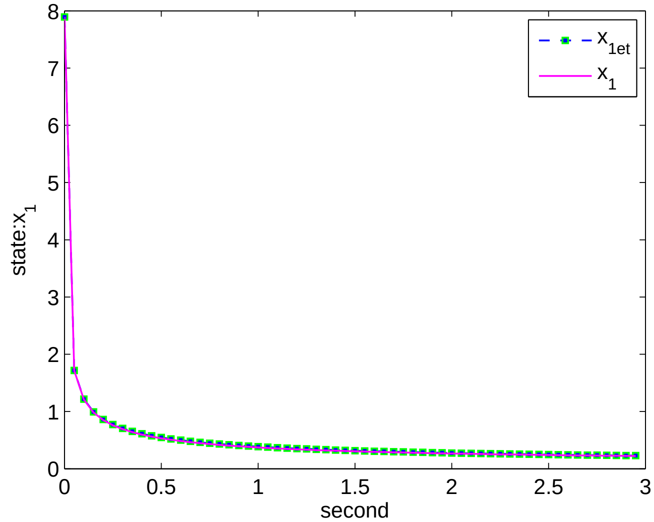


Fig. 15: evolution of the states

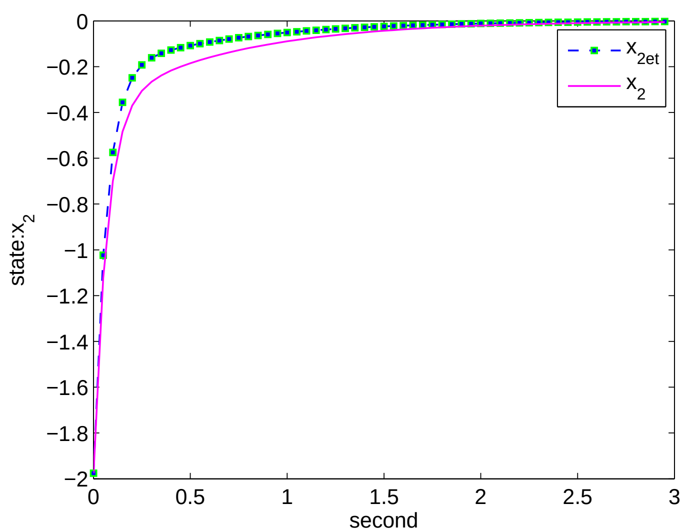


Fig. 16: evolution of the states

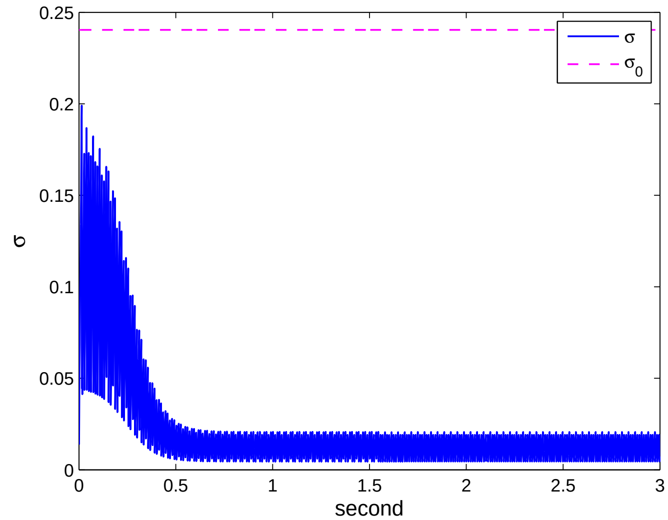


Fig. 17: evolution of $\frac{\|e(t)\|_2}{\|y(t_i)\|_2}$

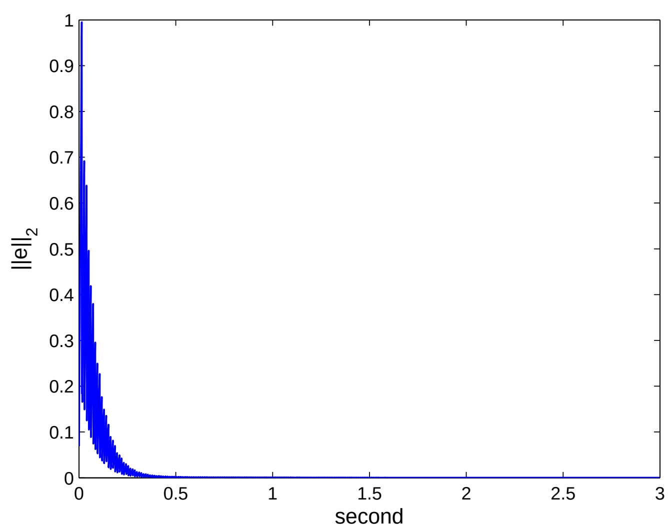


Fig. 18: evolution of $\|e(t)\|_2$

## VII. CONCLUSION

In this report, we propose a robust self-triggered real-time scheduling strategy for stabilization of passive/dissipative systems. We assume that the model of the plant is a passive or a disspative system, and we assume that the structure uncertainty is a $\mathcal{L}_2$ stable dynamic system in a feedback/feedforward interconnection with the model of the plant. We derived the self-triggered real-time scheduling strategies for both cases and we have also shown that the inter-execution time under the proposed scheduling strategy is non-trivial. Simulation results are also provided.

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