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A Necessary and Sufficient Condition for Robust Asymptotic Stabilizability of Continuous-Time Uncertain Switched Linear Systems

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***Abstract*— The main contribution of this paper is a necessary and sufficient rank condition derived for the robust asymptotic stabilizability of switched linear systems with time-variant parametric uncertainties, thus improving the sufficient only conditions found in the literature. The method is based on polyhedral Lyapunov-like functions for each unstable subsystems, which represent generalizations of the polytopic Lyapunov functions in the classical sense. Under certain rank condition, which is related to these Lyapunov-like functions, the stabilizing switching law is constructed and a global Lyapunov function is composed, which guarantees asymptotic stability for the closed-loop switched linear system. The rank condition is also proved to be necessary for the existence of an asymptotically stabilizing switching control law for the switched linear systems.**

I. INTRODUCTION

A switched system is a dynamical system that consists of a finite number of subsystems described by differential or difference equations and a logical rule that orchestrates switching between these subsystems. Properties of this type of model have been studied for the past fifty years to consider engineering systems that contain relays and/or hysteresis. Recently, there has been increasing interest in the stability analysis and switching control design of switched systems, see for example [17], [11], [4], [16], [18] and the references cited therein. The motivation for studying such switched systems comes partly from the realization that there exists a large class of nonlinear systems which can be stabilized by switching control schemes, but cannot be stabilized by any continuous static state feedback control law [9]. In addition, switched systems and switched multi-controller systems have numerous applications in control of mechanical systems, process control, automotive industry, power systems, aircraft and traffic control, and many other fields. Switched systems with all subsystems described by linear differential or difference equations are called piecewise linear/ affine systems or switched linear systems, and have gained the most attention [16], [5], [4], [2]. Recent efforts in switched linear system research typically focus on the analysis of the dynamic behaviors, like stability [16], [17], [11], controllability and observability [5] etc., and aim to design controllers with guaranteed stability and performance [4], [16], [2].

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The stability issues of switched systems have been of increasing interest in the recent decade, see for example the survey papers [17], [11], the recent book [18] and the references cited therein. The stability study of switched systems can be roughly divided into two kinds of problems. One is the stability analysis of switched systems under given switching signals (maybe arbitrary, slow switching etc.), and the other is the synthesis of stabilizing switching signals for a given collection of dynamical systems.

For the stability analysis problem, the first question is whether the switched system remains stable when there is no restriction (or no *a priori* knowledge) on the switching signals. This problem is usually called stability analysis under arbitrary switching, and is typically being dealt with by constructing a common Lyapunov function. For example, various attempts has been made [24], [30], [19], [20] to find a common quadratic Lyapunov function for the family of systems, ensuring the asymptotic stability of switched systems for any switching signal. In [19] and [1], Lie algebra conditions were given, which imply the existence of a common quadratic Lyapunov function. It is worth pointing out that a converse Lyapunov theorem was derived in [10] for the globally asymptotic stability of arbitrary switching systems. This converse Lyapunov theorem justifies the common Lyapunov method which was pursued in the literature. However, most of the work was restricted to the case of quadratic Lyapunov function, which only gave sufficient stability test criteria. Usually, given switched systems may fail to preserve the stability under arbitrary switchings. Therefore, a natural question is what if we restrict the switching signal to some constrained subclass of switchings. It is shown in [13], [34], [14] that the stability and performance could be preserved under certain constrained switchings, for example slow switching with certain average dwell time. The stability analysis with constrained switchings has been usually pursued in the framework of multiple Lyapunov functions (MLF), see for example [28], [8], [33], [11], [17], [18] and references therein.

The other basic problem for switched systems is the synthesis of stabilizing switching laws for a given collection of dynamical systems, which is called switching stabilization problem. In the switching stabilization literature, most of the work focused on quadratic stabilization for certain classes of systems. For example, a quadratic stabilization switching law between two LTI systems was considered in [29], in which it was shown that the existence of a stable convex combination of the two subsystem matrices implies the existence of a state-dependent switching rule that stabilizes the switched system along with a quadratic Lyapunov function. A generalization to more than two LTI subsystems was suggested in [26] by using a "min-projection strategy". In [12], it was shown that the stable convex combination condition is also necessary for the quadratic stabilizability of two mode switched LTI system. However, it is only sufficient for switched LTI systems with more than two modes. A necessary and sufficient condition for quadratic stabilizability of switched controller systems was derived in [31]. There are extensions of [29] to output-dependent switching and discrete-time case [17], [35]. For robust stabilization of polytopic uncertain switched systems, a quadratic stabilizing switching law was designed for polytopic uncertain switched linear systems based on LMI techniques in [35]. All of these methods guarantee stability by using a common quadratic Lyapunov function, which is conservative in the sense that there are switched systems that can be asymptotically (or exponentially) stabilized without using a common quadratic Lyapunov function. There have been some results in the literature that propose constructive synthesis methods to switched systems using multiple Lyapunov functions [11]. Exponential stabilization for switched LTI systems was considered in [27] based on piecewise quadratic Lyapunov functions, and the synthesis problem was formulated as a bilinear matrix inequality (BMI) problem. In [15], a probabilistic algorithm was proposed for the synthesis of an asymptotically stabilizing switching law for switched LTI systems along with a piecewise quadratic Lyapunov function. Notice that these stabilizability conditions, which may be expressed as the feasibility of certain LMIs or BMIs, in the existing literature are basically sufficient only, except for certain cases of quadratic stabilization. A necessary and sufficient condition for asymptotic stabilizability of second-order switched LTI systems was derived in [32] by detailed vector field analysis. However, it was not apparent how to extend the method to either higher dimensions or to the parametric uncertainty case.

This paper will focus on this switching stabilization problem, and derive necessary and sufficient conditions for asymptotic stabilizability of switched linear systems with time-variant parametric uncertainties. In particular, we will focus on globally asymptotic stability and derive a rank condition, which is necessary and sufficient for the asymptotic stabilizability of the uncertain switched linear systems. Similar uncertain switched system models have been considered in our previous work. In [21], a class of uncertain switched linear systems affected by both parameter variations and exterior disturbances was considered, and the uniformly ultimate boundedness control problem was studied for both discrete-time and continuous-time case. Under the assumption that each subsystem admits a finite persistent disturbance attenuation level, it was shown in [21] that, by proper switching, the closed-loop switched systems can reach a better disturbance attenuation level than any single subsystem. The determination of optimal disturbance attenuation property for uncertain switched systems and its decidability issue was discussed in [22].

The rest of the paper is organized as follows. In Section II, mathematical models for the uncertain switched linear system are described, and the robust stabilizability problem is formulated. The main result is presented in Section IV. The sufficiency and the necessity of the rank condition are proved. The techniques are based on polyhedral Lyapunov-like functions, which are introduced in Section III. Finally, concluding remarks are presented and future work is proposed.

In this paper, we use the letters $\mathcal{E}, \mathcal{P}, \mathcal{S} \cdots$ to denote sets. $\partial\mathcal{P}$ stands for the boundary of set $\mathcal{P}$, and $int(\mathcal{P})$ its interior. For any real $\lambda \geq 0$, the set $\lambda\mathcal{S}$ is defined as $\{x = \lambda y,\ y \in \mathcal{S}\}$. The term C-set stands for a convex and compact set containing the origin in its interior.

II. Problem Formulation

We consider a collection of continuous-time linear systems represented by the differential equations with parametric uncertainties

$$\dot{x}(t) = A_q(w)x(t), \quad t \in \mathbb{R}^+, \quad q \in Q = \{1, \cdots, N\} \tag{1}$$

where $\mathbb{R}^+$ denotes non-negative real numbers. In the above uncertain continuous-time state equations, the state variable $x(t) \in \mathbb{R}^n$. Note that the origin $x_e = 0$ is an equilibrium (maybe unstable) for the systems described in (1). The finite set Q stands for the collection of discrete modes. In particular, for all $q \in Q$, $A_q(w) : \mathcal{W} \to \mathbb{R}^{n\times n}$, and the entries of $A_q(w)$ are assumed to be continuous functions of $w \in \mathcal{W}$, where $\mathcal{W} \subset \mathbb{R}^v$ is an assigned compact set.

Because the minimum and maximum of a continuous function on a compact set is well-defined, so the entries of $A_q(w)$ can be bounded in an interval, that is $\underline{a_{i,j}} \leq a_{i,j}(w) \leq \overline{a_{i,j}}$. Therefore, the above uncertain time-variant state equation can be written as a time-invariant differential inclusion:

$$\dot{x}(t) \in F_q(x) \tag{2}$$

where the multivalued vector-function $F_q(x)$ is defined for all $x \in \mathbb{R}^n$ by

$$F_q(x) = \{f \ : \ f = A_q x, \quad A_q \in \mathcal{A}_q\} \tag{3}$$

and the set $\mathcal{A}_q \subset \mathbb{R}^{n\times n}$ is the collection of all constant matrices with entries satisfying $\underline{a_{i,j}} \leq a_{i,j} \leq \overline{a_{i,j}}$. The equivalence is regarded in the sense of coincidence of the sets of absolutely continuous solutions of the system (1) and of the differential inclusion (2) and (3) [3], [23]. It is known that the robust absolute stability problem of the differential inclusion (2)-(3) can be reduced to an equivalent problem of robust stability of the following linear time-varying system [23].

$$\dot{x}(t) = A_q(t)x(t) \tag{4}$$

where

$$A_q(t) = \sum_{j=1}^{v} w_j(t) A_q^j, \quad w_j(t) \geq 0, \ \sum_{j=1}^{v} w_j(t) = 1 \quad (5)$$

and the constant matrices $A_q^j \in \mathbb{R}^{n\times n}$ are the extreme points of $\mathcal{A}_q$. Note that $\mathcal{A}_q$ has finite number (up to 2^n) of such extreme points. This gives us an equivalent linear time-varying system with polytopic uncertainties.

Therefore, without loss of generality, let us assume polytopic uncertainty in (1). In particular, $A_q(w) = \sum_{j=1}^{v} w_j A_q^j$, $w_j \geq 0$ and $\sum_{j=1}^{v} w_j = 1$. Notice that the coefficients w_j are unknown and possibly time varying.

Combine the family of continuous-time uncertain linear systems (1) with a class of piecewise constant functions, $\sigma : \mathbb{R}^+ \to Q$, which serves as the switching signal between the collection of continuous-time systems (1). The continuous-time switched linear system can be described as

$$\dot{x}(t) = A_{\sigma(t)}(w)x(t), \quad t \in \mathbb{R}^+ \quad (6)$$

and the switching signal is generated by

$$\sigma(t) = \delta(\sigma(t^-), x(t)) \quad (7)$$

where $\delta : Q \times \mathbb{R}^n \to Q$ and $t^- = \lim_{\tau\to 0, \tau\geq 0}(t-\tau)$. The discrete mode is determined by the current continuous state $x(t)$ and the previous mode $\sigma(t^-)$.

For this uncertain continuous-time switched system (6)-(7), we are interested in the following problem.

Problem: Given the continuous-time switched system (6)-(7), derive a necessary and sufficient condition, under which there exist switching control laws to make the closed-loop switched system globally asymptotically stable.

The following assumption is made for each unstable subsystems (1).

Assumption: It is assumed that there exists a full row rank matrix $L_q \in \mathbb{R}^{m_q\times n}$, where $m_q < n$, such that the auxiliary system for the q-th subsystem (1)

$$\dot{\xi}(t) = L_q A_q(w) R_q \xi(t), \quad t \in \mathbb{R}^+ \quad (8)$$

is asymptotically stable. Here $R_q \in \mathbb{R}^{n\times m_q}$ is the right inverse of L_q.

The above auxiliary system is derived through the generalized similarity transformation $R_q\xi = x$. Notice that there may exist more than one pair of the matrices L_q and R_q that satisfy the above assumption.

Intuitively, the above assumption can be interpreted as representing part of the states (or their linear combinations) of the original system (1) that is asymptotically stable. The auxiliary system implies the lower dimensional subspace to which the original system can be projected for stability. Note that even when all parts of the states of the original system (1) are unstable, there still may exist L to satisfy the assumption. For example,

Example 1: Consider a continuous-time linear system,

$$\dot{x}(t) = \begin{bmatrix} 0.5 & w \\ 0 & 1 \end{bmatrix} x(t)$$

where the uncertain parameter $1 \leq w \leq 2$. The above continuous-time system is obviously unstable. However, we may select $L = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to obtain

$$LA(w)R = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & w \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0.5 - w < 0,$$

for all $w \in [1,\ 2]$. Therefore, the auxiliary system

$$\dot{\xi}(t) = (0.5 - w)\xi(t)$$

is asymptotically stable.

Remark 1: It can be shown that for the LTI case[1], $\dot{x}(t) = Ax(t)$, there always exist L and R satisfying the above assumption, except for the case when all the eigenvalues of A are the same positive real number $\lambda > 0$ and the geometric multiplicity of the eigenvalue λ equals to n. The proof of this claim just explores the structure of the Jordan canonical form of A and uses straight-forward computation. Details are omitted here.

Remark 2: For the case that there does not exist L to satisfy the above assumption for a particular subsystem, we simply set L as a null row vector, which implies that the corresponding subsystem makes no contribution for the stabilization of the switched system. To justify this, note that in this case the matrix A is similar to the matrix λI for some positive real number $\lambda > 0$. Here I stands for the identity matrix. If we look at the phase plane of the LTI system, $\dot{x}(t) = \lambda I x(t)$, all the field vectors points to infinity along the radial direction. Intuitively speaking, the dynamics are explosive and do nothing but drag all part of the states to infinity, which we would like to avoid. Therefore, setting L to be a null vector and contribute nothing to the rank condition in Theorem 1.

III. Polyhedral Lyapunov-like Functions

It was shown in [23] that the asymptotic stability of the auxiliary system (1) in $\mathbb{R}^{m_q}$ implies the existence of a polytopic Lyapunov function $\Phi_q(\xi)$, which can be constructed by either algebraic or numerical methods, see [23], [7]. Assume that the polytopic Lyapunov function for (1) is represented as follows:

$$\Phi_q(\xi) = \max_{1\leq i\leq s_q} \{f_i \xi\}.$$

Let $F_q \in \mathbb{R}^{s_q\times m_q}$ ($s_q \geq m_q$) be the matrix with $f_i \in \mathbb{R}^{1\times m_q}$ as its i-th row vector. It was shown in [7] that the Lyapunov level set

$$\mathcal{P}_q = \{\xi \in \mathbb{R}^{m_q} :\ \Phi_q(\xi) \leq 1\} = \{\xi \in \mathbb{R}^{m_q} :\ F_q\xi \leq \bar{1}\} \quad (9)$$

is an invariant set, where $\bar{1}$ stands for a column vector in $\mathbb{R}^{m_q}$ with all elements being 1 and $\leq$ is component-wise. In the previous example, one may pick the interval $\mathcal{P} = \{\xi :\ -1 \leq \xi \leq 1\}$, which is an invariant set for the auxiliary system.

[1]This corresponds to the uncertain parameter set $\mathcal{W}$ being a singleton.

The next step is to shape the polytopic Lyapunov function $\Psi_q(\xi)$ for the auxiliary system (8) into a polyhedral Lyapunov-like function for the original system (1).

For such purpose, we need to introduce the Euler Approximate System (EAS) for the auxiliary system (8) as:

$$\xi[k+1] = L_q[I + \tau A_q(w)]R_q\xi[k], \quad k \in \mathbb{Z}^+. \qquad (10)$$

The connection between the continuous-time system (8) and its corresponding discrete-time EAS (10) is based on the concept of a contractive set.

Definition 1: Given a positive scalar λ, $0 < \lambda < 1$, a set $\mathcal{P}$ is said λ-*contractive* with respect to the discrete-time EAS (10), if, for any $\xi \in \mathcal{P}$, all the possible next step states through the transition (10) are contained in the set $\lambda\mathcal{P}$, that is

$$L_q[I + \tau A_q(w)]R_q\xi \in \lambda\mathcal{P},$$

holds for any $w \in \mathcal{W}$.

It is shown in [7], that for an asymptotically stable system (8), there always exists a positive scalar $\bar{\tau} > 0$ such that, for all $0 < \tau \leq \bar{\tau}$, the level set $\mathcal{P}_q = \{\xi \ : \ F_q\xi \leq \bar{1}\}$ in (9) is λ-contractive for the corresponding discrete-time EAS (10).

Therefore, we obtain that

$$F_qL_q[I + \tau A_q(w)]R_q\xi \leq \lambda\bar{1}$$

holds for all $\xi \in \mathcal{P}_q = \{\xi \ : \ F_q\xi \leq \bar{1}\}$, for all $w \in \mathcal{W}$ and for τ small enough.

Note that $R_q\xi = x$ and $\xi = L_qx$, then for all $x \in \{x \ : \ F_qL_qx \leq \bar{1}\}$,

$$F_qL_q[I + \tau A_q(w)]x \leq \lambda\bar{1}$$

holds for all $w \in \mathcal{W}$ and for all $0 < \tau \leq \bar{\tau}$. This means that the set $\mathcal{S}_q = \{x \ : \ F_qL_qx \leq \bar{1}\}$, which may be an unbounded polyhedral set, is a λ-contractive set for the discrete-time system

$$x[k+1] = [I + \tau A_q(w)]x[k] \qquad (11)$$

by definition.

Notice that the above discrete-time system (11) is the EAS of the original subsystem (1). In the following, we will show that the existence of such contractive set $\mathcal{S}_q$ for the EAS (11) implies a polyhedral Lyapunov-like function for the continuous-time subsystem (1).

Denote $F_qL_q \in \mathbb{R}^{s_q \times n}$ as H_q, and h_i as the i-th row vector of H_q. Then the polyhedral Lyapunov-like function candidate from the polyhedron $\mathcal{S}_q$ can be defined as

$$\Psi_q(x) = \max_{1 \leq i \leq s_q} \{h_ix, 0\}. \qquad (12)$$

It is straightforward to verify that $\Psi_q(x) \geq 0$ for all $x \in \mathbb{R}^n$, and that $\Psi_q(x)$ is convex, continuous and piecewise linear for x. However, $\Psi_q(x) = 0$ does not imply x is the origin. In fact, for all x contained in the convex cone

$$\mathcal{C}_q = \{x \ : \ H_qx \leq \bar{0}\},$$

we have $\Psi_q(x) = 0$. This is one of the main differences from the classical Lyapunov function, so we call it Lyapunov-like function.

Next, we will show that the Dini derivative of Ψ_q along the trajectory of the continuous-time system (1) is negative for all x outside the cone $\mathcal{C}_q$, where the Dini derivative $\mathcal{D}^+\Psi_q(x(t))$ is defined as

$$\mathcal{D}^+\Psi_q(x(t)) = \lim \sup_{\tau \to 0, \tau \geq 0} \frac{\Psi_q(x(t+\tau)) - \Psi_q(x(t))}{\tau}.$$

It was shown in [7] that the Dini derivative of Ψ_q at the time instant t, for $x(t) = x$, and $w(t) = w$, can be calculated as

$$\mathcal{D}^+\Psi_q(x(t)) = \lim \sup_{\tau \to 0, \tau \geq 0} \frac{\Psi_q(x + \tau A_q(w)x) - \Psi_q(x)}{\tau}.$$

The following property of the contractive sets for EAS (11) is essential to prove that the Dini derivative of Ψ_q is negative along the trajectory of (1).

Lemma 1: If $\mathcal{S}$ is a λ-contractive set for the EAS (11), then $\mu\mathcal{S}$ is so for all $\mu > 0$.

A similar result for a bounded polyhedron appeared in [6], and the proof can be extended to the unbounded case without difficulties. So the details are omitted here due to space limitation.

The next lemma shows that the contractiveness of the polyhedral set $\mathcal{S}_q$ for the EAS (11) implies the negativeness of the Dini derivative of Ψ_q for (1).

Lemma 2: If there exist scalars $0 < \lambda < 1$ and $\bar{\tau} > 0$, such that the polyhedral set $\mathcal{S}_q = \{x \ : \ H_qx \leq \bar{1}\}$ is a λ-contractive set for the EAS (11) with all $0 < \tau < \bar{\tau}$, then the Dini derivative $\mathcal{D}^+\Psi_q(x(t))$ for all $x(t)$ outside the cone $\mathcal{C}_q$ is negative along the trajectory of the continuous-time system (1).

Proof: For any $x \in \mathbb{R}^n$ outside the cone $\mathcal{C}_q$, there exists a positive scalar $\mu > 0$ such that x lies on the boundary of the polyhedron $\mu\mathcal{S}_q$, that is $x \in \partial\mu\mathcal{S}_q$. From Lemma 1, the the polyhedron $\mu\mathcal{S}_q$ is a λ-contractive set ($0 < \lambda < 1$) for the EAS (11). Therefore,

$$\Psi_q(x + \tau A_q(w)x) \leq \mu\lambda,$$

holds for $\tau > 0$ and $0 < \lambda < 1$. Subtract $\Psi_q(x) = \mu$ to both sides and divide by τ, then

$$\frac{\Psi_q(x + \tau A_q(w)x) - \Psi_q(x)}{\tau} \leq \frac{\mu\lambda - \mu}{\tau}.$$

Notice that $\frac{\mu\lambda - \mu}{\tau} = \frac{\mu}{\tau}(\lambda - 1) < 0$, so

$$\frac{\Psi_q(x + \tau A_q(w)x) - \Psi_q(x)}{\tau} < 0.$$

In addition, the difference quotient is a nondecreasing function for τ for a convex Ψ_q, we conclude that

$$\mathcal{D}^+\Psi_q(x(t)) < 0. \qquad \square$$

In summary, we started from the assumption and proved the existence of a polyhedral Lyapunov-like function

$\Psi_q(x(t))$ for the subsystems (1). The determination of such polyhedral Lyapunov-like function can be reduced to the determination of a polytopic Lyapunov function of the auxiliary system (8) if the matrix L_q is given. It was shown in [7] that polytopic Lyapunov function may be derived by numerically efficient algorithms involving polyhedral sets. To calculate the matrix L_q that satisfies the assumption, a systemic method can be developed for the LTI case by exploring the Jordan canonical form. In the next section, we will construct the stabilizing switching law and compose a global Lyapunov function for the switched system based on these polyhedral Lyapunov-like function $\Psi_q(x(t))$ for the subsystems.

IV. Main Results

A necessary and sufficient condition for the robust asymptotic stabilizability of the uncertain switched linear systems (6)-(7) can be stated as the following rank condition.

Theorem 1: The switched linear system (6)-(7) with time-variant uncertainties can be globally asymptotically stabilized by a switching law, if and only if there exist matrices L_q, which satisfies the assumption (8) for each subsystem respectively, and the following matrix

$$\begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_N \end{bmatrix} \in \mathbb{R}^{\sum_q m_q \times n}, \tag{13}$$

has n linear independent row vectors.

A. Sufficiency

First, we give a necessary and sufficient condition for a polyhedron to be bounded. Note that a bounded polyhedron is usually called a polytope.

Lemma 3: [25] A non-empty polyhedral set

$$\mathcal{P} = \{x \in \mathbb{R}^n \ : \ Hx \leq g\}$$

is bounded, if and only if the cone

$$\mathcal{C} = \{x \in \mathbb{R}^n \ : \ Hx \leq \bar{0}\}$$

only contains the null vector $\bar{0}$.

For 0-symmetric polyhedrons, the boundedness checking can be reduced to the following simple rank condition.

Lemma 4: A non-empty 0-symmetric polyhedral set

$$\mathcal{P} = \{x \in \mathbb{R}^n \ : \ |Hx| \leq g\}$$

is bounded, if and only if the matrix $H \in \mathbb{R}^{s\times n}$ $(s \geq n)$ has n linear independent row vectors, or equivalently the rank of H equals to n.

In particular, for the intersection of $\mathcal{S}_q$ we have the following corollary.

Corollary 1: If all the polyhedral sets $\mathcal{P}_q$ in $\mathbb{R}^{m_q}$ are 0-symmetric, then so are the $\mathcal{S}_q$ in $\mathbb{R}^n$, for all $q \in Q = \{1, 2, \cdots, N\}$. In addition, the intersection of all the polyhedral sets $\mathcal{S}_q$ is bounded, if and only if the matrix (13) has n linear independent row vectors, or equivalently it has rank n.

The proof of this corollary uses Lemma 4 and explores the algebraic structure of $H_q = F_q L_q$. It is not difficult, so details are omitted here.

To prove the sufficiency of the main theorem, it is assumed that there exist matrices L_q, for $q \in Q$, satisfying the assumption and the rank condition (13). This implies that the intersection of the polyhedron $\mathcal{S}_q$ is a bounded set, denoted as $\mathcal{S}$. It is straightforward to verify that $\mathcal{S}$ is convex and compact, and the origin is contained in the interior of $\mathcal{S}$. Hence, $\mathcal{S}$ is a polyhedral C-set.

In our recent work [21], we constructed a switching control law to guarantee that the switched system is uniformly ultimate bounded in a polyhedral C-set. The techniques can be used here without significant change to construct switching laws that attract all the state trajectories into $\mathcal{S}$ within a finite time interval. We will sketch the switching law synthesis procedure and verify the ultimate boundedness here.

First, we introduce some notations. Given a polyhedral C-set $\mathcal{S}$, let $vert(\mathcal{S}) = \{v_1, v_2, \cdots, v_e\}$ stand for its finite vertices, while $face(\mathcal{S}) = \{F_1, F_2, \cdots, F_M\}$ denote its facets. The hyperplane that corresponds to the k-th facet F_k is defined by

$$H_k = \{x \in \mathbb{R}^n : f_k x = 1\} \tag{14}$$

where $f_k \in \mathbb{R}^{1\times n}$ is the corresponding gradient vector of facet F_k. The set of vertices of F_k can be found as $vert(F_k) = vert(\mathcal{P}) \cap F_k$. Finally, we denote the cone generated by the vertices of F_k by $cone(F_k) = \{x \in \mathbb{R}^n : \sum_i \alpha_i v_{k_i},\ \alpha_i \geq 0,\ v_{k_i} \in vert(F_k)\}$.

Note that if $\mathcal{S}$ is a polyhedral C-set then

$$\bigcup_{k=1,\cdots,M} cone(F_k) = \mathbb{R}^n.$$

Hence, we obtain a conic partition of the whole state space $\mathbb{R}^n$, from which we induce a stabilizing switching law as follows.

For each facet of $\mathcal{S}$, F_k, there exists at least one mode q such that the gradient vector of facet F_k, namely f_k, is one of the (non-redundant) row vectors of H_q. Collect all such modes q and call them active modes for $cone(F_k)$, denoted as $Act\{cone(F_k)\}$.

It can be shown that these active modes in $cone(F_k)$ have the following properties. First, for two different modes q_1, $q_2 \in Act\{cone(F_k)\}$, the equality $\Psi_{q_1}(x) = \Psi_{q_2}(x)$ holds for all $x \in cone(F_k)$. Secondly, for any $q \in Act\{cone(F_k)\}$ and any another mode $q' \in Q$, $\Psi_q(x) \geq \Psi_{q'}(x)$ holds for all $x \in cone(F_k)$. The results are not surprising, by considering the geometric interpretation of the Lyapunov-like function which is basically a distance measure from a point to the boundaries of a polyhedral set.

For any convex cone $cone(F_k)$, we simply pick one mode q from its active modes set $Act\{cone(F_k)\}$ and associate

q with the cone by relabelling the cone as Ω^q. After this relabelling process, the whole state space $\mathbb{R}^n$ is partitioned into a finite number of conic cones Ω^q, that is

$$\bigcup_{q \in Q} \Omega^q = \mathbb{R}^n.$$

In addition, for all the $x \in \Omega^q \bigcap \Omega^{q'}$, the equality $\Psi_q(x) = \Psi_{q'}(x)$ holds.

Based on the conic partition of the state space given by Ω^q, $q \in Q$, we define the switching law as:

$$x \in \Omega^q \Rightarrow \delta(\cdot, x) = q \tag{15}$$

It can be shown that the switching law defined above can guarantee the uniformly ultimate boundedness (UUB) for the uncertain switched system (6)-(7) into $\mathcal{S} = \bigcap_{q \in Q} \mathcal{S}_q$.

Proposition 1: Consider the class of switching laws defined by $\delta(\cdot, x) = q$ if x is contained in Ω^q. Then, the uncertain continuous-time switched system (6)-(7) is UUB in the polyhedral C-set $\bigcap_{q \in Q} \mathcal{S}_q$.

Proof : Define the function $V(x) = \max_{q \in Q} \Psi_q(x)$. For all $x(t) \notin \bigcap_{q \in Q} \mathcal{S}_q$, $V(x(t)) = \max_{q \in Q} \Psi_q(x) > 1$. Assume that $x(t) \in \Omega^q$ and current mode $\sigma(t) = q$. If no switching occurs at t, then there exists $\bar{\tau} > 0$ such that $\forall 0 < \tau \leq \bar{\tau}$, $x(t+\tau) \in \Omega^q$ and $x(t+\tau) \notin int(\mathcal{S}_q)$. Then $V(x(t)) = \max_{q \in Q} \Psi_q(x(t)) = \Psi_q(x(t))$ and $V(x(t+\tau)) = \Psi_q(x(t+\tau))$. Then we derive that

$$\mathcal{D}^+ V(x(t)) = \mathcal{D}^+ \Psi_q(x(t)) < 0$$

Else, if switching occurs at time t, then there exists $\bar{\tau} > 0$ such that $\forall 0 < \tau \leq \bar{\tau}$, $x(t+\tau) \in \Omega_{q'}$ and $x(t+\tau) \notin int(\mathcal{S}_{q'})$. Then $V(x(t)) = \max_{q \in Q} \Psi_q(x(t)) = \Psi_q(x(t)) = \Psi_{q'}(x(t))$ and $V(x(t+\tau)) = \Psi_{q'}(x(t+\tau))$. Therefore,

$$\mathcal{D}^+ V(x(t)) = \lim \sup_{\tau \to 0^+} \frac{\Psi_{q'}(x(t+\tau)) - \Psi_{q'}(x(t))}{\tau} < 0.$$

Therefore, the uncertain switched system (6)-(7) is UUB with respect to the region $\bigcap_{q \in Q} \mathcal{S}_q$. □

Because of Lemma 1 and the above UUB result, the switching control law can drive all the state trajectories into $\mu\mathcal{S}$ for all $\mu > 0$ within a finite time interval. Pick a decreasing sequence of μ_k with $\lim_{k \to \infty} \mu_k = 0$, then all the trajectories will finally be driven to the origin. This implies globally asymptotic stability for the switched linear system (6)-(7). In summary, we have the following result.

Proposition 2: Under the assumption, if the matrix (13) has n linear independent row vectors, then there exists a switching control law to asymptotically stabilize the uncertain switched system (6)-(7).

Proof: It was shown in [23] that the asymptotic stability of the q-th auxiliary system (8) implies the existence of a polytopic Lyapunov function Ψ_q in the following infinite norm form

$$\Psi_q(\xi) = \|F_q \xi\|_\infty.$$

By the arguments in the previous section, the polytopic Lyapunov function Ψ_q implies a contractive polytope for the EAS (10), which can be represented as

$$\mathcal{P}_q = \{\xi \in \mathbb{R}^{m_q} \ : \ |F_q \xi| \leq \bar{1}\}.$$

So the generated contractive polyhedral for the original system (1) in $\mathbb{R}^n$ can be represented as

$$\mathcal{S}_q = \{x \in \mathbb{R}^n \ : \ |F_q L_q x| \leq \bar{1}\},$$

which is 0-symmetric. With the satisfaction of the rank condition (13), we know that $\mathcal{S} = \bigcap_q \mathcal{S}_q$ is a C-set by Corollary 1. Therefore, a conic partition based stabilizing switching law can be constructed, and a global Lyapunov function $V(x)$ can be composed. □

This completes the sufficiency proof of Theorem 1.

B. Necessity

To discuss the necessity of the stabilizability of switched systems, we need the following lemma.

Lemma 5: The uncertain switched linear system (6)-(7) can be globally asymptotically stabilized by a switching law if and only if it can be asymptotically stabilized by a conic partition switching law.

Proof : Because of the fact that a conic partition switching law is a specific class of switching law, the necessity is obvious.

To prove sufficiency, it is assumed that the switched system can be globally asymptotically stabilized by a properly designed switching law. Therefore, there exists a switching signal $\sigma(t)$ such that the closed-loop switched system

$$\dot{x}(t) = A_{\sigma(t)}(w)x(t)$$

is globally asymptotically stable. Therefore, there exists a polytopic Lyapunov function $\Psi(x)$ [23] for the closed-loop switched linear systems. Note that the level set

$$\mathcal{P} = \{x \in R^n \ : \ \Psi(x) \leq 1\}$$

is a C-set [7].

For any $x \in \partial\mathcal{P}$, according to the above asymptotic stabiity assumption, there exists at least one mode q such that the Dini derivative of $\Psi(x)$ is negative along the dynamics of mode q. Similar to the arguments in [7], there exists a positive constant $\bar{\tau} > 0$ and a scalar $0 < \lambda < 1$, such that

$$[I + \tau A_q(w)]x \subset \lambda\mathcal{P},$$

holds for all $0 < \tau \leq \bar{\tau}$. In addition, for any positive scalar $\mu > 0$,

$$[I + \tau A_q(w)]\mu x \subset \lambda\mu\mathcal{P},$$

holds for all $0 < \tau \leq \bar{\tau}$. This implies that the Dini derivative of $\Psi(\mu x)$ is negative along the dynamics of mode q [7].

Because of the continuity, there exists a small neighborhood of x, $B_r(x)$, such that for all $y \in \partial\mathcal{P} \bigcap B_r(x)$,

$$[I + \tau A_q(w)]y \subset \lambda\mathcal{P},$$

holds for all $0 < \tau \leq \bar{\tau}$. Note that

$$\partial\mathcal{P} \subseteq \bigcup_{x\in\partial\mathcal{P}} \partial\mathcal{P} \cap B_r(x),$$

and the fact that $\partial\mathcal{P}$ is closed and bounded in $\mathcal{R}^n$, namely compact. Therefore, there exists a finite cover for $\partial\mathcal{P}$, which induces a finite partition of the faces of $\mathcal{P}$. With each partition of $\partial\mathcal{P}$, a conic cone can be generated and the union of these cones is the whole state space. Within each cone, following the previous arguments, one mode q can be selected to make the Dini derivative of $\Psi(x)$ negative along the dynamics of mode q. This generates a conic switching law which globally asymptotically stabilizes the switched system. $\square$

Because of the above lemma, the existence of asymptotically stabilizing switching law for the switched system (6)-(7) implies the existence of a polytopic Lyapunov function $\Psi(x)$ and a conic partition based switching law which globally asymptotically stabilizes the closed-loop switched system. Denote such conic partition as Ω^q, such that

$$\bigcup_{q\in Q} \Omega^q = \mathbb{R}^n.$$

As proved in the above lemma, within the cone Ω^q, the Dini derivative of $\Psi(x)$ is negative along the dynamics of the mode q. This implies that the cone

$$\mathcal{C} = \bigcap_q \mathcal{C}_q$$

only contains the null vector, which implies that the intersection of $\mathcal{S}_q$ is bounded. By Corollary 1, the rank condition is obtained.

This completes the necessity proof. $\square$

To illustrate the results, let us consider the following example, which was considered in [11] used primarily as a counter-example to show that switching between two unstable systems may exhibit stable behavior.

Example 2: Consider the following continuous-time switched linear system:

$$\dot{x}(t) = \begin{cases} A_1(w)x(t), & \sigma(t) = q_1 \\ A_2(w)x(t), & \sigma(t) = q_2 \end{cases}$$

In this example the mode set $Q = \{q_1, q_2\}$, and the corresponding state matrices for each subsystem are given as

$$A_1(w) = \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix}, \; A_2(w) = \begin{bmatrix} 1.5 & 2 \\ -2 & -0.5 \end{bmatrix}$$

Select $L_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $R_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then

$$L_1A_1R_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -10.$$

The auxiliary system for mode q_1 can be written as

$$\dot{\xi}(t) = -10\xi(t),$$

which is asymptotically stable. It is easy to verify that the interval $[-1, \ 1]$ is contractive for the first auxiliary system. Then, we obtain the polyhedral region $\mathcal{S}_1$ as

$$\mathcal{S}_1 = \{x \in \mathbb{R}^2 \mid -1 \leq \begin{bmatrix} 1 & 0 \end{bmatrix} x \leq 1\}.$$

For the second subsystem, pick $L_2 = \begin{bmatrix} 1 & -8 \end{bmatrix}$ and $R_2 = \begin{bmatrix} -1 \\ -0.25 \end{bmatrix}$, then

$$L_2A_2R_2 = \begin{bmatrix} 1 & -8 \end{bmatrix} \begin{bmatrix} 1.5 & 2 \\ -2 & -0.5 \end{bmatrix} \begin{bmatrix} -1 \\ -0.25 \end{bmatrix} = -19.$$

Similarly, the auxiliary system for mode q_2 can be written as

$$\dot{\xi}(t) = -19\xi(t),$$

which is asymptotically stable as well. The interval $[-3, \ 3]$ is contractive for the second auxiliary system, which induce the polyhedral region $\mathcal{S}_2$ as

$$\mathcal{S}_2 = \{x \in \mathbb{R}^2 \mid -3 \leq \begin{bmatrix} 1 & -8 \end{bmatrix} x \leq 3\}.$$

Note that the rank condition (13) is satisfied, namely

$$rank \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = rank \begin{bmatrix} 1 & 0 \\ 1 & -8 \end{bmatrix} = 2.$$

Therefore, the switched linear systems can be globally asymptotically stabilized through proper switching. From the sufficiency proof in Section IV-A, the stabilizing switching control law can be constructed as follows:

First take the intersection of $\mathcal{S}_1$ and $\mathcal{S}_2$ as shown in Figure 1. Then, the state space $\mathbb{R}^2$ can be partitioned into four conic regions by the radii that starting from the origin and going through the four vertices of $\mathcal{S} = \mathcal{S}_1 \bigcap \mathcal{S}_2$ respectively. Within each conic partition, we associate either q_1 or q_2 to it, as shown in Figure 1, and if the state x is within that region then the associated mode becomes active.

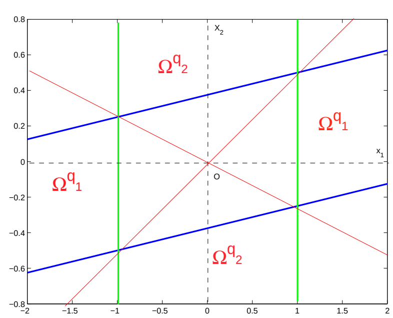

Fig. 1. The intersection of $\mathcal{S}_1$ and $\mathcal{S}_2$ and its induced conic partition based stabilizing switching law.

This gives us exactly the same switching control law as the one proposed in [11]; for simulations see [11]. Of course, it is possible to get different stabilizing switching law for this example, if we choose different L_q and/or different contractive regions for the auxiliary systems. Theoretically speaking, we may obtain a whole class of switching control laws that asymptotically stabilize the switched system.

V. Concluding Remarks

In this paper, continuous-time switched linear systems affected by parameter variations were considered. The robust stabilizability problem for such uncertain switched linear systems was investigated. A necessary and sufficient rank condition for the existence of a switching control law to assure the asymptotic stability of the closed-loop switched systems was derived. The sufficiency proof also proposed a constructive method for the stabilizing switching law synthesis, which is characterized by conic partition of the state space.

In our future work, the nature of the generalized similarity transformation, especially for parametric uncertainty case, needs to be better understood. In fact, two problems are of particular interest for this rank test. One is the existence of the stable generalized similar system for (1) with uncertainties. The second problem is how to develop an efficient method to determine or to parameterize the similarity transformation matrix L_q for a given subsystem.

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