Uniformly Ultimate Boundedness Control for Uncertain Switched Linear Systems*

Hai Lin Panos J. Antsaklis†

Department of Electrical Engineering,
University of Notre Dame, Notre Dame, IN 46556, USA
E-mail: {`hlin1, antsaklis.1`}`@nd.edu`

Abstract: In this paper, piecewise linear switched systems affected by both parameter variations and exterior disturbances are considered. The problem of synthesis of switching laws, which assure that the system state is ultimately bounded within a given compact set containing the origin with an assigned rate of convergence, is investigated. Given an uncertain switched linear system, we present a systematic methodology for computing switching laws that guarantee ultimate boundedness. The method is based on set-induced Lyapunov functions. For systems with linearly constrained uncertainties, it is shown that such a function may be derived by numerically efficient algorithms involving polyhedral sets. Based on these Lyapunov functions, we compose global Lyapunov functions that guarantee ultimate boundedness for the switched linear system. The switching laws are characterized by computing conic partitions of the state space.

Keywords: Switched Systems, Uncertainty, Persistent Disturbance, Uniformly Ultimate Boundedness, Set-Induced Lyapunov Functions, Invariant Sets

1 Introduction

A switched system is a dynamical system that consists of a finite number of subsystems described by differential or difference equations and a logical rule that orchestrates switching

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†Corresponding author. Tel: +1(574)631-5792, Fax: +1(574)631-4393.

between these subsystems. Properties of this type of model have been studied for the past fifty years to consider engineering systems that contain relays and/or hysteresis. Recently, there has been increasing interest in the stability analysis and switching control design of switched systems, see for example [9, 6, 2, 3, 16] and the references cited therein. The motivation for studying such switched systems comes partly from the discovery that there exist large class of nonlinear systems which can be stabilized by switching control schemes, but cannot be stabilized by any smooth state feedback control law. In addition, switched systems and switched multi-controller systems have numerous applications in control of mechanical systems, process control, automotive industry, power systems, aircraft and traffic control, and many other fields. Switched systems with all subsystems described by linear differential or difference equations are called piecewise linear/ affine systems or switched linear systems, and have gained the most attention [7, 3, 2, 1]. Recent efforts in switched linear system research typically concentrate on the analysis of the dynamic behaviors, like stability [7, 9, 6], controllability and observability [3, 13] etc., and aim to design controllers with guaranteed stability and performance [2, 13, 7, 1].

In this paper, we will concentrate on robust stabilization problem for the switched linear systems affected by both parameter variations and exterior disturbances. The stability issues of switched systems have been studied extensively in the literature [9, 6], and can be roughly divided into two kinds of problems. One is the stability analysis of switched systems under given switching signals (maybe arbitrary, slow switching etc.), and the other is the synthesis of stabilizing switching signals for a given collection of dynamical systems. The first stability analysis problem is usually dealt with using Lyapunov method, such as common Lyapunov function, multiple Lyapunov functions, see [6, 9] and references therein. Notice that usually (piecewise) quadratic Lyapunov(-like) functions were considered, because of comparable simplicity for calculation by employing LMI techniques. There are less results for the second problem, stabilization switching control for switched systems. Quadratic stabilization for LTI systems was considered in [14], in which it was shown that the existence of a stable convex combination of the subsystem matrices implies the existence of a state-dependent switching rule that stabilizes the switched system along with a quadratic Lyapunov function. There are extensions of [14] to the case of output-dependent switching and discrete-time case [9, 17]. The switching stabilization of second-order LTI systems was considered in [15]. For robust stabilization of polytopic uncertain switched systems, a quadratic stabilizing switching law was designed for polytopic uncertain switched systems based on LMI techniques in [17].

Because of parameter variations and exterior disturbances considered in this paper, it is only reasonable to stabilize the system within a neighborhood region of the equilibrium, which is the so called practical stabilization or ultimate boundedness control in the literature.

In [4], the ultimate boundedness control problem for uncertain discrete-time linear systems was studied based on set-induced Lyapunov functions, and the methods were extended to the continuous-time case in [5]. The problem studied here is *uniformly ultimate boundedness switching control*, that is, to synthesize switching control laws assuring that the system state will be ultimately bounded within a given compact set containing the origin with an assigned rate of convergence. The motivation for considering this problem comes from the following fact. As explained in [8], switching control design methods have become more and more popular. However, switching among these multi-controllers, which are designed with respect to different performance criteria, may leads to undesirable or even unbounded trajectories [6]. Therefore, the stabilizing switching sequences design is not a trivial task and is the central problem in switching control design method, even when all the subsystems are all stable. In addition, by switching among multi-controllers, we can achieve better closed-loop performance than a single controller.

This paper is an extension of our group's recent work [8] to uncertain switched systems[1]. In [8], a class of stabilization switching law for switched autonomous linear time-invariant systems is considered. In the present paper, not only parameter uncertainties in the state matrices but also exterior persistent disturbances are considered in the piecewise linear systems. The rest of the paper is organized as follows. In Section 2, mathematical models for discrete-time and continuous-time switched linear system affected by both parameter variations and exterior disturbances are described, and the ultimate boundedness control problem is formulated. An efficient approach for coping with problems of this kind is based on Lyapunov theory. Section 3 presents the necessary background for set-induced Lyapunov functions. For systems with linearly constrained uncertainties, it is shown that such set-induced Lyapunov functions may be derived by numerically efficient algorithms involving polyhedral sets. Based on these Lyapunov functions, we compose a global Lyapunov function which guarantees ultimate boundedness of the switched linear system. The switching laws are characterized by computing conic partitions of the state space. The technical results for the characterization of stabilizing switching laws are presented in Section 4 & 5, and the approach is illustrated with examples. Finally, concluding remarks are presented and future work is proposed.

In this paper, we use the letters $\mathcal{E}, \mathcal{P}, \mathcal{S} \cdots$ to denote sets. $\partial\mathcal{P}$ stands for the boundary of set $\mathcal{P}$, and $int\{\mathcal{P}\}$ its interior. For any real $\lambda \geq 0$, the set $\lambda\mathcal{S}$ is defined as $\{x = \lambda y,\ y \in \mathcal{S}\}$. The term C-set stands for a convex and compact set containing the origin in its interior.

[1]Previous work along this line has appeared in [10], in which ultimate bounded switching control laws were designed for discrete-time uncertain switched linear systems.

2 Problem Formulation

In this paper, we consider a collection of discrete-time linear systems described by the difference equations with parametric uncertainties

$$x(t+1) = A_q(w)x(t) + E_q d(t), \quad t \in \mathbb{Z}^+, \quad q \in Q = \{1, \cdots, N\} \tag{2.1}$$

where $\mathbb{Z}^+$ stands for non-negative integers. We also consider continuous-time linear systems represented by the differential equations with parametric uncertainties

$$\dot{x}(t) = A_q(w)x(t) + E_q d(t), \quad t \in \mathbb{R}^+, \quad q \in Q = \{1, \cdots, N\} \tag{2.2}$$

where $\mathbb{R}^+$ denotes non-negative real numbers. In the above uncertain discrete-time and continuous-time state equations, the state variable $x(t) \in \mathbb{R}^n$ and the disturbance input $d(t) \in \mathcal{D} \subset \mathbb{R}^r$. Note that the origin $x_e = 0$ is an equilibrium for the systems described in (2.1) and (2.2).

Assume that $\mathcal{D}$ is a C-set, and that the entries of $A_q(w)$ are continuous functions of $w \in \mathcal{W}$, where $\mathcal{W} \subset \mathbb{R}^v$ is an assigned compact set. In particular, for all $q \in Q$, $A_q(w) : \mathcal{W} \to \mathbb{R}^{n\times n}$. Without loss of generality, we assume that $E_q \in \mathbb{R}^{n\times r}$ is a constant matrix. The motivation for considering parametric uncertainty and exterior disturbance in the switched linear system model partially comes from the fact that model uncertainty and exterior disturbances are common in practice and almost unavoidable. In addition, such uncertain switched linear systems may serve as a good candidate for studying uncertain nonlinear systems in a systematic way.

Combine the family of discrete-time uncertain linear systems (2.1) with a class of piecewise constant functions, $\sigma : \mathbb{Z}^+ \to Q$. Then we can define the following linear time-varying system as a discrete-time switched linear system

$$x(t+1) = A_{\sigma(t)}(w)x(t) + E_{\sigma(t)}d(t), \quad t \in \mathbb{Z}^+ \tag{2.3}$$

The signal $\sigma(t)$ is called a *switching signal.* The particular value of the switching signal $\sigma(t)$ at any given time instant t may be generated by a decision-making process. One desirable form of the decision-making process is state feedback based transition law, which can be represented as follows

$$q(t) = \delta(q(t-1), x(t)) \tag{2.4}$$

The discrete mode is determined by the current continuous state $x(t)$ and the previous mode $q(t-1)$[2].

[2]It turns out later that the UUB switching law designed in this paper only depends on the $x(t)$, which is referred to as state space partition based switching law.

Similarly, we introduce a class of piecewise constant functions, $\sigma : \mathbb{R}^+ \to Q$, which serves as the switching signal between the class of continuous-time systems (2.2). The continuous-time switched linear system can be described as

$$\dot{x}(t) = A_{\sigma(t)}(w)x(t) + E_{\sigma(t)}d(t), \quad t \in \mathbb{R}^+ \tag{2.5}$$

and the switching law is determined by

$$q(t) = \delta(q(t^-), x(t)) \tag{2.6}$$

where $t^- = \lim_{\tau\to 0, \tau \geq 0}(t - \tau)$.

For this uncertain discrete-time switched system (2.1)-(2.4) and continuous-time switched system (2.2)-(2.6), we are interested in characterizing the switching law $\delta(\cdot)$ such that the state $x(t)$ asymptotically converges to the equilibrium, $x_e = 0$. Because of the uncertainty and disturbance, we can not drive the state $x(t)$ to the origin exactly, and it is only reasonable to converge into a neighborhood region of the origin. In particular, we introduce the following definition for uniformly ultimate boundedness (UUB).

Definition 2.1 The discrete-time switched system (2.1)-(2.4), or continuous-time switched system (2.2)-(2.6), with the switching law $\delta(\cdot)$ is *Uniformly Ultimately Bounded (UUB)* in the C-set $\mathcal{P}$ if for every initial condition $x(0) = x_0$, there exists $T(x_0) > 0$, such that for $t \geq T(x_0)$, we have $x(t) \in \mathcal{P}$.

The problems being addressed in this paper can be formulated as follows:

Problem : Given the discrete-time switched linear systems (2.1)-(2.4), or continuous-time switched system (2.2)-(2.6), synthesize switching law $\delta(\cdot)$ to assure that the system state $x(t)$ is uniformly ultimately bounded within a given compact set containing the origin with an assigned rate of convergence.

Our methodology for computing switching laws that guarantee ultimate boundedness is based on *set-induced Lyapunov functions*, which will be derived in the next section. For systems with linearly constrained uncertainties, it is shown that such functions may be derived by numerically efficient algorithms involving polyhedral sets. Based on these Lyapunov functions, we compose global Lyapunov functions that guarantee ultimate boundedness for the switched linear system.

3 Set-Induced Lyapunov Functions

In this section, we briefly present some background material necessary for the set-induced Lyapunov functions for uncertain discrete-time and continuous-time systems.

Following the notation of [4], we call a function $\Psi : \mathbb{R}^n \to \mathbb{R}$ a *gauge function* if $\Psi(x) \geq 0$, $\Psi(x) = 0 \Leftrightarrow x = 0$; for $\mu > 0$, $\Psi(\mu x) = \mu\Psi(x)$; and $\Psi(x + y) \leq \Psi(x) + \Psi(y)$, $\forall x, y \in \mathbb{R}^n$. A gauge function is convex and it defines a distance of x from the origin which is linear in any direction. If Ψ is a gauge function, we define the closed set (possibly empty) $\bar{N}[\Psi, \xi] = \{x \in \mathbb{R}^n : \Psi(x) \leq \xi\}$. It is easy to show that the set $\bar{N}[\Psi, \xi]$ is a C-set for all $\xi > 0$. On the other hand, any C-set $\mathcal{S}$ induces a gauge function $\Psi_{\mathcal{S}}(x)$ (Known as Minkowski function of $\mathcal{S}$), which is defined as $\Psi(x) \doteq \inf\{\mu > 0 : x \in \mu\mathcal{S}\}$. Therefore a C-set $\mathcal{S}$ can be thought of as the unit ball $\mathcal{S} = \bar{N}[\Psi, 1]$ of a gauge function Ψ and $x \in \mathcal{S} \Leftrightarrow \Psi(x) \leq 1$.

3.1 Discrete-time System

Consider the subsystem of mode q for the discrete-time uncertain switched linear systems (2.1)-(2.4) as

$$x(t + 1) = A_q(w)x(t) + E_q d(t) \tag{3.1}$$

for which the UUB in a C-set $\mathcal{S}$ is guaranteed by the existence of a Lyapunov function outside $\mathcal{S}$ [5].

In particular, a Lyapunov function outside $\mathcal{S}$ for the subsystem (3.1) can be defined as a continuous function $\Psi : \mathbb{R}^n \to \mathbb{R}^+$ such that $\bar{N}[\Psi, \kappa] \subseteq \mathcal{S}$, for some $\kappa > 0$, for which the following conditions hold:

if $x \notin \bar{N}[\Psi, \kappa]$ then there exists $\beta > 0$ such that

$$\Psi(A(w)x + Ed) - \Psi(x) \leq -\beta;$$

if $x \in \bar{N}[\Psi, \kappa]$ then

$$\Psi(A(w)x + Ed) \leq \kappa.$$

Lemma 3.1 [5] If there exists a Lyapunov function outside $\mathcal{S}$ for the system (3.1), then it is uniformly ultimately bounded (UUB) in $\mathcal{S}$.

In the following, we will assume that for each subsystem (3.1) there exist a corresponding Lyapunov function Ψ_q, with $\bar{N}[\Psi_q, 1] \subseteq \mathcal{S}$. Under this assumption, we will review the procedure for the construction of such Lyapunov function Ψ_q for each subsystem (3.1). For notational simplicity, we will drop the subscript q in this subsection.

It can be derived from the definition of the Lyapunov function Ψ that

$$\Psi(x(t)) \leq \min\{\lambda^t\Psi(x(0)), 1\}$$

for some λ with $0 < \lambda < 1$. This property motivates the following concept of contractive set.

Definition 3.1 Given λ, $0 < \lambda < 1$, a set $\mathcal{S}$ is said λ-*contractive* with respect to subsystem (3.1), if for any $x \in \mathcal{S}$ such that $post_q(x, \mathcal{W}, \mathcal{D}) \subseteq \lambda\mathcal{S}$. Here $post_q(\cdot)$ is defined as

$$post_q(x, \mathcal{W}, \mathcal{D}) = \{x' : x' = A_q(w)x + E_q d;\ \forall w \in \mathcal{W}, d \in \mathcal{D}\}, \tag{3.2}$$

which represents all the possible next step states of system (3.1), given current state $x(t)$.

Let $\mathcal{S}$ be an assigned C-set in $\mathbb{R}^n$. We say that a λ-contractive set $\mathcal{P}_m \subseteq \mathcal{S}$ is *maximal* in $\mathcal{S}$ if and only if every λ-contractive set $\mathcal{P}$ contained in $\mathcal{S}$ is also contained in $\mathcal{P}_m$. Because of the fact that the union of two λ-contractive subsets of $\mathcal{S}$ is also λ-contractive, the existence and uniqueness of $\mathcal{P}_m$ can be easily shown.

Consider the following sequence of sets:

$$\{\mathcal{X}_k\} : \mathcal{X}_0 = \mathcal{S}, \quad \mathcal{X}_k = pre_q(\lambda\mathcal{X}_{k-1}) \cap \mathcal{S}; \quad k = 1, 2, \cdots \tag{3.3}$$

where $pre_q(\mathcal{S})$ is defined as

$$pre_q(\mathcal{S}) = \{x \in \mathbb{R}^n : post_q(x, \mathcal{W}, \mathcal{D}) \subseteq \mathcal{S}\}. \tag{3.4}$$

Then the maximal λ-contractive set $\mathcal{P}_m \subseteq \mathcal{S}$ is given by $\mathcal{P}_\lambda = \bigcap_{k=0}^{\infty} \mathcal{X}_k$.

Proposition 3.1 If $\mathcal{P}_\lambda = \bigcap_{k=0}^{\infty} \mathcal{X}_k$ is nonempty, then the system (3.1) is uniformly ultimately bounded (UUB) in $\mathcal{S}$.

Proof : It can be shown that $\mathcal{P}_\lambda$ is a C-set, when it is nonempty. Let $\psi(x) = \Psi_{\mathcal{P}_\lambda}(x)$ be its Minkowski functional. We have $\psi(x(t+1)) \leq \lambda\psi(x(t))$ for all $x(t) \notin int\{\mathcal{P}_\lambda\}$, and $\bar{N}[\psi, 1] \subset \mathcal{S}$. Then ψ is a Lyapunov function outside $\mathcal{S}$ for the system (3.1). By Lemma 3.1, the existence of a Lyapunov function outside $\mathcal{S}$ implies the UUB of (3.1) in $\mathcal{S}$.

□

Lyapunov function ψ is uniquely generated from the target set $\mathcal{S}$ for any fixed λ. Such a function has been named Set-induced Lyapunov Function (SILF) in the literature [4].

3.2 Continuous-time System

We now consider continuous-time q-th subsystems of the form

$$\dot{x}(t) = A_q(w)x(t) + E_q d(t) \tag{3.5}$$

Parallel to the discrete-time case, we give a definition of the Lyapunov function outside $\mathcal{S}$ for the continuous-time system (3.5).

Definition 3.2 [5] A locally Lipschitz function $\Psi : \mathbb{R}^n \to \mathbb{R}^+$ is a Lyapunov function outside the C-set $\mathcal{S}$ for the continuous-time system (3.5) if $\bar{N}[\Psi, \kappa] \subseteq \mathcal{S}$, for some $\kappa > 0$, and if $x(t) \notin \bar{N}[\Psi, \kappa]$ then there exists $\beta > 0$ such that the Dini derivative of Ψ along the trajectory of the continuous-time system (3.5) satisfies

$$\mathcal{D}^+\Psi(x(t)) \leq -\beta,$$

where the Dini derivative $\mathcal{D}^+\Psi(x(t))$ is defined as

$$\mathcal{D}^+\Psi(x(t)) = \lim \sup_{\tau \to 0, \tau \geq 0} \frac{\Psi(x(t+\tau)) - \Psi(x(t))}{\tau}.$$

It was shown in [5] that the Dini derivative of Ψ at the time instant t, for $x(t) = x$, $d(t) = d$ and $w(t) = w$, can be calculated as

$$\mathcal{D}^+\Psi(x(t)) = \lim \sup_{\tau \to 0, \tau \geq 0} \frac{\Psi(x + \tau[A(w)x + Ed]) - \Psi(x)}{\tau}$$

The existence of the Lyapunov function outside a C-set $\mathcal{S}$ guarantees that the continuous-time system (3.5) is UUB in $\mathcal{S}$. The next question is how to determine such Lyapunov function for the continuous-time system (3.5).

The use of contractive sets allows us to extend results for the discrete-time case to continuous-time systems by introducing the Euler approximating system (EAS), as follows:

$$x(t+1) = [I + \tau A_q(w)]x(t) + \tau E_q d(t), \ \tau > 0 \tag{3.6}$$

In [5], the connection between the continuous-time Lyapunov functions for the continuous-time system (3.5) and the problem of finding a contractive set in $\mathcal{S}$ for the discrete-time Euler approximating system (3.6) is established as the following lemma.

Lemma 3.2 [5] There exists a Lyapunov function Ψ outside a C-set $\mathcal{S}$ for the continuous-time system (3.5) if and only if $\exists \bar{\tau} > 0$ and for all $0 < \tau \leq \bar{\tau}$, there exists a positive scalar $\lambda < 1$, such that $\bar{N}[\Psi, 1]$ is a λ-contractive C-set for the discrete-time EAS (3.6), and $\bar{N}[\Psi, 1] \subseteq \mathcal{S}$. Moreover, the Dini derivative of Ψ satisfies

$$\mathcal{D}^+\Psi(x(t)) < -\beta \tag{3.7}$$

where $\beta = \frac{1-\lambda}{\tau}$.

This lemma shows that the same set-induced Lyapunov function of the EAS (3.6) solves the continuous-time UUB problem.

Therefore, the set-induced Lyapunov function outside $\mathcal{S}$ for the continuous-time system (3.5) can be determined by applying the Procedure (3.3) to its EAS (3.6). And it suggested that a small $\tau > 0$ be first chosen and fix a positive $\lambda < 1$ sufficiently close to 1. If the procedure fails to converge to a C-set, one may reduce τ and reset $\lambda(\tau) = 1 - \rho\tau^2$, which ρ is an arbitrary positive constant [5].

3.3 Linearly Constrained Case

In practice, uncertainties often enter linearly in the system model and they are linearly constrained. To handle this particular but interesting case, we consider the class of polyhedral sets. Such sets have been considered in the literature concerning the control of systems with input and state constraints [4]. Their main advantage is that they are suitable for computation. In the sequel, let us assume polytopic uncertainty in $A_q(w)$. In particular,

$$A_q(w) = \sum_{j=1}^{v} w_j A_q^j, \quad w_j \geq 0, \ \sum_{j=1}^{v} w_j = 1 \tag{3.8}$$

which provides a classical description of model uncertainty. Notice that the coefficients w_j are unknown and possibly time varying.

For computational efficiency, we assume that $\mathcal{D}$ and $\mathcal{S}$ to be polyhedral C-sets, convex and compact polyhedrons containing the origin, and in addition, $\mathcal{S}$ contains the origin in its interior. A convex polyhedral set $\mathcal{S}$ in $\mathbb{R}^n$ can be represented by a set of linear inequalities

$$\mathcal{S} = \{x \in \mathbb{R}^n : f_i x \leq g_i, \ i = 1, \cdots, m\} \tag{3.9}$$

and for brevity, we denote $\mathcal{S}$ as $\{x : Fx \leq g\}$, where $\leq$ is with respect to componentwise. The set $\lambda\mathcal{S}$, $\lambda > 0$, is given by $\{x : Fx \leq \lambda g\}$. Consider the vector δ whose components are

$$\delta_i = \max_{d \in \mathcal{D}} f_i E_q d \geq 0, \quad i = 1, \cdots, m \tag{3.10}$$

The vector δ incorporates the effects of the disturbance $d(t)$. For $\lambda > 0$, we have $post_q(x, \mathcal{W}, \mathcal{D}) \subseteq \lambda\mathcal{S}$ iff $FA_q(w)x \leq \lambda g - \delta$, for all $w \in \mathcal{W}$, which is equivalent to:

$$FA_q^j x \leq \lambda g - \delta, \quad j = 1, \cdots, v \tag{3.11}$$

The above constraints define a convex polyhedron in the space $\mathbb{R}^n$ which is exactly the set $pre_q(\lambda\mathcal{S})$ by definition. Note that the intersection of finite convex polyhedra produces a convex polyhedron. Therefore, the set $\mathcal{X}_1 = pre_q(\lambda\mathcal{S}) \cap \mathcal{S}$ is a convex polyhedron, which is denoted as $\mathcal{X}_1 = \{x : \ F^{(1)}x \leq g^{(1)}\}$. Following the procedure described in (3.3), the set $\mathcal{X}_{k+1} = \{x : F^{(k+1)}x \leq g^{(k+1)}\}$ can be generated inductively as the intersection of $pre_q(\lambda\mathcal{X}_k)$ with $\mathcal{S}$. In view of the convergence of the sequence $\mathcal{X}_k$, $k = 0, 1, \cdots$, we may derive an arbitrarily close external polyhedral approximation of $\mathcal{P}_\lambda$ by $\mathcal{X}_k$ as follows. For every $\lambda^* : \lambda < \lambda^* < 1$, a λ^*-contractive polyhedral C-set $\mathcal{P}_{\lambda^*}$ can be obtained as $\mathcal{P}_{\lambda^*} = \mathcal{X}_k$ for a finite k [4]. Therefore, we can always determine a λ^*-contractive polyhedral C-set $\mathcal{P}_{\lambda^*} \subseteq \mathcal{S}$ in finite number of steps for all λ^*, $\lambda < \lambda^* < 1$, if $\mathcal{S}$ has nonempty λ-contractive subsets. The Minkowski function of a polyhedral C-set $\mathcal{P}$, which can be canonically represented by

$$\mathcal{P} = \{x \in \mathbb{R}^n : f_i x \leq 1, \ i = 1, \cdots, m\}, \tag{3.12}$$

has the following expression

$$\Psi_{\mathcal{P}}(x) = \max_{1 \leq i \leq m} \{f_i x\}. \tag{3.13}$$

In this case, the Minkowski function $\Psi_{\mathcal{P}}$ of $\mathcal{P}$ is called as polyhedral Lyapunov function or piecewise-linear Lyapunov function in the literature, see for example [11, 12, 5] and references therein.

The above procedure to determine a polyhedral Lyapunov function can be immediately extended to the continuous-time case by employing EAS (3.6). We omit the detail here, because of space limitation.

In [5], it was shown that if a Lyapunov function exists and solves the uniform ultimate boundedness problem in a certain convex neighborhood of the origin then there exists a polyhedral Lyapunov function that solves the problem in the same neighborhood. In other words, the polyhedral Lyapunov function is universal. Therefore, without loss of generality, we will restrict to polyhedral Lyapunov functions in the sequel. Another advantage of the polyhedral Lyapunov functions is that it can be determined by numerical methods within finite number of iterations under mild assumption. In addition, the polyhedral Lyapunov functions is suitable for control design, which will be explored in the following sections.

4 Ultimate Boundedness Switching Law

It is known that the stability (or UUB) of all the subsystems can not guarantee the stability (or UUB) of the switched system. Such a switched system might become unbounded for certain switching laws [6, 9]. Therefore, it is important to characterize switching laws that result in ultimately bounded trajectories. In this section, we will present an approach to design the ultimately bounded switching laws for the uncertain discrete-time switched system (2.1)-(2.4) and continuous-time switched system (2.2)-(2.6). This method is based on set-induced Lyapunov functions derived in the previous section.

Recall that the problem we are concerned with is to synthesize switching law $\delta(\cdot)$ so as to assure that the system state $x(t)$ is uniformly ultimately bounded within a given compact set containing the origin, say a polyhedral C-set $\mathcal{T}$, with an assigned rate of convergence, say $0 < \lambda < 1$ (or $\beta > 0$). It is assumed that each individual discrete-time (or continuous-time) subsystem admits a polyhedral Lyapunov function ψ_q outside $\mathcal{T}$, which may be generated by using the procedures described in the previous section. Denote $\mathcal{P}_q = \bar{N}[\psi_q, 1]$, which is a polyhedral C-set contained in $\mathcal{T}$ and can be described as

$$\mathcal{P}_q = \{x \in \mathbb{R}^n : F^q x \leq \bar{1}\} \subseteq \mathcal{T} \tag{4.1}$$

where $F^q \in \mathbb{R}^{m_q \times n}$, $\bar{1} = [1, \cdots, 1]^T \in \mathbb{R}^{m_q}$ and "$\leq$" is with respect to componentwise. It is assumed that $\mathcal{P}_q$ is λ_q-contractive set for the q-th subsystem (or its EAS for some τ_q), where $\lambda_q \leq \lambda$ (or $\beta_q = \frac{1-\lambda_q}{\tau_q} \geq \beta$). We denote the rows of the matrix F^q by $f_i^q \in \mathbb{R}^{1\times n}$, $i = 1, \cdots, m_q$. By Equation (3.13), the Lyapunov function induced by the polyhedral C-set $\mathcal{P}_q$ can be described by $\psi_q(x) = \max_{1\leq i \leq m_q}\{f_i^q x\}$.

First, we briefly describe the necessary notation from convex analysis. Given a polyhedral C-set $\mathcal{P}$, let $vert(\mathcal{P}) = \{v_1, v_2, \cdots, v_N\}$ stands for the vertices of a polytope $\mathcal{P}$, while $face(\mathcal{P}) = \{F_1, F_2, \cdots, F_M\}$ denotes its faces. The hyperplane that corresponds to the k-th face F_k is defined by

$$H_k = \{x \in \mathbb{R}^n : f_k x = 1\} \tag{4.2}$$

where $f_k \in \mathbb{R}^{1\times n}$ is the corresponding gradient vector of face F_k. The set of vertices of F_k can be found as $vert(F_k) = vert(\mathcal{P}) \cap F_k$. Finally, we denote the cone generated by the vertices of F_k by $cone(F_k) = \{x \in \mathbb{R}^n : \sum_i \alpha_i v_{k_i},\ \alpha_i \geq 0,\ v_{k_i} \in vert(F_k)\}$. The $cone(F_k)$ has the property that $\forall x \in cone(F_k)$, $\psi(x) = f_k x$. In Figure 1, illustrations for these concepts are given.

Next we will characterize a conic partition of the state space based on these polyhedral Lyapunov functions $\psi_q(x)$. Consider any pair of subsystems with modes q_1 and q_2, with $q_1 \neq q_2 \in Q$, we want to compute the region

$$\Omega_{q_1}^{q_2} = \{x \in R^n : \psi_{q_1}(x) \leq \psi_{q_2}(x)\} \tag{4.3}$$

For this purpose, we first consider a pair of faces $F_{i_1}^{q_1}$ and $F_{i_2}^{q_2}$ of the polyhedral C-sets $\mathcal{P}_{q_1}$ and $\mathcal{P}_{q_2}$ respectively and consider

$$C_{q_1,i_1}^{q_2,i_2} = cone(F_{i_1}^{q_1}) \cap cone(F_{i_2}^{q_2}) \tag{4.4}$$

The set $C_{q_1,i_1}^{q_2,i_2}$ is either empty or a polyhedral cone. If $C_{q_1,i_1}^{q_2,i_2} \neq \emptyset$, then all the state $x \in C_{q_1,i_1}^{q_2,i_2}$ has the property that, $\psi_{q_1}(x) = f_{i_1}^{q_1} x$ and $\psi_{q_2}(x) = f_{i_2}^{q_2} x$. Next, we intersect the set $C_{q_1,i_1}^{q_2,i_2}$

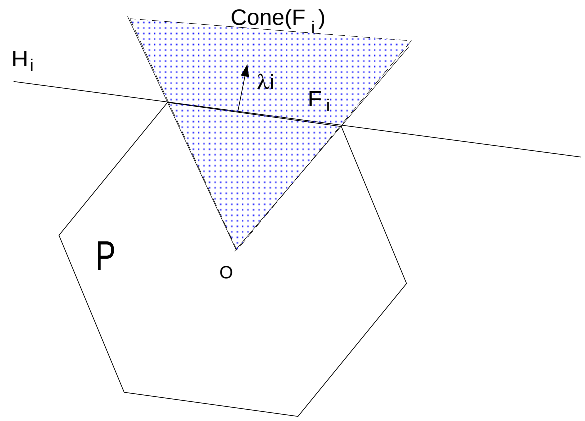

Figure 1: A polyhedral C-set $\mathcal{P}$, its face F_i, the face's corresponding hyperplane H_i and its corresponding polyhedral cone $cone(F_i)$.

with the half-space defined by

$$HF^{q_2,i_2}_{q_1,i_1} = \{x \in R^n : (f^{q_1}_{i_1} - f^{q_2}_{i_2})x \leq 0\} \tag{4.5}$$

and get the set $\Omega^{q_2,i_2}_{q_1,i_1} = C^{q_2,i_2}_{q_1,i_1} \cap HF^{q_2,i_2}_{q_1,i_1}$. The reason for specifying the region $\Omega^{q_2,i_2}_{q_1,i_1}$ can be clarified by the following lemma [8].

Lemma 4.1 For every $x \in \Omega^{q_2,i_2}_{q_1,i_1}$, we have that $\psi_{q_1}(x) \leq \psi_{q_2}(x)$.

Proof : By definition, $\Omega^{q_2,i_2}_{q_1,i_1} = C^{q_2,i_2}_{q_1,i_1} \cap HF^{q_2,i_2}_{q_1,i_1}$, where $C^{q_2,i_2}_{q_1,i_1} = cone(F^{q_1}_{i_1}) \cap cone(F^{q_2}_{i_2})$. The $cone(F^{q_1}_{i_1})$ and $cone(F^{q_2}_{i_2})$ have the property that $\forall x \in cone(F^{q_1}_{i_1})$, $\psi_{q_1}(x) = f^{q_1}_{i_1}x$, and $\forall x \in cone(F^{q_2}_{i_2})$, $\psi_{q_2}(x) = f^{q_2}_{i_2}x$. Note that $\forall x \in HF^{q_2,i_2}_{q_1,i_1}$, $f^{q_1}_{i_1}(x) \leq f^{q_2}_{i_2}(x)$. Therefore, for all $x \in \Omega^{q_2,i_2}_{q_1,i_1}$, we have that $\psi_{q_1}(x) \leq \psi_{q_2}(x)$.

$\square$

The illustration of the conic region $\Omega^{q_2,i_2}_{q_1,i_1}$ is shown in Figure 2. Notice that the hyperplane $H^{(q_2,i_2)}_{(q_1,i_1)} = \{x \in R^n : (f^{q_2}_{i_2} - f^{q_1}_{i_1})x = 0\}$ goes through the origin and the intersection of the faces $F^{q_1}_{i_1}$ and $F^{q_2}_{i_2}$. This comes from the fact that $\psi_{q_1}(0) = \psi_{q_2}(0) = 0$, and for $x \in F^{q_1}_{i_1} \bigcap F^{q_2}_{i_2} \Rightarrow \psi_{q_1}(x) = \psi_{q_2}(x) = 1$. We will show later that this observation simplifies the design procedure for conic partition based switching law.

Based on the above lemma, we have

$$\Omega^{q_2}_{q_1} = \bigcup_{i_1,i_2} \Omega^{q_2,i_2}_{q_1,i_1} \tag{4.6}$$

where i_1 and i_2 go through all the faces' index of $\mathcal{P}_{q_1}$ and $\mathcal{P}_{q_2}$ respectively. And the following corollary holds.

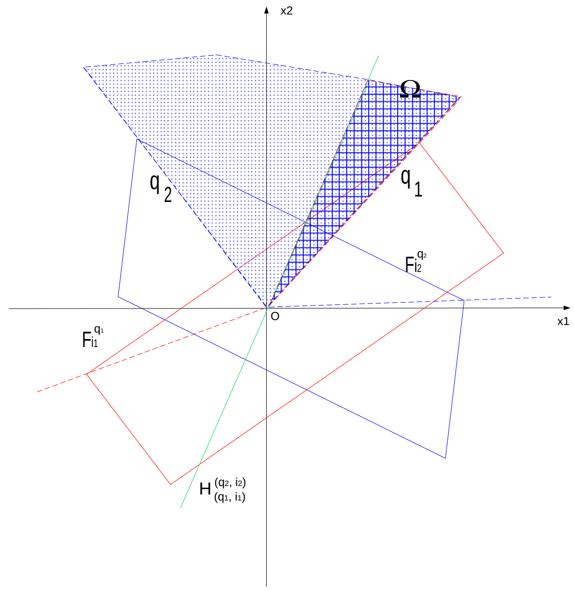

Figure 2: The conic region of Ω.

Corollary 4.1 For every $x \in \Omega_{q_1}^{q_2}$, we have that $\psi_{q_1}(x) \leq \psi_{q_2}(x)$.

Because $\Omega_{q_1,i_1}^{q_2,i_2}$ is an intersection of a polyhedral cone with a half-space, so it is either an empty set or a polyhedral cone. Hence $\Omega_{q_1}^{q_2}$ is finite union of polyhedral cones. And due to the fact that $\bigcup_{i_2} cone(F_{i_1}^{q_1}) = \bigcup_{i_2} cone(F_{i_2}^{q_2}) = \mathbb{R}^n$, it is obvious that for $x \notin \Omega_{q_1}^{q_2}$, we have that $\psi_{q_1}(x) \geq \psi_{q_2}(x)$. Therefore, $\Omega_{q_1}^{q_2} \bigcup \Omega_{q_2}^{q_1} = \mathbb{R}^n$.

Finally, define

$$\Omega_q = \bigcap_{q_i \in Q,\ q_i \neq q} \Omega_q^{q_i}, \tag{4.7}$$

which has the property as follows.

Lemma 4.2 For every $x \in \Omega_q$, we have that $\psi_q(x) \leq \psi_{q_i}(x)$, $\forall q_i \in Q$ and $q_i \neq q$.

Proof : For every $x \in \Omega_q = \bigcap_{q_i \in Q,\ q_i \neq q} \Omega_q^{q_i}$, then $x \in \Omega_q^{q_i}$ for all $q_i \in Q$ and $q_i \neq q$. Therefore, $\psi_q(x) \leq \psi_{q_i}(x)$, $\forall q_i \in Q$ because of Corollary 4.1.

$\square$

Remark 1 Some observations about Ω_q are important for the following design procedure. First, in the region of Ω_q, $q \in \arg\min_{q \in Q} \psi_q(x)$. Secondly, Ω_q is finite union of polyhedral cones. Finally, for $x \in \Omega_q \cap \Omega_{q'}$, $\psi_q(x) = \psi_{q'}(x)$, and $\bigcup_{q \in Q} \Omega_q = \mathbb{R}^n$, so Ω_q, $q \in Q$, serves as a conic partition of the state space.

Based on the above conic partition of the state space given by Ω_q, $q \in Q$, we may define the following switching law:

$$x \in \Omega_q \Rightarrow \delta(\cdot, x) = q \tag{4.8}$$

For the case $x \in \Omega_q \cap \Omega_{q'}$, one simply remains the mode as its previous value, i.e. $\delta(q, x) = q$.

It can be shown that the switching law defined as above can guarantee the UUB for the uncertain discrete-time switched system (2.1)-(2.4).

Theorem 4.1 Consider the class of switching laws defined in (4.8). Then, the uncertain discrete-time switched system (2.1)-(2.4) is UUB in $\bigcup_{q\in Q} \mathcal{P}_q \subseteq \mathcal{T}$ with convergence rate $\lambda = \max_{q\in Q}\{\lambda_q\}$.

Proof : Define the function $V(x) = \min_{q\in Q} \psi_q(x)$. In the following, we will prove that such $V(x)$ is a Lyapunov function for the switched system (2.1)-(2.4) with the specified switching law, $x \in \Omega_q \Rightarrow \delta(\cdot, x) = q$. First, it is straightforward to verify that $V(x)$ is positive definite, $V(x) = 0$ iff $x = 0$ etc. The key point is to show that $V(x)$ decreases along all the trajectories of the switched systems under above switching law. First, for the case of $x \notin int(\bigcup_{q\in Q} \mathcal{P}_q)$. Assume that at time t, $x(t) \in \Omega_q$ and current mode $q(t) = q$. If no switching occur, i.e. $x(t + 1) \in \Omega_q$, then $V(x(t)) = \min_{q\in Q} \psi_q(x(t)) = \psi_q(x(t))$ and $V(x(t + 1)) = \psi_q(x(t + 1)) \leq \lambda_q \psi_q(x(t)) \leq \lambda V(x(t))$. Else, if switching occur at time t, say $x(t + 1) \in \Omega_{q'}$, then $V(x(t + 1)) = \min_{q\in Q} \psi_q(x(t + 1)) = \psi_{q'}(x(t + 1)) \leq \psi_q(x(t + 1)) \leq \lambda_q \psi_q(x(t)) \leq \lambda V(x(t))$. Therefore, for $x \notin int(\bigcup_{q\in Q} \mathcal{P}_q)$, we have $V(x(t + 1)) \leq \lambda V(x(t))$.

Similarly, it can be shown that for $x \in int(\bigcup_{q\in Q} \mathcal{P}_q)$, we have $V(x(t+1)) \leq \lambda$. Therefore, by definition, the uncertain switched system (2.1)-(2.4) is UUB with convergence index λ with the class of switching law defined by $\delta(\cdot) = q$ for $x \in \Omega_q$.

$\square$

Remark 2 Stabilizing switching control laws based on the conic partitions of the state space were previously proposed for second-order linear time invariant switched systems in [15]. Note that the method developed in this paper is for robust stabilization and not restricted to second-order switched systems.

Similarly, it can be shown that the switching law defined as above can guarantee the UUB for the uncertain continuous-time switched system (2.2)-(2.6).

Theorem 4.2 Consider the class of switching laws defined in (4.8). Then, the uncertain continuous-time switched system (2.2)-(2.6) is UUB.

Proof : Define the function $V(x) = \min_{q\in Q} \psi_q(x)$. First, for the case of $x \notin int(\bigcup_{q\in Q} \mathcal{P}_q)$. Assume that at time t, $x(t) \in \Omega_q$ and current mode $q(t) = q$. If no switching occur, then there exist $\bar{\tau} > 0$ such that $\forall 0 < \tau \leq \bar{\tau}$, $x(t+\tau) \in \Omega_q$. Then $V(x(t)) = \min_{q\in Q} \psi_q(x(t)) = \psi_q(x(t))$ and $V(x(t+\tau)) = \psi_q(x(t+\tau))$. Then we derive that

$$\begin{aligned}\mathcal{D}^+V(x(t)) &= \limsup_{\tau\to 0^+} \frac{V(x(t+\tau)) - V(x(t))}{\tau} = \limsup_{\tau\to 0^+} \frac{\psi_q(x(t+\tau)) - \psi_q(x(t))}{\tau} \\ &= \mathcal{D}^+\psi_q(x(t)) \leq -\beta_q\end{aligned}$$

Else, if switching occur at time t, then there exist $\bar{\tau} > 0$ such that $\forall 0 < \tau \leq \bar{\tau}$, $x(t+\tau) \in \Omega_{q'}$. Then $V(x(t)) = \min_{q\in Q} \psi_q(x(t)) = \psi_q(x(t))$ and $V(x(t+\tau)) = \psi_{q'}(x(t+\tau))$. Therefore,

$$\mathcal{D}^+V(x(t)) = \limsup_{\tau\to 0^+} \frac{V(x(t+\tau)) - V(x(t))}{\tau} = \limsup_{\tau\to 0^+} \frac{\psi_{q'}(x(t+\tau)) - \psi_q(x(t))}{\tau}.$$

Note that, $x(t)$ is at the common boundary of Ω_q and $\Omega_{q'}$, i.e. $x(t) \in \Omega_q \cap \Omega_{q'}$, so $\psi_q(x(t)) = \psi_{q'}(x(t))$. Therefore,

$$\mathcal{D}^+V(x(t)) = \limsup_{\tau\to 0^+} \frac{\psi_{q'}(x(t+\tau)) - \psi_{q'}(x(t))}{\tau} \leq -\beta_{q'}.$$

Therefore, $V(x(t))$ is a Lyapunov function outside $\bigcup_{q\in Q} \mathcal{P}_q \subseteq \mathcal{T}$, which implies the UUB of the uncertain continuous-time switched system (2.2)-(2.6) under the above switching law.

$\square$

5 Improved Disturbance Attenuation Property

We have presented a methodology for the partition of the state space into conic regions which are used to characterize a class of stabilizing switching laws. However, as is shown in the proof of Theorem 4.1 and 4.2, the region, to which all the trajectories converge, is the union of the subsystems' contractive sets, i.e. $\bigcup_{q\in Q} \mathcal{P}_q$. As a main motivation to study switched systems, the multi-modal controller can achieve better performance level than single-modal controller. Therefore, the question left is whether we can improve the performance in the sense of converging to a smaller region by refining the previous switching control law. The answer is positive. In the following, we will show that by refining the previous switching law, all the trajectories will finally converge to the intersection of $\mathcal{P}_q$, i.e. $\bigcap_{q\in Q} \mathcal{P}_q$.

To explain how the refinement works, we define

$$\Omega^q = \bigcap_{q_i\in Q,\ q_i\neq q} \Omega^q_{q_i}, \tag{5.1}$$

which has the property as follows.

Lemma 5.1 For every $x \in \Omega^q$, we have that $\psi_q(x) \geq \psi_{q_i}(x)$, $\forall q_i \in Q$ and $q_i \neq q$.

Proof : By definition, $\Omega^q = \bigcap_{q_i \in Q,\ q_i \neq q} \Omega^q_{q_i}$. Therefore, $\forall x \in \Omega^q = \bigcap_{q_i \in Q,\ q_i \neq q} \Omega^q_{q_i}$, then $x \in \Omega^q_{q_i}$, for all $q_i \in Q$, $q_i \neq q$. Note that $\forall x \in \Omega^q_{q_i}$, $\psi_{q_i}(x) \leq \psi_q(x)$, which is from the definition of $\Omega^q_{q_i}$ and Corollary 4.1. Hence, for every $x \in \Omega^q$, $\psi_q(x) \geq \psi_{q_i}(x)$, $\forall q_i \in Q$ and $q_i \neq q$.

$\square$

Similar to the geometric structure of Ω_q, Ω^q is finite union of polyhedral cones and form a conic partition of the state space. In addition, in the region of Ω^q, $q \in \arg\max_{q \in Q} \psi_q(x)$. Based on the conic partition of the state space given by Ω^q, $q \in Q$, we define another switching law:

$$x \in \Omega^q \Rightarrow \delta(\cdot, x) = q \tag{5.2}$$

Also, when $x \in \Omega^q \cap \Omega^{q'}$, simply remain the previous mode.

It can be shown that the switching law defined as above can guarantee the UUB for the uncertain switched system (2.2)-(2.6) in $\bigcap_{q \in Q} \mathcal{P}_q$.

Theorem 5.1 Consider the class of switching law defined by $\delta(\cdot, x) = q$ if x is contained in Ω^q (5.1). Then, the uncertain continuous-time switched system (2.2)-(2.6) is UUB in the polyhedral C-set $\bigcap_{q \in Q} \mathcal{P}_q$.

Proof : Define the function $V(x) = \max_{q \in Q} \psi_q(x)$. For all $x(t) \notin \bigcap_{q \in Q} \mathcal{P}_q$, $V(x(t)) = \max_{q \in Q} \psi_q(x) > 1$. Assume that $x(t) \in \Omega^q$ and current mode $q(t) = q$. If no switching occurs at t, then there exists $\bar{\tau} > 0$ such that $\forall 0 < \tau \leq \bar{\tau}$, $x(t+\tau) \in \Omega^q$ and $x(t+\tau) \notin int(\mathcal{P}_q)$. Then $V(x(t)) = \max_{q \in Q} \psi_q(x(t)) = \psi_q(x(t))$ and $V(x(t+\tau)) = \psi_q(x(t+\tau))$. Then we derive that

$$\mathcal{D}^+ V(x(t)) = \mathcal{D}^+ \psi_q(x(t)) \leq -\beta_q$$

Else, if switching occurs at time t, then there exists $\bar{\tau} > 0$ such that $\forall 0 < \tau \leq \bar{\tau}$, $x(t+\tau) \in \Omega_{q'}$ and $x(t+\tau) \notin int(\mathcal{P}_{q'})$. Then $V(x(t)) = \max_{q \in Q} \psi_q(x(t)) = \psi_q(x(t)) = \psi_{q'}(x(t))$ and $V(x(t+\tau)) = \psi_{q'}(x(t+\tau))$. Therefore,

$$\mathcal{D}^+ V(x(t)) = \lim_{\tau \to 0^+} \sup \frac{\psi_{q'}(x(t+\tau)) - \psi_{q'}(x(t))}{\tau} \leq -\beta_{q'}.$$

Therefore, the uncertain switched system (2.2)-(2.6) is UUB with respect to the region $\bigcap_{q \in Q} \mathcal{P}_q$.

$\square$

Unfortunately, the refined switching law can not guarantee the discrete-time switched system (2.1)-(2.4) UUB in $\bigcap_{q\in Q}\mathcal{P}_q$. In particular, it is possible that for some $x\in\bigcap_{q\in Q}\mathcal{P}_q$ there may exists w and d, which drives x outside $\bigcap_{q\in Q}\mathcal{P}_q$. This is partially because for the discrete-time case the switching usually doesn't occur exactly at the boundary of the conic region Ω^q.

6 Illustrative Examples

6.1 Simplified Design Procedure

It has been pointed out that some geometric characteristics can be used to simplify the determination of the conic partition Ω_q (or Ω^q). In the following, we will describe the simplified design procedure through an example.

Consider a second order three mode discrete-time switched system, and assume that the target region is given as a polyhedral C-set $\mathcal{T}$, and the assigned rate of convergence is $0<\lambda<1$. Assume that each individual subsystem admits a λ_{q_i}-contractive polyhedral C-set $\mathcal{P}_{q_i}$, $\lambda_{q_i}\le\lambda$ for $i=1,2,3$. Such $\mathcal{P}_{q_i}$ can be generated by the procedure described in (3.3). In Figure 3, the two dimensional case $\mathcal{P}_{q_i}$, for $i=1,2,3$, is plotted.

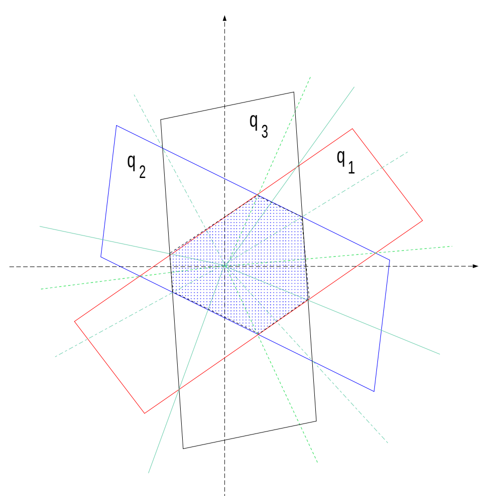

Figure 3: The λ_{q_i}-contractive polyhedral C-set $\mathcal{P}_{q_i}$, $\lambda_{q_i}\le\lambda$ for $i=1,2,3$.

Next, in order to calculate the region $\Omega^{q_2}_{q_1}$, we simply draw the radii that star from the origin and go through the intersection points of faces of $\mathcal{P}_{q_1}$ and $\mathcal{P}_{q_2}$. These radii partition the state space into a finite union of conic regions. Notice that on any such radii, $\psi_{q_1}(x)=\psi_{q_2}(x)$, and that within each conic region generated by these radii either

$\psi_{q_1}(x) \geq \psi_{q_2}(x)$ or $\psi_{q_1}(x) \leq \psi_{q_2}(x)$ holds. Therefore, $\Omega_{q_1}^{q_2}$ is just the union of some of these conic regions. To determine whether one of these polyhedral cones is contained in $\Omega_{q_1}^{q_2}$, one simply checks whether there exists one point in this cone which is on the edge of $\mathcal{P}_{q_1}$ but not contained in $int(\mathcal{P}_{q_2})$. If such points exist in the cone, then this cone is included into the region $\Omega_{q_1}^{q_2}$ (from the geometric interpretation of Minkowski function). The region $\Omega_{q_1}^{q_2}$ is just the union of such cones. Similarly, we obtain $\Omega_{q_1}^{q_3}$. And the region $\Omega_{q_1} = \Omega_{q_1}^{q_2} \bigcap \Omega_{q_1}^{q_3}$, which is illustrated in the leftmost plot in Figure 4. The middle plot of Figure 4 illustrates the region Ω_{q_2}, while Ω_{q_3} is the rightmost plot of Figure 4. And the conic partition of the state space is plotted in Figure 5. From this conic partition, the UUB switching law, $\delta(\cdot) = q_i$ for $x \in \Omega_{q_i}$, can be easily implemented.

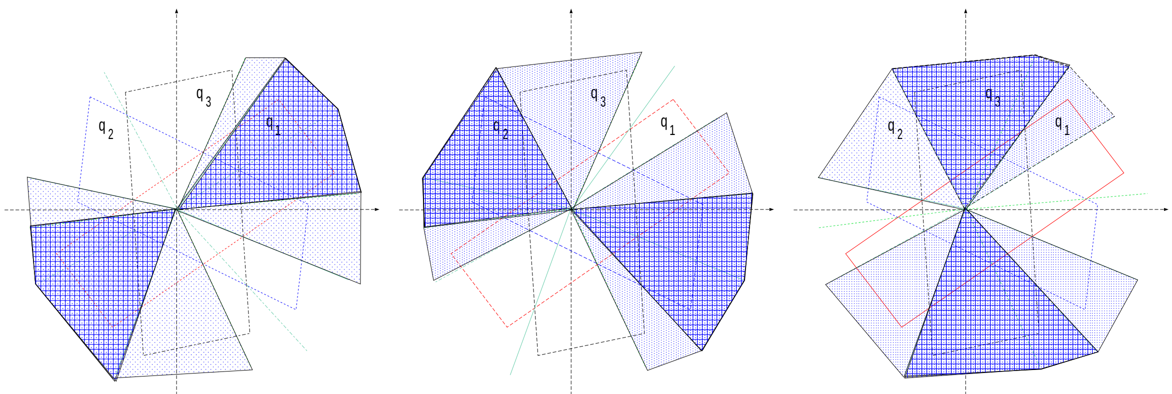

Figure 4: Determine the region of Ω_q as finite union of polyhedral cones.

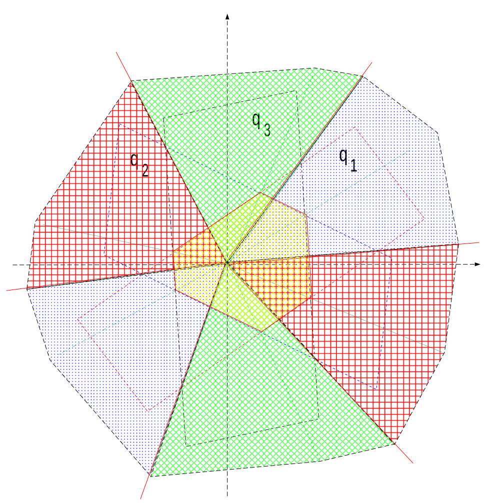

Figure 5: Conic partition based switching law.

Remark 3 In [8], $\Omega_q^{q'}$ was obtained based on the computation of $\Omega_{q,i}^{q',j}$ of all possible pairs of faces, F_i^q and $F_j^{q'}$, of $\mathcal{P}_q$ and $\mathcal{P}_{q'}$ respectively. Therefore, it may be computationally expensive to calculate $\Omega_q^{q'}$. In the present paper, a simplified method is developed to obtain the conic partition $\Omega_q^{q'}$ by employing geometric characteristics of $\mathcal{P}_q$ and $\mathcal{P}_{q'}$ as explained above. In addition, the stabilization switching sequences in [8] is based on partition $\Omega_q^{q'}$, which leads to possibly nondeterministic switching law. However, in this paper the UUB switching law is based on the conic partition Ω_q or Ω^q of the state space, and switching is deterministic.

6.2 Numerical Example

Consider the following continuous-time uncertain switched linear system:

$$\dot{x}(t) = \begin{cases} A_1(w)x(t) + E_1 d(t), & \sigma(t) = q_1 \\ A_2(w)x(t) + E_2 d(t), & \sigma(t) = q_2 \end{cases}$$

In this example the mode set $Q = \{q_1, q_2\}$, and the corresponding state matrices for each subsystem are given as

$$\begin{aligned} A_1(w) &= \begin{bmatrix} -0.9+w & 0.7 \\ -0.7 & -0.9+w \end{bmatrix}, \; E_1 = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix} \\ A_2(w) &= \begin{bmatrix} -0.9+w & 1 \\ 0 & -0.5-w \end{bmatrix}, \; E_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \end{aligned}$$

We assume that the time varying uncertain parameter w is subjected to the constraint $-0.2 \leq w \leq 0.2$, and the continuous variable disturbance $d(t)$ is bounded by $d \in \mathcal{D} = \{d : \|d\|_{l^\infty} \leq 1\} = \{d : -1 \leq d \leq 1\}$.

Assume that the target set $\mathcal{T}$ is given as $\mathcal{T} = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 1\}$, the unit square. We are interested in synthesizing a switching law $\delta(\cdot)$ to assure that the system state $x(t)$ is uniformly ultimately bounded within $\mathcal{T}$.

First, we introduce EAS for each continuous-time subsystems as in (3.6). For example, for subsystem q_1, we may obtain the EAS system with $\tau = 1$:

$$x(t+1) = \begin{bmatrix} 0.1+w & 0.7 \\ -0.7 & 0.1+w \end{bmatrix} x(t) + \begin{bmatrix} 0.3 \\ 0 \end{bmatrix} d(t)$$

Then, by employing the Procedure (3.3) to the above EAS, we derive a λ-contractive set, P_{q_1}, in $\mathcal{T}$, which induces a Lyapunov function $\Psi_{P_{q_1}}$ outside $\mathcal{T}$ for the corresponding continuous-time subsystem. Similarly, we may obtain a set-induced Lyapunov function $\Psi_{P_{q_2}}$ outside $\mathcal{T}$ for the q_2 continuous-time subsystem.

Finally, in order to obtain the conic partition based UUB switching law in $P_{q_1} \cap P_{q_2}$, we simply draw radii starting from the origin and across the intersection points of the edges of P_{q_1} and P_{q_2}. These radii partition the state space into a finite number of conic regions.

The Ω^{q_1} (Ω^{q_2}) is just the union of some of these cones. To determine whether one of these polyhedral cones is contained in Ω^{q_1} (Ω^{q_2}), one simply checks whether there exists one point in this cone which is on the edge of $\mathcal{P}_{q_2}$ ($\mathcal{P}_{q_1}$) but not contained inside $\mathcal{P}_{q_1}$ ($\mathcal{P}_{q_2}$). In Figure 6, the P_{q_1} and P_{q_2} are illustrated, and the conic partition Ω^{q_2} is highlighted. Also a plot of the closed-loop trajectory simulation from initial state $x_0 = [3, -5]^T$ is given under the assumption that $w = 0$ and $\mathcal{D} = \{0\}$.

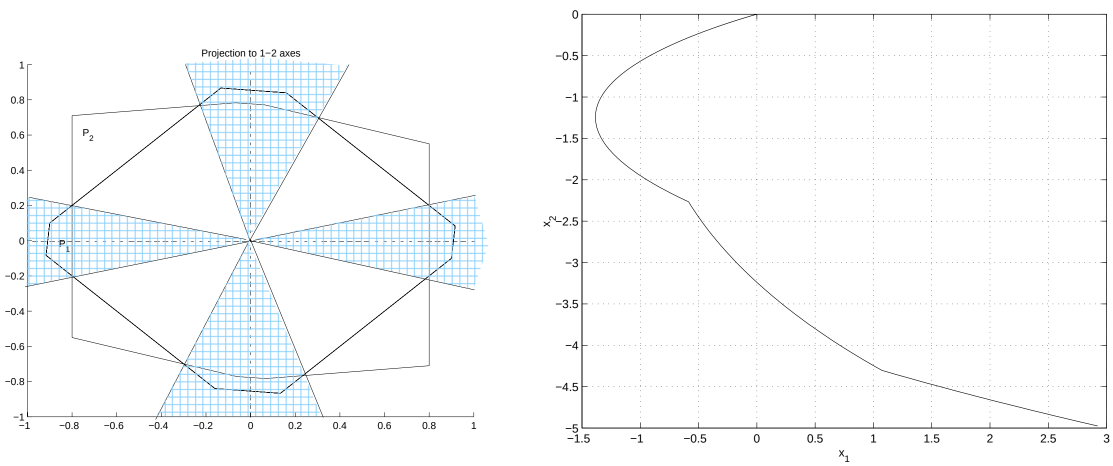

Figure 6: Conic partition based switching law and plot of the closed-loop trajectory simulation from initial state $x_0 = [3, -5]^T$.

7 Conclusion

In this paper, discrete-time and continuous-time switched linear systems affected by both parameter variations and exterior disturbances were considered. The problem of synthesis of switching control law, assuring that the system state is ultimately bounded within a given compact set containing the origin with an assigned rate of convergence, was investigated. Given an uncertain switched linear system, a systematic method for computing UUB switching control laws was proposed. The method was based on set-induced Lyapunov functions. For systems with linearly constrained uncertainties, it was shown that such a function could be derived by numerically efficient algorithms within finite number of iterations. Based on these set-induced Lyapunov functions, a procedure to construct UUB switching control laws based on the conic partition of the state space was presented. The main advantage of the approach is that the methodology for computing switching laws that guarantee stability is based on the parameters of the system, therefore trajectories for particular initial conditions do not need to be calculated. Therefore, the proposed approach can be used very efficiently to investigate the stability properties of practical hybrid/ switched systems.

In this paper, we assume that each individual subsystem is practically stable and admits a set-induced Lyapunov function. This assumption may not be true in some cases, for example when a failure occurs. This is the main drawback of the method developed here. Therefore, it is important to consider the case that not all subsystems are practically stable or all the subsystems are unstable in our future work.

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