Robust Tracking and Regulation Control of Uncertain Piecewise Linear Hybrid Systems *

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Abstract: In this paper, a class of uncertain, discrete-time, piecewise linear hybrid systems affected by both parameter variations and exterior disturbances is introduced. The robust tracking and regulation control problem for such uncertain piecewise linear hybrid systems is investigated. The main question is whether there exists a controller such that the closed-loop system exhibits desired behavior under dynamic uncertainties and exterior disturbances. If a specified behavior can be forced on the plant by a control mechanism, then it is called attainable. We present a method for checking attainability that employs the predecessor operator and backward reachability analysis, and introduce a procedure for robust tracking and regulation controller design that uses linear programming techniques. This procedure is then used to solve the robust stabilization problem for uncertain piecewise linear hybrid systems.

Keywords: Tracking and Regulation Control, Constrained Regulation, Uncertainty, Piecewise Linear/Affine Systems, Safety, Reachability, Controller Synthesis, Persistent Disturbance

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1 Introduction

Hybrid Systems are heterogeneous dynamical systems of which the behavior is determined by interacting continuous variable and discrete event dynamics. Hybrid systems have been identified in a wide variety of applications in control of mechanical systems, process control, automotive industry, power systems, aircraft and traffic control, among many other fields. Piecewise linear (affine) systems are recognized as an important model frame for hybrid systems and have been widely studied in the literature; see for example [26, 13, 6, 16] and the references therein. The issues studied include modeling [5, 26], stability [17, 10, 13], observability and controllability [6, 26] primarily. Piecewise linear systems arise often from linearizations of nonlinear systems. Notice, however, that for an important class of practical nonlinear systems with parameter variations the number of piecewise linear approximations may become prohibitively large. One way to study such uncertain nonlinear systems in a systematic way is to use a bundle of parameterized linearizations, which cover the original uncertain nonlinear dynamics within a region. The model mismatching may be tackled by introducing additive disturbances. This motivates us to study uncertain piecewise linear systems with disturbances as described in Section 2.1.

The dynamic uncertainty and robust control of hybrid/switched systems is a highly promising and challenging field. However, the literature on this topic is relatively sparse. Some of the contributions include modelling uncertain hybrid/switched systems, reachability analysis, stability analysis and so on. For example, impulse differential inclusions were proposed as a modeling framework for uncertain hybrid/switched systems in [4], where some theoretic results for viability and invariance analysis in classical differential inclusions were extended to the impulse differential inclusions. Reachability analysis results for uncertain hybrid/switched systems have appeared in [14, 19]. In addition, the safety and reachability problems for uncertain linear hybrid systems were studied in [20] based on backward reachability analysis techniques. There are also a few related publications on the robust controller design. In [24], the authors gave an abstract algorithm, based on modal logic formalism, to design the switching among a finite number of continuous variable systems. It was shown that the closed-loop system formed a hybrid automata and satisfied certain specifications robustly. In [12], the authors proposed logically supervised switching multiple controllers to control uncertain dynamical systems, while the closed-loop system forming a class of uncertain switched systems. The advantages of switching controllers over classical adaptive controllers were discussed in [12] as well. These advantages partially explain the increasing interest in switched systems during the past decade. For robust stabilization of uncertain switched systems, a quadratic stabilizing switching law was designed for polytopic uncertain switched systems based on LMI techniques in [28]. Some related work on the induced gain analysis for switched linear systems has appeared for example in [11, 27].

In this paper, we investigate the tracking and regulation control problem for a class of discrete-time uncertain piecewise linear hybrid systems, which is affected by both parameter variations and exterior disturbances. The control objective is for the closed-loop system to exhibit certain desired behavior despite the uncertainties and disturbances. Specifically, given finite number of regions $\{\Omega_0,\ \Omega_1, \cdots,\ \Omega_M\}$ in the state space, our goal is for the closed-loop system trajectories, starting from the given initial region Ω_0, to go through the sequence of finite number of regions $\Omega_1,\ \Omega_2, \cdots,\ \Omega_M$ in the desired order and finally reach the final region Ω_M and then remain in Ω_M. This kind of specifications is analogous to the ordinary tracking and regulation problem in pure classical continuous variable dynamical control systems. In addition, it also reflects the qualitative ordering of event requirements along trajectories. In Section 2, discrete-time uncertain piecewise linear systems are defined, and the robust tracking and regulation control problem for such uncertain piecewise linear system is qualitatively formulated. One of the main questions is to determine whether there exist admissible control laws such that the region-sequence can be followed. If such an admissible control law exists, the region-sequence specification $\{\Omega_0,\ \Omega_1, \cdots,\ \Omega_M\}$ is called *attainable*. The attainability checking is based on backward reachability analysis and symbolic model checking method discussed in Section 3. In Section 4, necessary and sufficient conditions for checking attainability are given. An optimization based method is proposed in Section 5 to design such admissible control laws. In Section 6, the robust stabilization control problem for uncertain piecewise linear systems is formulated and solved as a specific application of the robust tracking and regulation control method developed here. Finally, concluding remarks are given. Note that some preliminary results were presented in [19, 20]. However, in this paper we propose a method for specification refinement and explicitly deal with non-convexity in the controller design.

Notation : A polyhedron in $\mathbb{R}^n$ is a (convex) set given by the intersection of a finite number of open and/or closed half-spaces in $\mathbb{R}^n$. A polytope is a closed and bounded (i.e. compact) polyhedron. A polyhedral set $\mathcal{P}$ will be presented either by a set of linear inequalities $\mathcal{P} = \{x \mid F_i x \leq g_i,\ i = 1, \cdots, s\}$, compactly $\mathcal{P} = \{x \mid Fx \leq g\}$, or by the dual representation in terms of its vertex set $\{x_j\}$, denoted as $vert\{\mathcal{P}\}$. A piecewise linear set consists of a finite union of polyhedra.

2 Problem Formulation

In this section, the discrete-time uncertain piecewise linear systems are defined and the robust tracking and regulation control problem is formulated.

2.1 Uncertain Piecewise Linear Systems

We consider discrete-time uncertain piecewise linear systems of the form

$$x(t+1) = A_q(w(t))x(t) + B_q(w(t))u(t) + E_q d(t), \quad t \in \mathbb{Z}^+, \text{ if } x \in \mathcal{P}_q \tag{2.1}$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$, $u(t) \in \mathcal{U}_q \subset \mathbb{R}^m$, $d(t) \in \mathcal{D}_q \subset \mathbb{R}^r$, $w(t) \in \mathcal{W} \subset \mathbb{R}^v$ are the system state, the control input, the disturbance input, and the uncertainty parameter respectively. Let the finite set Q stand for the collection of discrete modes q. It is assumed that $\mathcal{X}$, $\mathcal{U}_q$, $\mathcal{D}_q$ and $\mathcal{W}$ are assigned polytopes for each mode $q \in Q$, and that $\mathcal{D}_q$ contains the origin. The partition of the state space $\mathcal{X}$ is given as a finite set of polyhedra $\{\mathcal{P}_q : q \in Q\}$, where $\mathcal{P}_q \subseteq \mathcal{X}$ and $\bigcup_{q \in Q} \mathcal{P}_q = \mathcal{X}$. The continuous variable dynamics of each mode q is defined by the state matrices $A_q(w)$, $B_q(w)$ and E_q. It is assumed that the entries of $A_q(w)$ and $B_q(w)$ are continuous functions of w for every mode q.

A possible evolution of the uncertain piecewise linear systems from a given initial condition $x_0 \in \mathcal{X}$ can be described as follows. First, there exists at least one discrete mode $q_0 \in Q$ such that $x_0 \in \mathcal{P}_{q_0}$; the mode q_0 is then called feasible mode for state x_0 [1]. The next continuous variable state is given by the transition $x_1 = A_{q_0}(w)x_0 + B_{q_0}(w)u + E_{q_0}d$ for some $w \in \mathcal{W}$, $d \in \mathcal{D}_{q_0}$ and specific $u \in \mathcal{U}_{q_0}$. Then the above procedure is repeated for state x_1 to determine the next possible state x_2, and so on.

2.2 Robust Tracking and Regulation

In continuous variable dynamical control systems, the tracking and regulation problem is a classical control problem. It can be formulated as follows: Given an initial region, a target region and a pipe connecting these two regions in the state space, design a control law so that all the trajectories starting from this initial region will be driven to the target region through the pipe. In the case of uncertain piecewise linear systems, the tracking and regulation problem can be formulated in an analogous fashion. However instead of connecting the initial region and the target region by a pipe, we use a sequence of connected polyhedral regions Ω_i, which may be seen as inner approximations of the pipe specification (See Figure 1).

The problem considered in this paper is to select feasible modes $q(t)$ and to design admissible control signals $u(t) \in \mathcal{U}_{q(t)}$ such that the closed-loop piecewise linear systems' trajectories, starting from the given initial region Ω_0, go through the sequence of regions

[1] In the definition of uncertain piecewise linear systems, it is not required that the partition $\mathcal{P}_q$ has mutually empty intersections. Therefore, for the initial state x_0 there may exist more than one feasible discrete modes. In such cases, it is assumed that the current active mode, q_0, is randomly selected from these feasible modes.

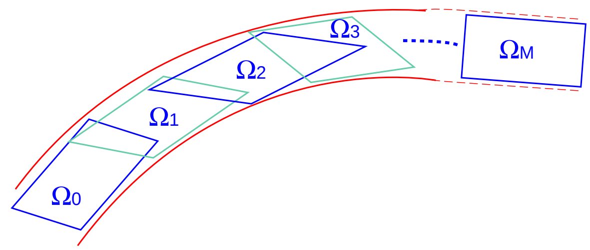

Figure 1: Tracking and regulation control specification and its polyhedral region sequence approximation.

$\Omega_1,\ \Omega_2, \cdots,\ \Omega_M$ in the desired order, finally reach the final region Ω_M and then remain in Ω_M, in spite of uncertainties and disturbances. Note that the region $\Omega_i \subseteq \mathcal{X}$ does not necessarily coincide with the partitions $\mathcal{P}_q$ in the definition of the uncertain piecewise linear systems (2.1).

We propose to solve the robust tracking and regulation problem for the uncertain piecewise linear systems (2.1) in three stages.

At the first stage, we check whether there exist feasible modes $q(t)$ and admissible control signals, $u(x(t)) \in \mathcal{U}_{q(t)}$, such that the region-sequence specification is satisfied despite the uncertainties and disturbances. If there exists such admissible control laws to satisfy the tracking and regulation specification, then the problem specification is called *attainable*. In order to check attainability, two different kinds of properties should be checked, namely the direct reachability between two successive regions, Ω_i and Ω_{i+1} for $1 \le i < M$, and the safety (or controlled invariance) for the final region Ω_M. The analysis problems for safety and direct reachability are formulated as follows.

- Safety: Given a region $\Omega \subset \mathcal{X}$, determine whether there exist admissible control laws such that the evolution of the system starting from Ω remains inside the region for all time, despite the presence of dynamic uncertainties and disturbances.

- Direct Reachability: Given two regions $\Omega_1,\ \Omega_2 \subset \mathcal{X}$, determine whether there exist admissible control laws such that all states in Ω_1 can be driven into Ω_2 in finite steps without entering a third region.

If the problem specification is attainable, an appropriated controller is designed in Stage three. Otherwise, if the problem specification, which may be given as an initial inner polyhedral approximation of the pipe specification (see Figure 1), is not attainable, we propose a procedure in Section 4 to refine the original specification. The refined sequence of regions, if non-empty, are guaranteed to be attainable.

The third stage is to design the controller to satisfy the tracking and regulation specification, if the specification is attainable or has been successfully refined. Similarly, the controller synthesis stage is also divided into two basic problems, that is safety control and direct reachability control. The two basic control problems are formulated as follows.

- Safety Controller Synthesis: Given a safe region $\Omega \subset \mathcal{X}$, determine the admissible control laws to make the region Ω safe (controlled invariant);
- Direct Reachability Controller Synthesis: Given two regions $\Omega_1,\ \Omega_2 \subset \mathcal{X}$, where Ω_2 is directly reachable from Ω_1, determine the admissible control laws such that all the states in Ω_1 can be driven into Ω_2 in finite steps without entering a third region.

In the following sections, we will solve the robust tracking and regulation problem stage by stage. First, necessary and sufficient conditions for safety and direct reachability are given in Section 4. The safety and direct reachability checking are based on backward reachability analysis. In the next section, we will briefly discuss the backward reachability analysis, which serves as one of the basic tools for the analysis that follows.

3 Robust Backward Reachability Analysis

This section describes the results of backwards reachability analysis for the uncertain piecewise linear systems, which server as the foundation for checking safety, reachability and attainability. Forward/Backward reachability analysis has been used widely in the study of hybrid systems [1, 24, 16], and a good deal of research effort has focused on developing sophisticated techniques drawn from optimal control, game theory, and computational geometry to calculate or approximate the reachable sets for various classes of hybrid systems [1, 9, 3, 16]. In this section, we will exactly calculate the robust one-step backward reachable set (defined below) for the uncertain piecewise linear systems (2.1).

3.1 Robust One-Step Predecessor Set

The basic building block to be used for backward reachability analysis is the *robust one-step predecessor operator*, which is defined below.

Definition 3.1 The *robust one-step predecessor set*, $pre(\Omega)$, is the set of states in $\mathcal{X}$, for which admissible control inputs exist and drive these states into Ω in one step, despite disturbances and uncertainties, i.e.

$$\begin{aligned} pre(\Omega) = \{x(t) \in \mathcal{X} \ \ | \ \ & \exists q(t) \in Q,\ u(t) \in \mathcal{U}_{q(t)} \ : \ x(t) \in \mathcal{P}_{q(t)}, \\ & A_{q(t)}(w)x(t) + B_{q(t)}(w)u(t) + E_{q(t)}d(t) \in \Omega,\ \forall d(t) \in \mathcal{D}_{q(t)},\ w \in \mathcal{W}\} \end{aligned}$$

We can also define the one-step predecessor set under the q-th mode, $pre_q(\Omega)$, as the set of all states $x \in \mathcal{P}_q$, for which an admissible control input $u \in \mathcal{U}_q$ exists and guarantees that the system will be driven to Ω by the transformation $A_q(w)x + B_q(w)u + E_q d$ for all allowable disturbances and uncertainties.

Proposition 3.1 The robust one-step predecessor set $pre(\Omega)$ for an uncertain piecewise linear system can be computed as follows:

$$pre(\Omega) = \bigcup_{q \in Q} pre_q(\Omega)$$

Proof : Assume that $\bigcup_{q\in Q} pre_q(\Omega)$ is not empty. For all the states $x \in \bigcup_{q\in Q} pre_q(\Omega)$, there exists at least one mode $q \in Q$ such that $x \in pre_q(\Omega) \subseteq \mathcal{P}_q$. Therefore, by the definition of $pre_q(\cdot)$, there exists control signal $u \in \mathcal{U}_q$ which drive the state x into the set Ω under the mode q despite the disturbances and uncertainties. So $pre(\Omega) \supseteq \bigcup_{q\in Q} pre_q(\Omega)$.

On the other hand, for the states $x \notin \bigcup_{q\in Q} pre_q(\Omega)$, x is not contained in any $pre_q(\Omega)$ for every $q \in Q$. Therefore, under all possible modes $q \in Q$, there always exists some disturbance or some uncertain parameter such that no admissible control signal exists to drive the state x into Ω. So $pre(\Omega) \subseteq \bigcup_{q\in Q} pre_q(\Omega)$.

□

Therefore, we only need to calculate the one-step predecessor set for each q-th subsystem.

3.2 Predecessor Sets for Subsystems

In the sequel, we will focus on the linear constrained case. It is known that often in practice uncertainties enter linearly in the system model and they are linearly constrained [8]. To handle this particular but interesting case, we consider the class of polyhedral sets. Such sets have been considered in previous works addressing the control of systems with input and state constraints. Their main advantage is that they are suitable for computation. Therefore, in the sequel, we turn to polytopic uncertainty in $A_q(w)$ and $B_q(w)$ for every mode $q \in Q$. Without loss of generality, we assume that

$$A_q(w) = \sum_{k=1}^{v_q} w_q^k A_q^k, \; B_q(w) = \sum_{k=1}^{v_q} w_q^k B_q^k,$$

where $w_q^k \geq 0$ and $\sum_{k=1}^{v_q} w_q^k = 1$. The pair $(A_q(w), B_q(w))$ represents the model uncertainty which belongs to the polytopic set $Conv\{(A_q^k, B_q^k), \; k = 1, \cdots, v_q\}$ for each mode $q \in Q$. This is referred to as polytopic uncertainty and provides a classical description of model uncertainty. Notice that the coefficients w_q^k are unknown and possibly time varying.

The difficulty in calculating $pre_q(\Omega)$ comes mainly from the fact that the region Ω is typically non-convex. Even if one starts with convex sets, the procedure deduces non-convex sets for piecewise linear systems after an one-step predecessor operation. Because of the non-convexity, some of the linearity and convexity arguments do not hold and extra care should be taken. However, under the polytopic uncertainty assumption, the calculation of the predecessor set for piecewise linear sets can be simplified, in view of the following proposition.

Proposition 3.2 For polytopic uncertain piecewise linear systems, the robust one-step predecessor set for an assigned piecewise linear set Ω (may be non-convex) under the q-th subsystem can be determined from

$$pre_q(\Omega) = \bigcap_{k=1}^{v_q} pre_q^k(\Omega),$$

where $pre_q^k(\Omega)$ stands for the one-step predecessor operator of the k-th vertex state matrix (A_q^k, B_q^k) for $1 \leq k \leq v_q$, i.e.

$$pre_q^k(\Omega) = \{x \in \mathcal{P}_q \mid \exists u \in \mathcal{U}_q \ : \ A_q^k x + B_q^k u + E_q d \in \Omega, \ \forall d \in \mathcal{D}_q\}.$$

Proof : See Appendix A.

□

Therefore, we derived the relationship between the robust one-step predecessor operator for the polytopic uncertain systems, $pre_q(\cdot)$, and the one-step predecessor set of the vertex dynamics, $pre_q^k(\cdot)$ for $k = 1, \cdots, v_q$. From Proposition 3.2, it turns out that the robust one-step predecessor set for a piecewise linear set Ω under polytopic uncertain linear dynamics can be reduced to the finite intersection of one-step predecessor sets corresponding to the dynamic matrix polytope vertices, which have no parametric uncertainty. The predecessor set under deterministic linear dynamics, $pre_q^k(\Omega)$, has been studied extensively in the literature and can be computed by Fourier-Motzkin elimination [25] and linear programming techniques; see for example [8, 15] and the references therein.

Proposition 3.3 The robust one-step predecessor set for a (non-convex) piecewise linear set Ω, $pre(\Omega)$, can be written as a finite union of polyhedra.

Proof : When Ω is convex polyhedral set, it is trivially true. When Ω is non-convex piecewise linear set, it was shown in [15] that $pre_q^k(\Omega)$ can be written as a finite union of polyhedra. Because $pre_q(\Omega) = \bigcap_k pre_q^k(\Omega)$, hence $pre_q(\Omega)$ can be written as a finite union of polyhedra by repeatedly applying the distributive law. Therefore, the one-step predecessor set of non-convex piecewise linear set Ω, $pre(\Omega) = \bigcup_q pre_q(\Omega)$, can be written as a finite union of polyhedra.

$\square$

Although the convexity is not preserved under the one-step predecessor operation, the piecewise linearity remains unchanged in view of Proposition 3.3. Therefore, one can apply the predecessor operation recursively, and this is explored in the next section.

4 Safety, Direct Reachability and Attainability

In this section, we first present necessary and sufficient conditions for checking the safety for a given region $\Omega \subset \mathcal{X}$ and the direct reachability between two given regions Ω_1 and Ω_2. Then a necessary and sufficient condition for checking the attainability of a given specification is presented. Finally, if the original specification can not be satisfied, a refinement procedure is given such that the refined specifications, if not empty, are guaranteed to be attainable.

4.1 Safety

The following is an important, well-known geometric condition for a set to be safe (controlled invariant) [8].

Theorem 4.1 The set Ω is safe *if and only if* $\Omega \subseteq pre(\Omega)$.

Proof : The proof follows immediately from the definition of the predecessor set $pre(\Omega)$.

$\square$

In general, a given set Ω is not safe. However, Ω may contain safety subsets. In addition, it follows immediately from the definition that the union of two safety sets is also safe. Therefore there exists a unique safety subset in Ω which contains all the safety subsets in Ω. It is called maximal safety set in Ω. In order to calculate the maximal safety set in Ω, we introduce the one-step safety set of Ω as

$$\mathcal{C}_1(\Omega) = pre(\Omega) \cap \Omega.$$

It follows from Proposition 3.3 that if Ω is a piecewise linear set then the one-step safety set $\mathcal{C}_1(\Omega)$ is also piecewise linear. Therefore, the one-step safety set operator can be used recursively to define i-step safety set $\mathcal{C}_i(\Omega)$ as follows.

$$\mathcal{C}_i(\Omega) = pre(\mathcal{C}_{i-1}(\Omega)) \cap \mathcal{C}_{i-1}(\Omega), \text{ for } i \geq 2$$

The sequence of finite-step safety sets $\mathcal{C}_i(\Omega)$ has the following property.

Proposition 4.1 The sequence of finite step safety sets $\mathcal{C}_i(\Omega)$ is decreasing in the sense of

$$\mathcal{C}_i(\Omega) \subseteq \mathcal{C}_{i-1}(\Omega),$$

for $i \geq 1$ and $\mathcal{C}_0(\Omega) = \Omega$. The maximal safety set in Ω for polytopic uncertain piecewise linear system (2.1) is given by

$$\mathcal{C}_\infty(\Omega) = \bigcap_{i=0}^{\infty} \mathcal{C}_i(\Omega).$$

The proof is similar to the proof of Theorem 3.1 in [8], and it is omitted here.

4.2 Direct Reachability

Here we study the reachability problem for uncertain piecewise linear systems. It should be emphasized that we are interested only in the case when reachability between two regions Ω_1 and Ω_2 is defined so that the state is driven to Ω_2 directly from the region Ω_1 in finite steps without entering a third region. This is a problem of practical importance in hybrid systems since it is often desirable to drive the state to a target region of the state space while satisfying constraints on the state and input during the operation of the system.

The problem of deciding whether a region Ω_2 is directly reachable from Ω_1 can be solved by recursively computing all the states that can be driven to Ω_2 from Ω_1 using the predecessor operator. Given Ω_1 and Ω_2, define the robust one-step controllable set as the set consisting of all the states in Ω_1 for which there exist feasible modes and admissible control signals to drive such states into Ω_2 in the next one step, for all allowable uncertainties and disturbances, i.e.

$$\begin{aligned}\mathcal{K}_1(\Omega_1, \Omega_2) = \{x(t) \in \Omega_1 \quad | \quad & \exists q(t) \in Q,\ u(t) \in \mathcal{U}_{q(t)},\ :\ x(t) \in \mathcal{P}_{q(t)}, \\ & A_{q(t)}(w)x(t) + B_{q(t)}(w)u(t) + E_{q(t)}d(t) \in \Omega_2,\ \forall d(t) \in \mathcal{D}_{q(t)},\ w \in \mathcal{W}\}\end{aligned}$$

The robust one-step controllable set $\mathcal{K}_1(\Omega_1, \Omega_2)$ can be computed as follows:

$$\mathcal{K}_1(\Omega_1, \Omega_2) = pre(\Omega_2) \cap \Omega_1$$

It follows from Proposition 3.3 that the robust one-step controllable set $\mathcal{K}_1(\Omega_1, \Omega_2)$ is piecewise linear if Ω_1 and Ω_2 are given as piecewise linear sets. Therefore, the robust one-step controllable set operator can be used recursively to define i-step controllable set $\mathcal{K}_i(\Omega_1, \Omega_2)$ as follows.

$$\mathcal{K}_i(\Omega_1, \Omega_2) = \mathcal{K}_1(\Omega_1, \mathcal{K}_{i-1}(\Omega_1, \Omega_2)),$$

where $i \geq 1$ and $\mathcal{K}_0(\Omega_1, \Omega_2) = \Omega_2$.

With the introduction of the finite step controllable set from Ω_1 to Ω_2, the geometric condition to check the direct reachability can be given as follows.

Theorem 4.2 Consider an uncertain piecewise linear systems and the regions Ω_1 and Ω_2. The region Ω_2 is directly reachable from Ω_1 in finite number of steps *if and only if* there exist finite integer N such that $\Omega_1 \subseteq \bigcup_{i=0}^{N} \mathcal{K}_i(\Omega_1, \Omega_2)$.

Proof : The proof follows immediately from the definition of the finite-step controllable set $\mathcal{K}_i(\Omega_1, \Omega_2)$.

$\square$

4.3 Attainability

Given a finite number of regions $\{\Omega_0,\ \Omega_1, \cdots,\ \Omega_M\}$, the attainability for this sequence of regions is equivalent to the following two different kinds of properties, namely the direct reachability from region Ω_i to Ω_{i+1} for $0 \leq i < M$ and the safety for the final region Ω_M. Therefore, the following necessary and sufficient condition for attainability is derived.

Theorem 4.3 The specification $\{\Omega_0,\ \Omega_1, \cdots,\ \Omega_M\}$ is attainable *if and only if* the following conditions hold: First, Ω_M is safe; and secondly the region Ω_{i+1} is directly reachable from Ω_i, for $i = 0, 1, \cdots, M-1$.

Proof : The proof follows immediately from the attainability definition of a specification $\{\Omega_0,\ \Omega_1, \cdots,\ \Omega_M\}$.

$\square$

Combined with Theorem 4.1 & 4.2, it is straight forward to derive corresponding geometric conditions for attainability. Let us see a numerical example for illustration.

Example 4.1 Consider the discrete-time uncertain piecewise linear hybrid systems,

$$x(t+1) = \begin{cases} A_1(w)x(t) + B_1(w)u(t) + E_1 d(t), & x \in \mathcal{P}_1 \\ A_2(w)x(t) + B_2(w)u(t) + E_2 d(t), & x \in \mathcal{P}_2. \end{cases}$$

where $\mathcal{P}_1 = \{x \in \mathbb{R}^2 |\ \|x\|_\infty \leq 100\}$ and $\mathcal{P}_2 = \{x \in \mathbb{R}^2 |\ -50 \leq x_1 \leq 100,\ -50 \leq x_2 \leq 100\}$. The vertex matrices of polytopic uncertain $A_1(w)$ and $B_1(w)$ are

$$\begin{aligned} A_1^1 &= \begin{pmatrix} 0.825 & 0.135 \\ 0.68 & 1 \end{pmatrix},\ A_1^2 = \begin{pmatrix} 1 & 0.35 \\ 0.068 & 0.555 \end{pmatrix} \\ B_1^1 &= \begin{pmatrix} 1.7 \\ 0.06 \end{pmatrix},\ B_1^2 = \begin{pmatrix} 1.9 \\ 0.08 \end{pmatrix},\ E_1 = \begin{pmatrix} 0.0387 \\ 0.3772 \end{pmatrix}, \end{aligned}$$

and the vertex matrices of polytopic uncertain $A_2(w)$ and $B_2(w)$ are

$$A_2^1 = \begin{pmatrix} -0.664 & 0.199 \\ 0.199 & 0.264 \end{pmatrix}, \; A_2^2 = \begin{pmatrix} -0.7 & 0.32 \\ 0.32 & 0.44 \end{pmatrix}$$
$$B_2^1 = \begin{pmatrix} 0.8 \\ 0.1 \end{pmatrix}, \; B_2^2 = \begin{pmatrix} 0.9 \\ 0.2 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0.1369 \\ 0.5363 \end{pmatrix}$$

Assume that the constraints of the continuous control signal are given as $\mathcal{U}_1 = \mathcal{U}_2 = [-1, 1]$, while the bound of disturbance is $d \in \mathcal{D}_1 = \mathcal{D}_2 = [-0.1, 0.1]$.

First, we consider a region $\Omega_1 = \{x \in \mathbb{R}^2 | \; \|x\|_\infty \leq 10\}$ and calculate the one-step predecessor set of such region, $pre(\Omega_1)$, following the procedure given in Section 3. The set $pre(\Omega_1)$ is plotted in the left part of Figure 2 as a union of two polytopes. It can be seen that $pre(\Omega_1) \supset \Omega_1$, therefore the region Ω_1 is safe by Theorem 4.1.

Next, we consider direct reachability to Ω_1 from another region Ω_0, given as $\Omega_0 = \{x \in \mathbb{R}^2 | \; -20 \leq x_1 \leq 20, \; -40 \leq x_2 \leq 40\}$. It is found that $\Omega_0 \subseteq \bigcup_{i=0}^{3} \mathcal{K}_i(\Omega_0, \Omega_1)$, therefore Ω_1 is directly reachable from Ω_0 in finite number of steps (less or equal to three steps), by Theorem 4.2. The three-step controllable set from Ω_0 to Ω_1 is calculated and plotted in the right part of Figure 2.

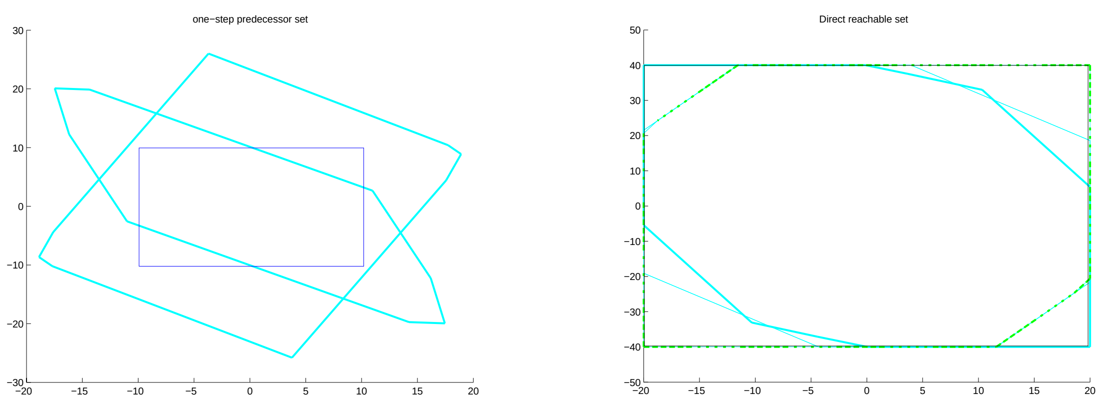

Figure 2: *Left:* The illustration of the one-step predecessor set $pre(\Omega_1)$. *Right:* Three-step controllable set from Ω_0 to Ω_1.

By Theorem 4.3, the attainability of the specification, $\{\Omega_0, \Omega_1\}$, is guaranteed.

4.4 Refinement of the Specification

We have discussed how to check the attainability of a given region-sequence specification $\{\Omega_0, \; \Omega_1, \cdots, \; \Omega_M\}$. However, a given specification $\{\Omega_0, \cdots, \; \Omega_M\}$ may be not attainable.

For example, if we set $\Omega_0 = \{x \in \mathbb{R}^2 | \; -40 \leq x_1 \leq 40, \; -40 \leq x_2 \leq 40\}$ and $N = 3$ in the previous example, then $\bigcup_{i=0}^{N} \mathcal{K}_i(\Omega_0, \Omega_1)$ does not cover Ω_0. Therefore, not every state in Ω_0 can be driven to Ω_1 in N or less steps according to Theorem 4.2. However, we still can specify a subset contained in Ω_0, such as $\bigcup_{i=0}^{N} \mathcal{K}_i(\Omega_0, \Omega_1) \subseteq \Omega_0$, for which admissible control laws exist and drive the states into Ω_1 in N or less steps. Based on this idea, we can determine a sequence of subsets of the previous unattainable region-sequence specification, and the obtained subset sequence is attainable by construction. This process is referred to as *refinement of the specification*. A refinement algorithm is now given:

Algorithma 4.1 Refinement of the Specifications

INPUT: $\{\Omega_1, \cdots, \Omega_M\}$;
$\tilde{\Omega}_M = \mathcal{C}_\infty(\Omega_M)$
for $i = M-1, \cdots, 1$
$\quad \tilde{\Omega}_i = \bigcup_{j=0}^{N} \mathcal{K}_j(\Omega_i, \tilde{\Omega}_{i+1})$
end
OUTPUT: $\{\tilde{\Omega}_1, \cdots, \tilde{\Omega}_M\}$

Here N stands for the upper bound of steps allowed to be taken to reach the successive region. The basic idea is to substitute the region Ω_M with $\mathcal{C}_\infty(\Omega_M)$ so as to satisfy the safety requirement of the final region. And Ω_i is replaced with a finite-step controllable subset to satisfy the reachability requirements. Therefore, the refined specification $\{\tilde{\Omega}_1, \; \tilde{\Omega}_2, \cdots, \; \tilde{\Omega}_M\}$ is attainable if the refinement process can terminate successfully and return non-empty $\tilde{\Omega}_i$ for all $1 \leq i \leq M$. Note that the determination of $\mathcal{C}_\infty(\Omega_M)$ algorithm may fail to terminate in finite number of steps, and that the refined specifications may contain non-convex piecewise linear sets.

5 Controller Synthesis

The hybrid tracking and regulation control problem considered in this section is to select feasible modes $q(t)$ and to design admissible control signals, $u(t) \in \mathcal{U}_{q(t)}$, such that the state trajectory $x(t)$ goes through the regions, namely Ω_0, Ω_1, $\Omega_2 \cdots$, in the specified order and the closed-loop system satisfies some requirements, such as sequencing of events and eventual execution of actions. In Section 4, we specified the conditions for existence of such control laws so that the closed-loop system satisfies safety, reachability and attainability specifications. Here we design such control laws based on optimization techniques. As it was discussed in the previous section, the attainability controller synthesis problem can be divided into two basic problems, that is safety controller synthesis and direct reachability controller synthesis. In the following, we first present a systematic procedure for the controller design for these two basic cases. Then a procedure for attainability controller synthesis is given.

5.1 Safety

First, we consider the safety controller synthesis for the terminating region, Ω_M. Without loss of generality, we assume that Ω_M is a connected piecewise linear set (may be non-convex), and we specify a polytope P_M contained in the interior of Ω_M. Let us assume that $P_M = \{x|G_M x \leq g_M\}$. We define the cost function $J_M : Q \times \mathcal{W} \times \mathcal{U}_q \to \mathbb{R}^+$ as

$$\begin{aligned} J_M(q,w,u) &= \|G_M(A_q(w)x + B_q(w)u)\|_\infty \\ &= \|\sum_{k=1}^{v_q} w_k[G_M(A_q^k x + B_q^k u)]\|_\infty \end{aligned}$$

where $\|\cdot\|_\infty$ stands for the infinite norm[2]. The cost function J_M is in fact the Minkowski function induced from the the convex polytope P_M [8].

Because Ω_M is assumed to be safe, $\Omega_M \subseteq pre(\Omega_M) = \bigcup_q pre_q(\Omega_M)$. Therefore, for any $x \in \Omega_M$, there exists at least one mode $q \in Q$ such that $x \in pre_q(\Omega_M)$. We call such mode feasible for state x. The control signal can be selected as the solution to the following min-max optimization problem for one of such feasible modes q:

$$\begin{aligned} &\min_{u\in\mathcal{U}_q} \max_{w\in\mathcal{W}} J_M(q,w,u) \\ &s.t.\ x \in pre_q(\Omega_M) \end{aligned}$$

The constraint "$x \in pre_q(\Omega_M)$" in the above optimization problem means that the admissible control signal $u \in \mathcal{U}_q$ must keep the state trajectory inside Ω_M despite the uncertainties and disturbances along the mode q. The existence of such control signals comes from the safety of Ω_M and the feasibility of mode q for current state x by assumption. Therefore, we can always select an admissible control signal for a feasible mode q. The optimal action of the controller is one that tries to minimize the maximum cost, to try to counteract the worst disturbance and the worst model uncertainty. In the following, we will describe step by step how to solve the above min-max optimization problem for the controller design.

First, we assume that Ω_M is convex, and it can be represented by $\Omega_M = \{x|F_M x \leq \theta_M\}$. In this case, the constraint "$x \in pre_q(\Omega_M)$" in the above optimization problem can be

[2] For a disconnected (non-convex) piecewise linear set, we may describe the set as a finite number of disjointed connected piecewise linear sets, and induce different cost functions for each of the disjointed piecewise linear sets.

represented as:

$$\begin{aligned} & A_q(w)x + B_q(w)u + E_q d \in \Omega_M \\ \Leftrightarrow \quad & F_M(A_q(w)x + B_q(w)u + E_q d) \leq \theta_M \\ \Leftrightarrow \quad & F_M(A_q(w)x + B_q(w)u) \leq \theta_M - \delta, \end{aligned}$$

where δ is a vector whose components are given by $\delta_j = \max_{d \in \mathcal{D}_q} F_j^T E_q d$, and F_j^T is the j-th row of matrix F_M. The constraint "$F_M(A_q(w)x + B_q(w)u) \leq \theta_M - \delta$", which holds for all $w \in \mathcal{W}$, contains infinite number of constraints. However, according to Proposition 3.2, these infinite number of constraints are in fact equivalent to a finite number of constraints corresponding to the vertex matrices of polytopic uncertain $A_q(w)$ and $B_q(w)$. Therefore, the previous min-max optimization problem can be rewritten as:

$$\min_{u \in \mathcal{U}_q} \max_{w \in \mathcal{W}} J_M(q, w, u)$$

$$s.t. \begin{cases} F_M[A_q^1 x + B_q^1 u] \leq \theta_M - \delta \\ F_M[A_q^2 x + B_q^2 u] \leq \theta_M - \delta \\ \cdots\cdots \\ F_M[A_q^{v_q} x + B_q^{v_q} u] \leq \theta_M - \delta \\ u \in \mathcal{U}_q \end{cases}$$

The min-max optimization problem is usually difficult to solve, so we go one step further to equivalently transform it into a pure minimization problem by introducing an auxiliary variable z [7],

$$\min_{u \in \mathcal{U}_q} z$$

$$s.t. \begin{cases} J_M(q, w, u) \leq z \\ F_M[A_q^1 x + B_q^1 u] \leq \theta_M - \delta \\ F_M[A_q^2 x + B_q^2 u] \leq \theta_M - \delta \\ \cdots\cdots \\ F_M[A_q^{v_q} x + B_q^{v_q} u] \leq \theta_M - \delta \\ u \in \mathcal{U}_q \end{cases}$$

Notice that the constraint "$J_M(q, w, u) \leq z$" should be hold for all parameter uncertainties, and it introduces infinite number of constraints again. Because of the linearity and convexity of J_M, the constraints "$J_M(q, w, u) \leq z$" can be reduced into a finite number of linear constraints based on similar arguments for constraints "$F_M(A_q(w)x + B_q(w)u) \leq \theta_M - \delta$" above. Then the original min-max optimization problem can be equivalently reduced to the following linear programming problem, if the control constraints $\mathcal{U}_q$ is given as a polytope.

$$\min_{u \in \mathcal{U}_q} z$$

$$s.t. \begin{cases} G_M[A_q^1 x + B_q^1 u] \leq \bar{z} \\ G_M[A_q^2 x + B_q^2 u] \leq \bar{z} \\ \cdots\cdots \\ G_M[A_q^{N_q} x + B_q^{N_q} u] \leq \bar{z} \\ -G_M[A_q^1 x + B_q^1 u] \leq \bar{z} \\ -G_M[A_q^2 x + B_q^2 u] \leq \bar{z} \\ \cdots\cdots \\ -G_M[A_q^{v_q} x + B_q^{v_q} u] \leq \bar{z} \\ F_M[A_q^1 x + B_q^1 u] \leq g_M - \delta \\ F_M[A_q^2 x + B_q^2 u] \leq g_M - \delta \\ \cdots\cdots \\ F_M[A_q^{v_q} x + B_q^{v_q} u] \leq g_M - \delta \\ u \in \mathcal{U}_q \end{cases}$$

where $\bar{z}$ stands for a column vector of proper dimension and with all elements being z. Hence the admissible control signal u can be designed for each feasible mode q of the current state x by solving a linear program of the above form. The feasibility of the linear program is guaranteed by the safety of Ω_M and the feasibility of mode q, i.e. $x \in pre_q(\Omega_M)$. Note that there may be more than one modes feasible for the current state x. For such case, a linear program is solved for each feasible mode, and the active mode and control action is selected as the pair that return the minimal value of the cost function. Notice that the values for cost functions of different mode q are comparable, because they are values of Mikowshki functions induced from the same convex set P_M. After the control action is applied and the system evolves along the active mode, the next step state x' is guaranteed to be contained inside Ω_M, namely the safety of Ω_M is guaranteed. Then the controller synthesis process repeated for the new state x' to design next step control signals.

In the following, we will deal with the case when Ω_M is a non-convex piecewise linear set. As it is shown in Proposition 3.3, $pre_q(\Omega_M)$ can be written as a finite union of polyhedra. Assume $pre_q(\Omega_M) = \bigcup_{i=1}^{s} \Phi_i$. For each feasible mode q, $x \in pre_q(\Omega_M) = \bigcup_{i=1}^{s} \Phi_i$, then there exists at least one i, for $1 \leq i \leq s$, such that $x \in \Phi_i$. The admissible control signal $u \in \mathcal{U}_q$, which keeps the state trajectory remaining in Ω_M despite the uncertainties and disturbances along the mode q, can be designed through the following min-max optimization problems.

$$\min_{u \in \mathcal{U}_q} \max_{w \in \mathcal{W}} J_M(q, w, u)$$
$$s.t. \ x \in \Phi_i$$

Because of the convexity of Φ_i, for $i = 1, \cdots, s$, each min-max optimization problem can be reduced into a linear programming problem as shown for the case of convex Ω_M. It is clear that at least one of these linear programs is feasible if q is a feasible mode for x, and the solution u is an admissible control that satisfies the safety specification. It is worth pointing out that the above partition of the non-convex set $pre_q(\Omega_M)$ into finite number of convex sets Φ_i does not imply that each Φ_i is safe. In fact, we only rely on the property that if $x \in \Phi_i \subset pre_q(\Omega_M)$ then admissible control signals to keep the next state x' remaining inside Ω_M (not Φ_i) along the mode q do exist.

Therefore, at each step the admissible hybrid control law is designed by solving a finite number of linear programs. The number of linear programs to be solved depends not only on the number of feasible modes for current state but also on how many polytopic cells Φ_i of the (non-convex) region $pre_q(\Omega_M)$ contains. The number of linear constraints for each linear programming problem is determined by the number of vertex matrices, namely v_q, and the number of faces of the polytopic subregion Φ_i. Hence, the online optimization based controller synthesis method may become complicated and inefficient for large dimensional practical hybrid systems. We will return to this issue in the conclusion part.

5.2 Direct Reachability

Next, we consider the reachability between two successive regions Ω_i and Ω_{i+1}. The control objective is to drive every state in Ω_i to Ω_{i+1} without entering a third region. Similarly, we specify a polytope $P_c \subseteq \Omega_{i+1}$, which can be represented as $P_c = \{x | G_C x \leq g_C\}$. We define the cost functional, $J_C : Q \times \mathcal{W} \times \mathcal{U}_q \to \mathbb{R}^+$ as

$$\begin{aligned} J_C(q, w, u) &= \|G_C(A_q(w)x + B_q(w)u)\|_\infty \\ &= \|\sum_{k=1}^{v_q} w_k[G_C(A_q^k x + B_q^k u)]\|_\infty \end{aligned}$$

Because Ω_{i+1} is direct reachable from Ω_i, so that for all states x in Ω_i, there always exist discrete mode q, and admissible control signal $u \in \mathcal{U}_q$ such that the next state remains in $\Omega_i \cup \Omega_{i+1}$, for all allowable disturbances and uncertainties. Therefore, there always exists $q \in Q$ to make the following min-max optimization problem feasible:

$$\begin{gathered} \min_{u \in \mathcal{U}_q} \max_{w \in \mathcal{W}} J_C(q, w, u) \\ s.t.\ x \in pre_q(\Omega_i \cup \Omega_{i+1}) \end{gathered}$$

This optimization problem is of the same type as the one studied in the safety controller synthesis. Based on similar arguments as in the safety controller synthesis, the above min-max optimization problem can be reduced into a finite number of linear programming problems.

5.3 Tracking Attainable Specifications

Assume that the given tracking and regulation specification $\{\Omega_0, \Omega_1, \cdots, \Omega_M\}$ is attainable. Our task in this section is to select feasible modes $q(t)$ and to design admissible control signals, $u(t) \in \mathcal{U}_{q(t)}$, such that for all the initial states x_0 contained in Ω_0 will be driven into Ω_1, without violating state and input constraints, then into Ω_2 and so on. Finally, the state trajectory will reach the final region Ω_M and stay there.

The controller design procedure is now described: For initial condition $x_0 \in \Omega_0$, determine the feasible modes for x_0 as $act(x_0) = \{q \in Q \mid x_0 \in pre_q(\Omega_0 \cup \Omega_1)\}$. Note that $act(x_0)$ is a non-empty finite set by reachability assumption. For each feasible mode, we employ the reachability controller design procedure for Ω_0 and Ω_1. This can be done by solving a finite number of linear programs. And for each feasible mode q, at least one of the linear programs is feasible and returns an admissible control signal $u \in \mathcal{U}_q$. This claim comes from the assumed direct reachability from Ω_0 to Ω_1. The active mode q and control signal $u(x_0)$ is selected among these admissible control signal pairs as the one that returns the smallest value of the cost function, which represents the best effort being taken to reach Ω_1. Then, the control signal is applied and its corresponding feasible mode q is followed, and the next state x_1 is guaranteed to be contained in Ω_0 or Ω_1, for all possible disturbances and uncertainties. If x_1 does not reach Ω_1, then the reachability controller design procedure for Ω_0 and Ω_1 is taken again to obtain the next step feasible mode and admissible control signal. If, after some steps, $x(t)$ is driven into Ω_1, then the reachability controller design procedure for Ω_1 to Ω_2 is invoked. This procedure is repeated and finally $x(t)$ reaches Ω_M, for which the safety controller synthesis procedure is followed to obtain the active modes and admissible control signals. Similar to the case of reachability controller synthesis, the non-emptiness of the feasible modes, and the feasibility of the finite number of linear programs can be guaranteed for the safety controller synthesis of Ω_M.

The following algorithm describes the controller design procedure that guarantees the attainability for an attainable specification described by $\{\Omega_0, \Omega_1, \cdots, \Omega_M\}$.

Algorithma 5.1 Attainability Control
INPUT: $\{\Omega_0,\ \Omega_1, \cdots, \Omega_M\}$;
for $i = 0,\ 1, \cdots,\ M-1$,
  while $x(t) \in \Omega_i$ and $x(t) \notin \Omega_{i+1}$
  Design reachability controller from Ω_{i} to $\Omega_{\mathrm{i+1}}$
  end
end
Design safety controller for Ω_{M}
OUTPUT: q^*, u^*

The example below illustrates the methods developed in this paper.

Example 5.1 Consider the same example setup as in the previous section. We have defined a specification, $\{\Omega_0, \Omega_1\}$, and checked its attainability. Here we will design the controller to satisfy the tracking and regulation specification. Assume initial condition $x_0 = [16, 36]^T$, which is contained in Ω_0 but not contained in Ω_1. First, design the direct reachability control signal for x_0. Select the cost function as the induced Minkowski function of the set $P_C = \{x \in \mathbb{R}^2 | \|x\|_\infty \leq 10\}$ for each mode $q = 1, 2$. Then solve the optimization problem w.r.t. the cost function for each mode under constraints "$x \in pre_q(\Omega_0 \cup \Omega_1)$". Following the procedure developed above, the active mode and control signal can be designed by solving a finite number of linear programs. In this example, we need to solve two linear programs with 26 linear inequality constraints in u and z each (16 of these constraints are induced from the cost function, 8 constraints come from predecessor constraint, and 2 are control constraints). Here it is shown that the active mode is $q = 2$ and the control signal $u = -1.00 \in \mathcal{U}_2$ return the best control effort. Therefore, if the control signal is applied and the mode $q = 2$ is followed, the next state x_1 is guaranteed to be contained in $\Omega_1 \bigcup \Omega_0$ despite uncertainties and disturbances. If we simulate the state evolution under nominal condition, i.e. setting $\mathcal{D} = \{0\}$ and choosing the epicenter of the state matrix $\frac{1}{2}(A_q^1 + A_q^2)$ for each mode, then we obtain the next step state $x_1 = [-2.42, 16.67]^T$, which is contained in $\Omega_0 \bigcup \Omega_1$ as expected. Because $x_1 \notin \Omega_1$, the reachability regulation design is repeated again, also by solving two linear programs with 26 linear inequality constraints. After one more step, we obtain $x_2 = [5.13, 5.09]^T$ which is contained in Ω_1. Then a safety regulation controller is designed. The cost function is induced from the region Ω_1, because of its convexity. Similarly, we obtain active modes and control signals by solving two linear programs at each step. Simulation results under nominal assumptions are shown in Figure 3. The sequence of selected active mode and admissible control signals of the controller is also illustrated in Figure 3.

6 Practical Stabilization

Several classes of control problems with practical importance can be studied in the framework of tracking and regulation control for uncertain hybrid systems. In this section, we will consider the stabilization problem for uncertain piecewise linear systems and solve it by transforming it into a tracking and regulation control problem.

First, we formulate the stabilization problem. Because of parameter variations and exterior disturbances, it is only reasonable to stabilize the uncertain piecewise linear system within a neighborhood region of the equilibrium. This is called *practical stabilization* or

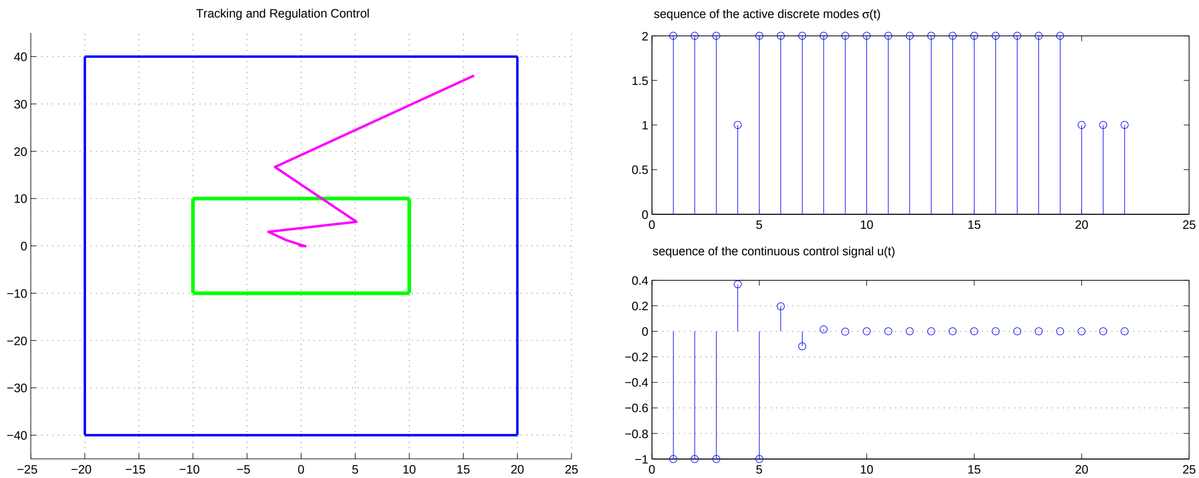

Figure 3: *Left:* Simulation for closed-loop nominal plant (assuming $d = 0$). *Right:* The active mode sequence q and control signals u of the controller.

ultimate boundedness control in the literature; see for example [8] and the references therein. Formally, practical stabilizability can be defined as follows.

Definition 6.1 The discrete-time uncertain piecewise linear system (2.1) is *practically stabilizable* or *uniformly ultimate bounded* in the polytope Ω, which contains the origin in its interior, *if and only if* for every possible initial condition $x(0) = x_0 \in \mathcal{X}$, there exists $T(x_0) > 0$ and admissible control laws, such that $x(t) \in \Omega$ for $t \geq T(x_0)$.

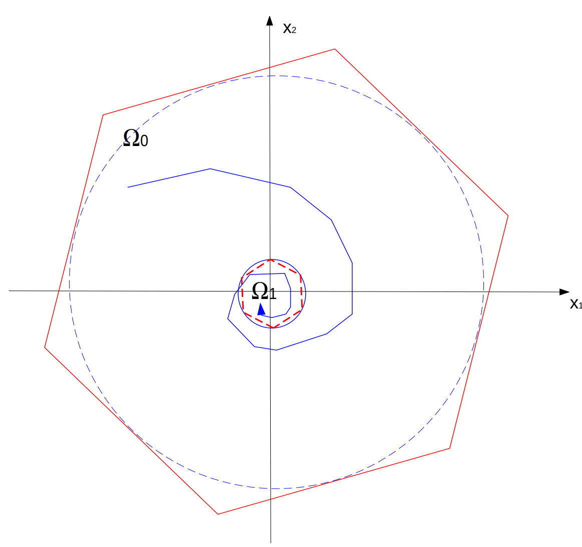

Figure 4: The hybrid systems' ultimate boundedness control as a regulation problem.

Figure 4 illustrates the definition of the practical stabilization above. It is worthy to point out that the region $\mathcal{X}$ may be obtained as an outer-approximation of the bounded region (Ω_0 in the figure) containing all the initial conditions for a semi-global asymptotic practical stabilization problem, while Ω inner-approximates the small neighborhood of the origin (Ω_1 in the figure) as illustrated in Figure 4. Therefore, any semi-global asymptotic practical stabilization problem can be studied in the framework of tracking and regulation control, and two kinds of problems are considered here:

- For a given polytope Ω with the origin contained in its interior, is the discrete-time uncertain piecewise linear system (2.1) practically stabilizable in Ω?
- If yes, then design the practical stabilization control law. If no, find the practically stabilizable region for Ω and design the stabilization control law.

In the following, we will show that the above two practical stabilization problems can be formulated and solved as tracking and regulation problems.

For the first problem, the uncertain piecewise linear system is practically stabilizable in Ω if Ω is a safety set (or has non-empty maximal safety subset $\mathcal{C}_\infty(\Omega)$) and Ω (or $\mathcal{C}_\infty(\Omega)$) is direct reachable from all possible initial states in $\mathcal{X}$, i.e. the region-sequence specification $\{\mathcal{X},\ \Omega\}$ (or $\{\mathcal{X},\ \mathcal{C}_\infty(\Omega)\}$) is attainable by definition. Therefore, the first problem can be solved by checking the attainability of $\{\mathcal{X},\ \Omega\}$ (or $\{\mathcal{X},\ \mathcal{C}_\infty(\Omega)\}$).

By Theorem 4.2, the direct reachability from $\mathcal{X}$ to the target region Ω (or $\mathcal{C}_\infty(\Omega)$) is verified if there exists a finite integer N such that the union of finite-step controllable set $\bigcup_{i=0}^{N}\mathcal{K}_i(\mathcal{X},\Omega)$ (or $\bigcup_{i=0}^{N}\mathcal{K}_i(\mathcal{X},\mathcal{C}_\infty(\Omega))$) equals to the set $\mathcal{X}$.

One interesting observation is that it is not necessary to take the union of the finite step controllable sets for direct reachability checking, because the i-step controllable $\mathcal{K}_i(\mathcal{X},\mathcal{C}_\infty(\Omega))$ has the following non-decreasing property.

Proposition 6.1 The sequence of multiple-step controllable sets $\mathcal{K}_i(\mathcal{X},\mathcal{C}_\infty(\Omega))$ is non-decreasing in the sense of

$$\mathcal{K}_i(\mathcal{X},\mathcal{C}_\infty(\Omega)) \supseteq \mathcal{K}_{i-1}(\mathcal{X},\mathcal{C}_\infty(\Omega)),$$

where $i \geq 1$ and $\mathcal{K}_0(\mathcal{X},\mathcal{C}_\infty(\Omega)) = \mathcal{C}_\infty(\Omega)$.

Proof : Prove by mathematical induction. First, $\mathcal{K}_0(\mathcal{X},\mathcal{C}_\infty(\Omega)) = \mathcal{C}_\infty(\Omega) \subseteq pre(\mathcal{C}_\infty(\Omega)) \cap \mathcal{X} = \mathcal{K}_1(\mathcal{X},\mathcal{C}_\infty(\Omega))$.

Secondly, if we assume $\mathcal{K}_i(\mathcal{X},\mathcal{C}_\infty(\Omega)) \subseteq \mathcal{K}_{i+1}(\mathcal{X},\mathcal{C}_\infty(\Omega))$. Then, $\mathcal{K}_{i+1}(\mathcal{X},\mathcal{C}_\infty(\Omega)) = pre(\mathcal{K}_i(\mathcal{X},\mathcal{C}_\infty(\Omega))) \cap \mathcal{X} \subseteq pre(\mathcal{K}_{i+1}(\mathcal{X},\mathcal{C}_\infty(\Omega))) \cap \mathcal{X} = \mathcal{K}_{i+2}(\mathcal{X},\mathcal{C}_\infty(\Omega))$.

□

The non-decreasing property of the sequence $\mathcal{K}_i(\mathcal{X},\Omega)$ also holds for a safety set Ω. Because of the above proposition and the results in Section 4, we can derive the following answer for the first problem.

Proposition 6.2 For an given polytope Ω with origin contained in its interior, the discrete-time uncertain piecewise linear system (2.1) is practically stabilizable in Ω, if and only if Ω has non-empty maximal safety subset $\mathcal{C}_\infty(\Omega)$, and there exist finite N such that the N-step controllable set $\mathcal{K}_N(\mathcal{X}, \mathcal{C}_\infty(\Omega)) \supseteq \mathcal{X}$.

Proof : To prove the sufficiency, it is assumed that Ω has non-empty maximal safety subset $\mathcal{C}_\infty(\Omega)$. Note that for safety set Ω, $\mathcal{C}_\infty(\Omega) = \Omega$. Also it is assumed that there exist a finite integer N such that the N-step controllable set $\mathcal{K}_N(\mathcal{X}, \mathcal{C}_\infty(\Omega)) \supseteq \mathcal{X}$. Therefore, for all the initial states $x_0 \in \mathcal{X} \subseteq \mathcal{K}_N(\mathcal{X}, \mathcal{C}_\infty(\Omega))$, there exist feasible mode q and admissible control signal u that drive x_0 into $\mathcal{C}_\infty(\Omega)$ with at most N steps according to the definition of $\mathcal{K}_N(\mathcal{X}, \mathcal{C}_\infty(\Omega))$. Hence, there exists $T(x_0) \leq N$, such that $x_{T(x_0)} \in \mathcal{C}_\infty(\Omega) \subseteq \Omega$. As long as the state trajectory starting from x_0 reaches $\mathcal{C}_\infty(\Omega) \subseteq \Omega$, feasible modes and control signals exist to make the trajectory remain inside $\mathcal{C}_\infty(\Omega) \subseteq \Omega$. This is because of the safety of $\mathcal{C}_\infty(\Omega)$. Therefore, the discrete-time uncertain piecewise linear system is practically stabilizable in Ω.

The necessity can be proven by contradiction. Assume that the discrete-time uncertain piecewise linear system is practically stabilizable in Ω. If Ω does not have any safe subsets, i.e. $\mathcal{C}_\infty(\Omega) = \emptyset$, then the state trajectory can not stay inside Ω, which is a contradiction. On the other hand, if there does not exist a finite integer N to make $\mathcal{K}_N(\mathcal{X}, \mathcal{C}_\infty(\Omega)) \supseteq \mathcal{X}$, then, according to Proposition 6.1 and Theorem 4.2, some initial states in $\mathcal{X}$ can not be driven into $\mathcal{C}_\infty(\Omega)$ in finite number of steps. In addition, even if these states might be driven into $\Omega \setminus \mathcal{C}_\infty(\Omega)$, the point is that these trajectories will eventually escape the set Ω in the future, which is implied by the maximality of the maximum safety set $\mathcal{C}_\infty(\Omega) \subseteq \Omega$. For either case, it always leads to a contradiction.

$\square$

If we fail to find a finite N such that $\mathcal{K}_N(\mathcal{X}, \mathcal{C}_\infty(\Omega)) \supseteq \mathcal{X}$, then we consider the second problem, i.e. determining the attractive region to Ω, in fact $\mathcal{C}_\infty(\Omega)$. Similar to the refinement of the specification in Section 4.4, the attractive region of the second problem can be given as $\mathcal{K}_N(\mathcal{X}, \mathcal{C}_\infty(\Omega))$ for N large enough. Therefore, the practical stabilization problem of the uncertain piecewise linear system can be transformed into a tracking and regulation problem. The stabilization control law maybe designed by solving the optimization based controller synthesis problem developed in the previous section.

Example 6.1 If we reformulate the previous two examples in the following way. Consider a semi-global asymptotic practical stabilization problem for the uncertain piecewise linear hybrid systems. Assume that all the initial conditions are contained inside the region $\mathcal{X} = \Omega_0$, and that the target origin neighborhood region is given by $\Omega = \Omega_1$.

First, calculate the maximal safety set in Ω. Because Ω is a safety set by itself, the maximal safety set $\mathcal{C}_\infty(\Omega) = \Omega$. Secondly, check whether all the bounded initial conditions in $\mathcal{X}$ can be driven into $\mathcal{C}_\infty(\Omega)$ in finite steps. Because the non-decreasing property, it is not necessary to take the union of these finite-step controllable sets. It was obtained that $\mathcal{X} \subseteq \mathcal{K}_3(\mathcal{X}, \mathcal{C}_\infty(\Omega))$. Therefore, according to Proposition 6.2, the discrete-time uncertain piecewise linear system is practically stabilizable in Ω. The stabilization control laws may be designed as the Example 5.1 shows.

However, as it has been pointed out, the termination of stabilization procedure described above is not guaranteed. To overcome such difficulties, in [22, 23], we focused on a subclass of such uncertain hybrid systems, uncertain switched systems, and proposed a new robust stabilization procedure based on set-induced Lyapunov functions and convex analysis.

7 Conclusions

In this paper, the tracking and regulation control problem for polytopic uncertain piecewise linear systems was formulated and solved. The existence of a controller such that the closed-loop system follows desired sequence of regions under uncertainties and disturbances was studied first. And a procedure for refinement of the specifications was presented. Then, based on the novel notion of attainability for the desired behavior of uncertain piecewise linear systems, we presented a systematic procedure for robust controller design by using linear programming techniques. The robust stabilization problem for uncertain piecewise linear systems was then formulated and solved in the framework of robust tracking and regulation control developed here.

The contributions of this paper primarily concern the explicit consideration of the time-variant parametric uncertainties and persistent exterior disturbances in hybrid systems. In addition, the exact calculation of the backward predecessor sets and controller design by linear programming techniques gives us necessary and sufficient conditions for safety, direct reachability and attainability analysis and effective methods for hybrid controller synthesis. The predecessor operator, attainability checking and controller design methods for piecewise linear hybrid systems have been implemented in a Matlab toolbox called HyStar [18], which is available upon request from the authors. Note that a new version capable of dealing with parameter uncertainties has also been developed.

As it has been pointed out, there is one difficulty of the online optimization-based controller, namely that the online computational burden may be very heavy for certain cases. Hence, an alterative method was developed in [21] to construct explicit state feedback control laws for safety controller synthesis. The feasible control law was given as piecewise linear state feedback control law, based on the partition of the safety region into finite polytopic subregions, which could be done off-line. More work along this line will be done in our future work. In addition, the controllers used in this paper rely on the accurate state information, which may not be available in practice. Our future work also includes development of set-valued state observers, and observer-based output feedback design methods for uncertain piecewise linear hybrid systems.

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Appendix A

Notations : Given two sets $Y \subset \mathbb{R}^n$ and $Z \subset \mathbb{R}^n$, the complement of Y is $Y^c = \{y \in \mathbb{R}^n : y \notin Y\}$, the Minkovski sum $Y \oplus Z = \{y + z : y \in Y, z \in Z\}$ and the Pontryagin difference $Y \sim Z = \{x \in R^n : x + z \in Y, \forall z \in Z\}$.

Proof for Proposition 3.2 :

For the discrete-time polytopic uncertain q-th subsystem

$$x(t+1) = A_q(w(t))x(t) + B_q(w(t))u(t) + E_q d(t), \quad t \in \mathbb{Z}^+,$$

the robust one-step predecessor set under mode q for a piecewise linear set $\Omega \subseteq \mathcal{X}$ can be derived as follows.

$$\begin{aligned}
pre_q(\Omega) &= \{x \in \mathcal{P}_q \mid \exists u \in \mathcal{U}_q, \ : \ A_q(w)x + B_q(w)u + E_q d \in \Omega, \ \forall w \in \mathcal{W}, d \in \mathcal{D}_q\} \\
&= \{x \in \mathcal{P}_q \mid \exists u \in \mathcal{U}_q \ : \ A_q(w)x + B_q(w)u \in \Omega \sim E_q\mathcal{D}_q, \ \forall w \in \mathcal{W}\} \\
&= Proj_x\{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid [A_q(w), B_q(w)] \begin{bmatrix} x \\ u \end{bmatrix} \in \Omega \sim E_q\mathcal{D}_q, \ \forall w \in \mathcal{W}\} \\
&= Proj_x\{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid \exists w \in \mathcal{W} \ : \ [A_q(w), B_q(w)] \begin{bmatrix} x \\ u \end{bmatrix} \in (\Omega \sim E_q\mathcal{D}_q)^c\}^c
\end{aligned}$$

where $Proj_x$ is the projection operator, which can be calculated by Fourier-Motzkin elimination method. It was shown in [15] that $(\Omega \sim E_q\mathcal{D}_q)^c$ can be written as finite union of convex polyhedral sets Φ_i, that is $(\Omega \sim E\mathcal{D}_q)^c = \cup_i \Phi_i$. Therefore,

$$\begin{aligned}
&\{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid \exists w \in \mathcal{W} \ : \ [A_q(w), B_q(w)] \begin{bmatrix} x \\ u \end{bmatrix} \in (\Omega \sim E_q\mathcal{D}_q)^c\} \\
= &\{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid \exists w \in \mathcal{W} \ : \ [A_q(w), B_q(w)] \begin{bmatrix} x \\ u \end{bmatrix} \in \bigcup_i \Phi_i\} \\
= &\bigcup_i \{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid \exists w \in \mathcal{W} \ : \ [A_q(w), B_q(w)] \begin{bmatrix} x \\ u \end{bmatrix} \in \Phi_i\} \\
= &\bigcup_i [\bigcup_{k=1}^{v_q} \{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid [A_q^k, B_q^k] \begin{bmatrix} x \\ u \end{bmatrix} \in \Phi_i\}]
\end{aligned}$$

And the last equality can be shown as follows. First, it is clear that

$$\{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid \exists w \in \mathcal{W} \ : \ [A_q(w), B_q(w)] \begin{bmatrix} x \\ u \end{bmatrix} \in \Phi_i\} \supseteq \bigcup_{k=1}^{v_q} \{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid [A_q^k, B_q^k] \begin{bmatrix} x \\ u \end{bmatrix} \in \Phi_i\}.$$

Secondly, for all the state and control $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q$, for which there exists w_q^k ($w_q^k \geq 0$ and $\sum_{k=1}^{v_q} w_q^k = 1$) and $\sum_{k=1}^{v_q} [w_q^k A_q^k, w_q^k B_q^k] \begin{bmatrix} x \\ u \end{bmatrix} \in \Phi_i$, there exists at least one k such that

$[A_q^k, B_q^k]\begin{bmatrix} x \\ u \end{bmatrix} \in \Phi_i$. Prove this claim by contradiction, assume that $[A_q^k, B_q^k]\begin{bmatrix} x \\ u \end{bmatrix} \notin \Phi_i$, for all $k = 1, \cdots, v_q$. Then, by convexity of Φ_i it can be shown that for all $w_q^k \geq 0$ and $\sum_{k=1}^{v_q}[w_q^k A_q^k, w_q^k B_q^k]\begin{bmatrix} x \\ u \end{bmatrix} \notin \Phi_i$. This leads to a contradiction. Therefore,

$$\{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid \exists w \in \mathcal{W} \ : \ [A_q(w), B_q(w)]\begin{bmatrix} x \\ u \end{bmatrix} \in \Phi_i\} \subseteq \bigcup_{k=1}^{v_q}\{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid [A_q^k, B_q^k]\begin{bmatrix} x \\ u \end{bmatrix} \in \Phi_i\}.$$

Therefore,

$$\begin{aligned}
&\{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid \exists w \in \mathcal{W} \ : \ [A_q(w), B_q(w)]\begin{bmatrix} x \\ u \end{bmatrix} \in (\Omega \sim E_q\mathcal{D}_q)^c\}^c \\
= \ & [\bigcup_i \{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid \exists w \in \mathcal{W} \ : \ [A_q(w), B_q(w)]\begin{bmatrix} x \\ u \end{bmatrix} \in \Phi_i\}]^c \\
= \ & \{\bigcup_i [\bigcup_{k=1}^{v_q} \{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid [A_q^k, B_q^k]\begin{bmatrix} x \\ u \end{bmatrix} \in \Phi_i\}]\}^c \\
= \ & \{\bigcup_{k=1}^{v_q} [\bigcup_i \{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid [A_q^k, B_q^k]\begin{bmatrix} x \\ u \end{bmatrix} \in \Phi_i\}]\}^c \\
= \ & [\bigcup_{k=1}^{v_q} \{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid [A_q^k, B_q^k]\begin{bmatrix} x \\ u \end{bmatrix} \in \bigcup_i \Phi_i\}]^c \\
= \ & \bigcap_{k=1}^{v_q} \{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid [A_q^k, B_q^k]\begin{bmatrix} x \\ u \end{bmatrix} \in (\bigcup_i \Phi_i)^c\} \\
= \ & \bigcap_{k=1}^{v_q} \{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid [A_q^k, B_q^k]\begin{bmatrix} x \\ u \end{bmatrix} \in \Omega \sim E\mathcal{D}_q\}
\end{aligned}$$

Therefore,

$$\begin{aligned}
pre_q(\Omega) &= Proj_x\{\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{P}_q \times \mathcal{U}_q \mid [A_q(w), B_q(w)]\begin{bmatrix} x \\ u \end{bmatrix} \in \Omega \sim E_q\mathcal{D}_q, \ \forall w \in \mathcal{W}\} \\
&= Proj_x(\bigcap_{k=1}^{v_q}\{\begin{bmatrix} x \\ u \end{bmatrix} \ : \ [A_q^k, B_q^k]\begin{bmatrix} x \\ u \end{bmatrix} \in \Omega \sim E_q\mathcal{D}\}) \\
&= \bigcap_{k=1}^{v_q}(Proj_x\{\begin{bmatrix} x \\ u \end{bmatrix} \ : \ [A_q^k, B_q^k]\begin{bmatrix} x \\ u \end{bmatrix} \in \Omega \sim E_q\mathcal{D}\}) \\
&= \bigcap_{k=1}^{v_q} pre_q^k(\Omega)
\end{aligned}$$

□