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Set-Valued Observer Design for a Class of Uncertain Linear Systems with Persistent Disturbance and Measurement Noise

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Abstract: In this paper, a class of linear systems affected by parameter variations, additive noise and persistent disturbances is considered. The problem of designing a set-valued state observer, which estimates a region containing the real state for each time instant, is investigated. The techniques for designing the observer are based on positive invariant set theory. By constructing a set-induced Lyapunov function, it is shown that the estimation error converges exponentially to a given compact set with an assigned rate of convergence.

Keywords: Set-Valued Observer, Uncertain Linear System, Invariant Sets, Set-Induced Lyapunov Function, Persistent Disturbance, Measurement Noise

1 Introduction

In control theory and engineering, it is often desirable to obtain full state information for control or diagnostic purpose. Therefore it is not surprising that the synthesis of a state observer has been of considerable interest in classical system theory, see for example O'Reilly (1983) and the references therein. The original theory of the state observer involves the asymptotic reconstruction of the state by using exact knowledge of inputs and outputs (Luenberger 1966). However, the real processes are often affected by disturbances and noise. Therefore, the design procedures of state observers were later extended to include the cases when disturbances and/or measurement noise were present. These generalizations may be roughly divided into two main groups. The first group relies on stochastic control approaches,

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which are based on probabilistic models of the disturbances and noise. The stochastic approach provides optimal state estimation based on the probabilistic models of the exogenous signals. Unfortunately, in many cases, no information about the disturbances or noise (in the deterministic or statistical sense) is available, and it can only be assumed that they are bounded in a compact set. Alternatively, disturbances and noise are dealt with in the framework of robust control. Under such framework, optimal state estimation that minimizes the induced-norm from exogenous disturbances and noise to estimation errors is often considered. In Shamma and Tu (1999), an l^1 optimal estimation problem was studied for a class of time varying discrete-time systems with process disturbance and measurement noise, and a set-valued observer, whose centers provided optimal estimates in the sense of l^∞-induced norm, was designed. The optimal l^∞-induced norm estimation problem was also considered in Voulgaris (1995). There also exist results for $\mathcal{H}^\infty$ optimal estimation problems (Nagpal and Khargonekar 1991).

In the previous work on observer design as mentioned above, deterministic dynamics were assumed, where there is no parameter variation in the model. However, it is known that we only have partial knowledge of almost all practical systems. In addition, the system parameters are often subject to unknown, possibly time-varying, perturbations. Therefore it is of practical importance to deal with systems with uncertain parameters. This consideration leads to the robust estimation problem, where robustness is with respect to not only exogenous signals but also model uncertainties. There are some results for the robust estimation problem from a variety of different approaches, see for example Bhattacharyya (1976), Akpan (2001), Collins and Song (2001) and references therein. In Bhattacharyya (1976), the structure features of robust observers in the presence of arbitrary small parameter perturbations were studied from a sensitivity standpoint. A similar problem was considered in Akpan (2001), where a technique for designing robust observers for perturbed linear systems was presented. In Collins and Song (2001), the robust l^1 estimation with plant uncertainties and external disturbance inputs was studied, and the estimator was applied to robust l^1 fault detection. The techniques in Collins and Song (2001) were based on the mixed structured singular value theory. There were also investigations into developing robust estimators using parametric quadratic Lyapunov theory (Haddad and Berstein 1995).

In this paper, we deal with a class of uncertain linear systems affected by both parameter variations and exterior disturbances. The problem studied is the design of a set-valued state observer, which constructs a set of possible state values based on measured outputs and inputs. The techniques used in this paper are based on positive invariant set theory and set-induced Lyapunov functions. By constructing a set-induced Lyapunov function, we can guarantee the ultimate boundedness and convergence rate of the estimation error. The work is inspired by the success of set-induced Lyapunov function together with positive invariant set theory in the fields of robust stability analysis, stabilization, constrained regulation etc, see Blanchini (1994), Bitsoris and Vassilaki (1995). Blanchini (1999) gave a general review of the set invariance theory.

This paper is organized as follows. In Section 2, a mathematical model for uncertain linear systems is described, and the observer design problem is formulated. Section 3 contains the necessary background from invariant set theory, and the definitions of positive $\mathcal{D}$-invariance and strong positive $\mathcal{D}$-invariance are introduced. The approaches to the observer design are developed in Section 4, and the techniques for the set-valued observers' implementation are described in Section 5. The convergence and ultimate boundedness of the estimation error are shown in Section 6. In Section 7, a numerical example is given. Finally, concluding remarks are presented. Note that some preliminary results were presented in Lin, Zhai and Antsaklis (2003). However, in this paper we explicitly deal with parametric variations and measurement noise in the measured output equation. And new techniques are employed to design and implement the set-valued observer for continuous-time systems.

In this paper, we use the letters $\mathcal{E}, \mathcal{P}, \mathcal{S} \cdots$ to denote sets. $\partial\mathcal{P}$ stands for the boundary of set $\mathcal{P}$, and $int\{\mathcal{P}\}$ its interior. For any real $\lambda \geq 0$, the set $\lambda\mathcal{S}$ is defined as $\{x = \lambda y,\ y \in \mathcal{S}\}$. The term C-set stands for a convex and compact set containing the origin in its interior.

2 Problem Formulation

In this paper, we consider linear discrete-time systems described by the difference equation

$$x(t+1) = A(w)x(t) + B(w)u(t) + Ed(t), \quad t \in \mathbb{Z}^+ \tag{2.1}$$

where $\mathbb{Z}^+$ stands for non-negative integers. We also consider linear continuous-time systems represented by the differential equation

$$\dot{x}(t) = A(w)x(t) + B(w)u(t) + Ed(t), \quad t \in \mathbb{R}^+ \tag{2.2}$$

where $\mathbb{R}^+$ denotes non-negative real numbers. Note that in the above uncertain discrete-time and continuous-time state equations, state variable $x(t) \in \mathbb{R}^n$, control input $u(t) \in \mathcal{U} \subset \mathbb{R}^m$, disturbance input $d(t) \in \mathcal{D} \subset \mathbb{R}^r$. Assume that $\mathcal{U}$ and $\mathcal{D}$ are C-sets, and that the entries of the state matrix $A(w)$ and $B(w)$ are continuous functions of $w \in \mathcal{W}$, where $\mathcal{W} \subset \mathbb{R}^v$ is an assigned compact set. In particular, $A(w) : \mathcal{W} \to \mathbb{R}^{n\times n}$ and $B(w) : \mathcal{W} \to \mathbb{R}^{n\times m}$. Without loss of generality, we assume that $E \in \mathbb{R}^{n\times r}$ is a constant matrix. Associated with the above uncertain discrete-time and continuous-time linear systems is the measured output

$$y(t) = C(w)x(t) + n(t) \tag{2.3}$$

where measurement noise $n(t) \in \mathcal{N} \subset \mathbb{R}^p$. Also assume that $\mathcal{N}$ is a C-set, and that the output matrix $C(w)$ is a continuous function from $\mathcal{W}$ to $\mathbb{R}^{p\times n}$.

For this parametric uncertain linear system, we are interested in determining the state $x(t)$ based on the measured output $y(t)$ and control signal $u(t)$. Because of the uncertainty, disturbance and noise, we can not estimate the state $x(t)$ exactly. Therefore, it is reasonable to estimate a region in which the real state is contained, which is called set-valued state estimation in the literature. The problem being addressed in this paper can be formulated as follows:

Problem: *Given the above discrete-time or continuous-time linear uncertain system with the measured output $y(t)$ and input $u(t)$, find $\mathcal{X}(t)$ such that $x(t) + e(t) \in \mathcal{X}(t)$, and assure that the estimation error $e(t)$ is uniformly ultimately bounded in a given C-set, $\mathcal{E}$, with an assigned rate of convergence.*

Here, *uniformly ultimately bounded* in $\mathcal{E}$ means that for any initial value of the estimation error $e(t_0) \notin \mathcal{E}$, $\exists T \geq t_0$ such that for all $t \geq T$, $e(t) \in \mathcal{E}$. The exact meaning of the convergence rate will be explained later in Section 6. Our methodology for designing the observer that guarantees uniformly ultimate boundedness of the estimation error is based on *positive invariant sets* and *set-induced Lyapunov functions*. For systems with linearly constrained uncertainties[1], it is shown that such method can be derived by numerically efficient algorithms involving polyhedral sets.

3 Positive Disturbance Invariance

Consider first the following discrete-time system

$$x(t+1) = A(w)x(t) + Ed(t) \tag{3.1}$$

where d(t) is assumed to be contained in a C-set $\mathcal{D}$.

Definition 3.1 A set $\mathcal{S}$ in the state space is said to be *positive $\mathcal{D}$-invariant (PDI)* for this system if for every initial condition $x(0) \in \mathcal{S}$, $x(t) \in \mathcal{S}$ for $t \geq 0$, for every admissible disturbance $d(t) \in \mathcal{D}$ and every admissible parameter variation $w(t) \in \mathcal{W}$.

In the particular case when $\mathcal{D} = \{0\}$, the positive $\mathcal{D}$-invariance is equivalent to the positive invariance (Blanchini 1999). For the counterpart of continuous-time systems, we have corresponding definitions for invariant set and positive $\mathcal{D}$-invariance.

We shall consider an index of the convergence speed of the state estimation error, and so we need to introduce the following definitions.

Definition 3.2 Let $\mathcal{S}$ be a compact set with nonempty interior in the state space. $\mathcal{S}$ is said to be *strongly positive $\mathcal{D}$-invariant* (SPDI) for system (3.1), if for every initial condition $x(0) \in \mathcal{S}$, for every disturbance sequence $d(t) \in \mathcal{D}$ and every admissible parameter variation $w(t) \in \mathcal{W}$ with $t = 0, 1, \cdots$, we have that $x(t) \in int\{\mathcal{S}\}$ for $t \geq 0$.

If no disturbance exists, namely $\mathcal{D} = \{0\}$, we shall refer to this property as strong positive invariance (SPI). In the discrete-time case, the strong positive invariance of $\mathcal{S}$ is equivalent to the contractivity; note that similar definitions are given in Blanchini (1990) for deterministic linear systems. Next, we introduce the following notation for system (3.1):

$$post(x, \mathcal{W}, \mathcal{D}) = \{x' : x' = A(w)x + Ed;\ \forall w \in \mathcal{W}, d \in \mathcal{D}\} \tag{3.2}$$

which represents all the possible next step states under the transition $A(w)x(t) + Ed(t)$, given current state $x(t)$. It can be shown that a set $\mathcal{S}$ is *strongly positive $\mathcal{D}$-invariant* if and only if $\exists\lambda$, $0 < \lambda < 1$, such that $\mathcal{S}$ is λ-*contractive*, i.e. for any $x \in \mathcal{S}$, $post(x, \mathcal{W}, \mathcal{D}) \subset \lambda\mathcal{S}$.

[1]By linearly constrained uncertainty is meant that the entries of state matrix $A(w)$, $B(w)$ and and output matrix $C(w)$ are linear or affine functions of $w \in \mathcal{W}$.

In the following, we shall assume that $\mathcal{D}$ and $\mathcal{S}$ are convex and compact polyhedrons containing the origin, and in addition, $\mathcal{S}$ contains the origin in its interior. A polyhedral set $\mathcal{S}$ in $\mathbb{R}^n$ can be represented by a set of linear inequalities

$$\mathcal{S} = \{x \in \mathbb{R}^n : f_i^T x \leq \theta_i,\ i = 1, \cdots, s\},$$

and for brevity, we denote $\mathcal{S}$ as $\{x : Fx \leq \theta\}$, where $\leq$ is with respect to componentwise. Let $vert\{\mathcal{S}\}$ stand for the vertices of a polytope $\mathcal{S}$. In the discrete-time case, the following results hold. Note that similar results were given in Blanchini (1990) for deterministic linear systems. The extensions to uncertain dynamics are not difficult, so the details of proof are omitted here for space limit.

Proposition 3.1 The polyhedral region $\mathcal{S} = \{x \in \mathbb{R}^n : Fx \leq \theta\}$ is PDI for system (3.1), *if and only if* for every vertex of $\mathcal{S}$, $v_j \in vert\{\mathcal{S}\}$, and for every vertex of $\mathcal{D}$, $d_h \in vert\{\mathcal{D}\}$, we have $A(w)v_j + Ed_h \in \mathcal{S}$ for all $w \in \mathcal{W}$, or equivalently,

$$FA(w)v_j + FEd_h \leq \theta, \quad \forall w \in \mathcal{W} \tag{3.3}$$

Similarly, we can derive the following result for SPDI.

Corollary 3.1 The polyhedral region $\mathcal{S} = \{x \in \mathbb{R}^n : Fx \leq \theta\}$ is SPDI for system (3.1), *if and only if* $\exists 0 < \lambda < 1$, such that $\forall v_j \in vert\{\mathcal{S}\}$ and $\forall d_h \in vert\{\mathcal{D}\}$, we have

$$FA(w)v_j + FEd_h \leq \lambda\theta, \quad \forall w \in \mathcal{W} \tag{3.4}$$

We now consider continuous-time systems of the form

$$\dot{x}(t) = A(w)x(t) + Ed(t) \tag{3.5}$$

Parallel to the discrete-time case, we can introduce PDI, SPDI concepts for the continuous-time system (3.5). The use of invariant sets allows us to extend results for the discrete-time case to continuous-time systems by introducing the Euler approximating system (EAS), as follows:

$$x(t+1) = [I + \tau A(w)]x(t) + \tau Ed(t) \tag{3.6}$$

It has been proven in Blanchini (1990) that: $\mathcal{S}$ is a SPDI region for a deterministic continuous-time system *if and only if* $\mathcal{S}$ is a SPDI region for its corresponding Euler approximating system for some $\tau > 0$. Therefore, we can derive the following proposition for uncertain continuous-time system (3.5) with polytopic constraints.

Proposition 3.2 The polyhedral region $\mathcal{S} = \{x \in \mathbb{R}^n : Fx \leq \theta\}$ is SPDI for system (3.5), *if and only if* there exists $0 < \lambda < 1$ and $\tau > 0$ such that $[I + \tau A(w)]v_j + \tau Ed_h \in \lambda\mathcal{S}$ holds for $\forall v_j \in vert\{\mathcal{S}\}$, $\forall d_h \in vert\{\mathcal{D}\}$, and $\forall w \in \mathcal{W}$. Synthetically, $\mathcal{S}$ is SPDI if and only if $\exists 0 < \lambda < 1$ and for some $\tau > 0$

$$Fv_j + \tau FA(w)v_j + \tau FEd_h \leq \lambda\theta \tag{3.7}$$

holds for all $w \in \mathcal{W}$.

In the above proposition, there are no indications on how to select τ, for which a small value is usually desirable. To overcome the problem of the choice of τ, we first introduce the following notation. Let C_j be the convex cone for a vertex v_j of $\mathcal{S}$, which is defined by the delimiting planes of $\mathcal{S}$. In particular,

$$C_j = \{f_i^T x \leq \theta_i,\ \theta_i > 0,\ \text{for every } f_i \text{ and } \theta_i,\ \text{s.t. } f_i^T v_j = \theta_i,\ v_j \in vert(\mathcal{S})\} \tag{3.8}$$

which is illustrated in Figure 1. Similar to the case of deterministic dynamics in Blanchini (1990), we derive the following result for uncertain continuous-time system (3.5) with polytopic constraints.

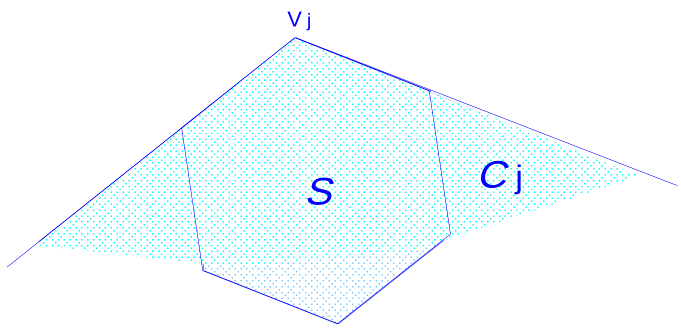

Figure 1: The illustration of convex cone C_j for vertex v_j.

Proposition 3.3 The polyhedral region $\mathcal{S} = \{x \in \mathbb{R}^n : f_i^T x \leq \theta_i,\ i = 1, \cdots, s\}$ is SPDI for system (3.5), *if and only if* for all $\tau > 0$, for every vertex of $\mathcal{S}$, $v_j \in vert\{\mathcal{S}\}$, and for every vertex of $\mathcal{D}$, $d_h \in vert\{\mathcal{D}\}$,

$$v_j + \tau(A(w)v_j + Ed_h) \in \lambda C_j, \quad \forall w \in \mathcal{W},\ \forall j = 1, \cdots, r \tag{3.9}$$

Notice that the inequalities (3.9), which have fewer constraints than inequalities (3.8), are valid for any $\tau > 0$. However, the inequalities (3.8) only hold for some τ. Therefore, we overcome the problem of the choice of τ by introducing the convex cone C_j and deriving conditions valid for any $\tau > 0$ (3.9). In the next section, we will design the set-valued observer based on the SPDI and its properties discussed in this section.

4 Observer Design

In this section, we will present the design procedure for set-valued observers. The method is based on set invariance theory. We will first consider the observer design for discrete-time case, namely for the system described by (2.1) and (2.3). The extension of these results to the continuous-time case will be discussed later in Section 4.2.

4.1 Discrete-Time Case

For discrete-time system described by (2.1) and (2.3), we consider a full state observer of the form

$$\hat{x}(t+1) = (A(w) - LC(w))\hat{x}(t) + B(w)u(t) + Ly(t) \tag{4.1}$$

Assume an admissible disturbance sequence $d_s(t) \in \mathcal{D}$, an admissible noise sequence $n_s(t) \in \mathcal{N}$, and an admissible parameter variation sequence $w_s(t) \in \mathcal{W}$. The corresponding real state trajectory is denoted as $x_s(t)$ for such $d_s(t)$, $n_s(t)$ and $w_s(t)$. At every time step t, the state region estimation of the observer (4.1), $\mathcal{X}(t)$, contains a state estimation $\hat{x}_s(t)$, which corresponds to the specified disturbance sequence $d_s(t)$, noise sequence $n_s(t)$, and parameter variation sequence $w_s(t)$. Then the estimation error for $x_s(t)$ is $e_s(t) = \hat{x}_s(t) - x_s(t)$ which satisfies $e_s(t+1) = (A(w_s) - LC(w_s))e_s(t) + Ln_s(t) - Ed_s(t)$. Considering all possible $w(t) \in \mathcal{W}$, $d(t) \in \mathcal{D}$ and $n(t) \in \mathcal{N}$, we can describe the behavior of the estimation error $e(t) = \hat{x}(t) - x(t)$ by the equation

$$e(t+1) = (A(w) - LC(w))e(t) + Ln(t) - Ed(t) \tag{4.2}$$

Our design objective is to ultimately bound the error $e(t)$ in a given compact set $\mathcal{E}$ for every admissible disturbance $d(t) \in \mathcal{D}$, noise $n(t) \in \mathcal{N}$ and parameter uncertainty $w(t) \in \mathcal{W}$.

Let $\mathcal{E} \subset \mathbb{R}^n$ be a given convex and compact polyhedral set containing the origin in its interior. We assume that $\mathcal{E}$ can be represented as $\mathcal{E} = \{e : Fe \leq \theta\}$, and also assume that the vertices of $\mathcal{E}$ are known. Otherwise a procedure is needed to calculate the vertices of $\mathcal{E}$, for example by solving some linear equations of the form $f_i^T v_j = \theta_i$.

In the discrete-time case, let us assume that $\mathcal{E}$ is SPDI for some given $0 < \lambda < 1$ with respect to state estimation error equation (4.2). Therefore, from Corollary 3.1, the matrix L satisfies the following constraints

$$[A(w) - LC(w)]v_j + Ln - Ed \in \lambda\mathcal{E}, \quad \forall v_j \in vert\{\mathcal{E}\},\ \forall w \in \mathcal{W}, n \in \mathcal{N}, d \in \mathcal{D} \tag{4.3}$$

It is known that in practice uncertainties often enter linearly in the system model and they are linearly constrained. To handle this particular but interesting case, we consider the class of polyhedral sets. Such sets have been considered in the literature addressing the control of systems with input and state constraints (Blanchini 1994, Blanchini 1999). Their main advantage is that they are suitable for computation. Therefore, in the sequel, we consider polytopic uncertainty in $A(w)$, $B(w)$ and $C(w)$. Without loss of generality, we assume that $A(w) = \sum_{k=1}^{v} w_k A_k$, $B(w) = \sum_{k=1}^{v} w_k B_k$, and $C(w) = \sum_{k=1}^{v} w_k C_k$, $w_k \geq 0$, $\sum_{k=1}^{v} w_k = 1$. Notice that the vertex matrices A_k, B_k and C_k are constant matrices of proper dimension respectively. Then the above constraints can be written as

$$f_i^T[\sum_{k=1}^{v} w_k A_k - L\sum_{k=1}^{v} w_k C_k]v_j + f_i^T L n_l \leq \lambda\theta_i - \delta_i$$

$$\forall v_j \in vert\{\mathcal{E}\},\ \forall n_l \in vert\{\mathcal{N}\},\ \forall i = 1, \cdots, s,\ \forall w_k \in [0,1],\ \text{and}\ \sum_{k=1}^{v} w_k = 1$$

where $\delta_i = \max_{d\in\mathcal{D}}(-f_i^T Ed)$, which incorporates the effects of the disturbance $d(t)$. Because of linearity and convexity, it is equivalent to only considering the vertices of $A(w)$ and $C(w)$, i.e.

$$f_i^T[A_k - LC_k]v_j + f_i^T Ln_l \leq \lambda\theta_i - \delta_i,\ \forall v_j \in vert\{\mathcal{E}\},\ n_l \in vert\{\mathcal{N}\},\ \forall i = 1, \cdots, s,\ \forall k = 1, \cdots, v \tag{4.4}$$

For brevity, we write

$$F[A_k - LC_k]v_j + FLn_l \leq \lambda\theta - \delta,\quad \forall v_j \in vert\{\mathcal{E}\},\ n_l \in vert\{\mathcal{N}\},\ \forall k = 1, \cdots, v \tag{4.5}$$

where δ has components as δ_i. We see that the observer design problem is solved if the sets of linear inequalities in the unknown L derived above have a feasible solution. The feasibility of the above linear inequalities is guaranteed by the assumption that $\mathcal{E}$ is SPDI for some given $0 < \lambda < 1$ with respect to system (4.2), and vise versa.

In conclusion, the existence of the set-valued state observer of the form (4.1), whose state estimation error is ultimately bounded[2] in a specified region $\mathcal{E}$, is equivalent to the feasibility of the linear inequalities in (4.5), and it is also equivalent to the condition that $\mathcal{E}$ is SPDI for some given $0 < \lambda < 1$ with respect to system (4.2).

4.2 Continuous-Time Case

For continuous-time system described by (2.2) and (2.3), we consider a full state observer of the form

$$\dot{\hat{x}}(t) = (A(w) - LC(w))\hat{x}(t) + B(w)u(t) + Ly(t) \tag{4.6}$$

The behavior of the estimation error $e(t) = \hat{x}(t) - x(t)$ is described by the equation

$$\dot{e}(t) = (A(w) - LC(w))e(t) + Ln(t) - Ed(t) \tag{4.7}$$

Our design objective is to ultimately bound the error $e(t)$ in a given compact set $\mathcal{E}$ for every admissible disturbance $d(t) \in \mathcal{D}$, noise $n(t) \in \mathcal{N}$ and parameter uncertainty $w(t) \in \mathcal{W}$.

Consider the Euler approximating system (EAS) for the estimation error (4.7) as follows:

$$e(t+1) = [I + \tau(A(w) - LC(w))]e(t) + \tau Ln(t) - \tau Ed(t) \tag{4.8}$$

In the continuous-time case, let us assume that $\mathcal{E}$ is SPDI for some given $0 < \lambda < 1$ with respect to state estimation error equation (4.7). From the discussion of the EAS in the previous section, we know that $\mathcal{E}$ is λ-contractive with respect to the above EAS for some $\tau > 0$. Therefore, from Proposition 3.2, the matrix L fulfills the following constraints for such τ

$$v_j + \tau[A(w) - LC(w)]v_j + \tau Ln - \tau Ed \in \lambda\mathcal{E},\ \forall v_j \in vert\{\mathcal{E}\},\ \forall w \in \mathcal{W}, n \in \mathcal{N}, d \in \mathcal{D} \tag{4.9}$$

The choice of the constant τ is not clear. However, in the previous section, we showed that the problem of selecting τ could be solved by the introduction of the convex cone C_j

[2]The convergence issue of the state estimation error will be discussed in the next section based on the set-induced Lyapunov functions.

corresponding to each vertex v_j of $\mathcal{E}$ (Proposition 3.3). In particular, the polyhedral region $\mathcal{E} = \{e \in \mathbb{R}^n : f_i^T e \leq \theta_i,\ i = 1, \cdots, s\}$ is SPDI for system (4.7), *if and only if* for all $\tau > 0$, and for every vertices of $\mathcal{E}$, $v_j \in vert\{\mathcal{E}\}$,

$$v_j + \tau[A(w) - LC(w)]v_j + \tau Ln - \tau Ed \in \lambda C_j, \quad \forall w \in \mathcal{W}, \tag{4.10}$$

which holds for all noise $n \in \mathcal{N}$ and disturbance $d \in \mathcal{D}$. The above constraints are also equivalent to consider the vertex of $\mathcal{N}$ and $\mathcal{D}$ only. $\mathcal{E}$ is SPDI for system (4.7), *if and only if* for all $\tau > 0$, for all vertices $n_l \in vert\{\mathcal{N}\}$ and $d_h \in vert\{\mathcal{D}\}$, the following constraints hold for every vertices of $\mathcal{E}$,

$$v_j + \tau[A(w) - LC(w)]v_j + \tau Ln_l - \tau Ed_h \in \lambda C_j, \quad \forall w \in \mathcal{W},\ \forall v_j \in vert\{\mathcal{E}\} \tag{4.11}$$

By incorporating the worst case of the disturbance, we further obtain

$$f_i^T v_j + \tau f_i^T [A(w) - LC(w)]v_j + \tau f_i^T Ln_l \leq \lambda\theta_i - \tau\delta_i$$

for every f_i and θ_i, such that $f_i^T v_j = \theta_i$, where $v_j \in vert\{\mathcal{E}\}$, and $\delta_i = \max_{d \in \mathcal{D}}(-f_i^T Ed)$.

If we also assume polytopic uncertainty, i.e. $A(w) = \sum_{k=1}^{v} w_k A_k$, $B(w) = \sum_{k=1}^{v} w_k B_k$, and $C(w) = \sum_{k=1}^{v} w_k C_k$, $w_k \geq 0$, $\sum_{k=1}^{v} w_k = 1$, then the above constraints can be written as

$$f_i^T v_j + \tau f_i^T [\sum_{k=1}^{v} w_k A_k - L \sum_{k=1}^{v} w_k C_k]v_j + \tau f_i^T Ln_l \leq \lambda\theta_i - \tau\delta_i$$

which holds for all $w_k \geq 0$, $\sum_{k=1}^{v} w_k = 1$, for all $n_l \in vert\{\mathcal{N}\}$, and for all $v_j \in vert\{\mathcal{E}\}$. By linearity and convexity, the above constraints hold if and only if they hold for each vertex state matrix A_k, B_k, and C_k,

$$f_i^T v_j + \tau f_i^T [A_k - LC_k]v_j + \tau f_i^T Ln_l \leq \lambda\theta_i - \tau\delta_i,\ \forall k = 1, \cdots, v$$

where $n_l \in vert\{\mathcal{N}\}$, $v_j \in vert\{\mathcal{E}\}$, and $f_i^T v_j = \theta_i$.

Notice that the above constraints hold for every $\tau > 0$. Without loss of generality, let $\tau = 1$. Therefore, for all $v_j \in vert(\mathcal{E})$, we get a collection of linear inequalities in L.

$$f_i^T v_j + f_i^T [A_k - LC_k]v_j + f_i^T Ln_l \leq \lambda\theta_i - \delta_i,\ \forall k = 1, \cdots, v \tag{4.12}$$

where $n_l \in vert\{\mathcal{N}\}$, $v_j \in vert\{\mathcal{E}\}$, and $f_i^T v_j = \theta_i$.

Solving the above linear inequalities in L for all $v_j \in vert(\mathcal{E})$, we get feasible solutions for L, which make the set $\mathcal{E}$ SPDI for the estimation error (4.7).

5 Implementation of the Observer

Note that the observer is set-valued, that is it estimates the region in which the real state resides. The observer maps the set $\mathcal{X}(t)$ to another set as time progresses. We will first consider the implementation of the discrete-time set-valued observer. It turns out that the observer maps a polytope to another polytope and only a finite number of vertex points are necessary to construct the set of state estimation $\mathcal{X}(t)$.

We consider a full state observer of the form

$$\hat{x}(t+1) = (A(w) - LC(w))\hat{x}(t) + B(w)u(t) + Ly(t) \tag{5.1}$$

for discrete-time system described by (2.1) and (2.3). Assume that the initial set $\mathcal{X}(t_0)$ is a polytope, whose vertices, $\hat{x}^i(t_0)$, $i = 1, \cdots, n$, are known. For any $\hat{x}(t_0) \in \mathcal{X}(t_0)$, we have $\hat{x}(t_0) = \sum_{i=1}^{n} \alpha_i \hat{x}^i(t_0)$, where $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. The next corresponding estimated state $\hat{x}(t_1)$, in fact a set, is given by:

$$\hat{x}(t_1) = (A(w) - LC(w))\hat{x}(t_0) + B(w)u(t_0) + Ly(t_0)$$

which is just the linear transformation of $\hat{x}(t_0)$. Note that the linear transformation of a polytope is still a polytope. In addition

$$\begin{aligned}
\hat{x}(t_1) &= (A(w) - LC(w))\hat{x}(t_0) + B(w)u(t_0) + Ly(t_0) \\
&= (A(w) - LC(w)) \sum_{i=1}^{n} \alpha_i \hat{x}^i(t_0) + B(w)u(t_0) + Ly(t_0) \\
&= \sum_{i=1}^{n} \alpha_i [(A(w) - LC(w))\hat{x}^i(t_0) + B(w)u(t_0) + Ly(t_0)]
\end{aligned}$$

If we assume polytopic uncertainty, i.e. $A(w) = \sum_{k=1}^{v} w_k A_k$, $B(w) = \sum_{k=1}^{v} w_k B_k$, and $C(w) = \sum_{k=1}^{v} w_k C_k$, $w_k \geq 0$, $\sum_{k=1}^{v} w_k = 1$, then the implementation of the set-valued observer can be further simplified as:

$$\begin{aligned}
\hat{x}(t_1) &= \sum_{i=1}^{n} \alpha_i [(A(w) - LC(w))\hat{x}^i(t_0) + B(w)u(t_0) + Ly(t_0)] \\
&= \sum_{i=1}^{n} \alpha_i \{ \sum_{k=1}^{v} w_k [A_k - LC_k, B_k] \begin{bmatrix} \hat{x}^i(t_0) \\ u(t_0) \end{bmatrix} + Ly(t_0) \} \\
&= \sum_{i=1}^{n} \sum_{k=1}^{v} \alpha_i \{ w_k [A_k - LC_k, B_k] \begin{bmatrix} \hat{x}^i(t_0) \\ u(t_0) \end{bmatrix} + w_k Ly(t_0) \} \\
&= \sum_{i,k=1}^{n,v} \alpha_i w_k \{ [A_k - LC_k, B_k] \begin{bmatrix} \hat{x}^i(t_0) \\ u(t_0) \end{bmatrix} + Ly(t_0) \} \\
&= \sum_{j=1}^{n \times v} \beta_j \hat{x}^j(t_1)
\end{aligned}$$

where $j = (i-1) \times n + k$, $\hat{x}^j(t_1)$ is the corresponding estimated state corresponding to the vertices $\hat{x}^i(t_0)$ under the vertices A_k, B_k and C_k. Also $\beta_j = (\alpha_i \times w_k) \geq 0$ and $\sum_{j=1}^{n \times v} \beta_j = 1$. Therefore for the case of polytopic uncertainty, the implementation of the observer only needs to consider the finite vertices of state matrices , i.e. (A_k, B_k, C_k) for $k = 1, \cdots, v$, and the finite vertices of the $\mathcal{X}(t)$, i.e. $\hat{x}^i(t)$ for $i = 1, \cdots, n$. In summary, the observer can be described by:

$$\begin{aligned}
\hat{x}^{(1,1)}(t+1) &= (A_1 - LC_1)\hat{x}^1(t) + B_1u(t) + Ly(t) \\
\hat{x}^{(1,2)}(t+1) &= (A_1 - LC_1)\hat{x}^2(t) + B_1u(t) + Ly(t) \\
&\cdots \\
\hat{x}^{(1,n)}(t+1) &= (A_1 - LC_1)\hat{x}^n(t) + B_1u(t) + Ly(t) \\
\hat{x}^{(2,1)}(t+1) &= (A_2 - LC_2)\hat{x}^1(t) + B_2u(t) + Ly(t) \\
&\cdots \\
\hat{x}^{(2,n)}(t+1) &= (A_2 - LC_2)\hat{x}^n(t) + B_2u(t) + Ly(t) \\
&\cdots\cdots \\
\hat{x}^{(v,1)}(t+1) &= (A_v - LC_v)\hat{x}^1(t) + B_ru(t) + Ly(t) \\
&\cdots \\
\hat{x}^{(v,n)}(t+1) &= (A_v - LC_v)\hat{x}^n(t) + B_ru(t) + Ly(t)
\end{aligned}$$

And $\mathcal{X}(t+1) = conv\{\hat{x}^{(1,1)}(t+1), \cdots, \hat{x}^{(v,n)}(t+1)\}$, where $conv\{\cdot\}$ stands for the convex hull. However, in the worst case, the number of the vertices of $\mathcal{X}(t)$ may increase geometrically as the time progresses. In order to deal with such problem, we may outer-approximate $\mathcal{X}(t)$ with, for example, a hyper-rectangle, when the number of the vertices of $\mathcal{X}(t)$ exceeds a threshold.

We now consider the implementation of the observer for continuous-time case. In particular,

$$\dot{\hat{x}}(t) = (A(w) - LC(w))\hat{x}(t) + B(w)u(t) + Ly(t)$$

where L is a feasible solution of the linear inequalities (4.12) for all $v_j \in vert(\mathcal{E})$. For given initial condition $\hat{x}(t_0) = \hat{x}_0$, the implementation of the observer, that is the calculation of $\mathcal{X}(t)$ for each time instant, is in fact an initial value problem for parameter uncertain ordinary differential equations. It should be pointed out that for general polytopic uncertainty there may not exist finite number of solutions which bound the evolution of $\mathcal{X}(t)$. This is due partially to the fact that the image of an interval (or a polytope) under a map is not an interval (or a polytope). Therefore, we have to restrict the class of models that can be dealt with. For example, a class of uncertain linear interval models, whose trajectory have interval boundaries corresponding to two extreme cases, were studied in Cugueró, Puig, Saludes and Escobet (2002). For more comprehensive review and development of interval analysis and computation, see Jaulin, Kieffer, Didrit and Walter (2001) and the references therein.

6 Convergence of the Estimation Error

In this section, we will study the uniformly ultimate boundedness of the estimation error $e(t) = \hat{x}(t) - x(t)$, which satisfies

$$e(t+1) = (A(w) - LC(w))e(t) + Ln(t) - Ed(t) \tag{6.1}$$

for discrete-time case, and satisfies

$$\dot{e}(t) = (A(w) - LC(w))e(t) + Ln(t) - Ed(t) \tag{6.2}$$

for continuous-time case. Our objective is to show that the error $e(t)$ is uniformly ultimately bounded in some C-set $\mathcal{E}$ for every admissible disturbance $d(t) \in \mathcal{D}$, measurement noise $n(t) \in \mathcal{N}$, and parameter uncertainty $w(t) \in \mathcal{W}$. For this purpose, we introduce the following concepts. Note that these concepts appeared previously in Blanchini (1990), Blanchini (1994) and also in some references therein.

A function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is said to be a *gauge function* if

1. $\Psi(x) \geq 0$, $\Psi(x) = 0 \Leftrightarrow x = 0$;
2. for $\mu > 0$, $\Psi(\mu x) = \mu \Psi(x)$;
3. $\Psi(x + y) \leq \Psi(x) + \Psi(y)$, $\forall x, y \in \mathbb{R}^n$.

A gauge function is convex and it defines a distance of x from the origin which is linear in any direction. A gauge function Ψ is 0-symmetric, that is $\Psi(-x) = \Psi(x)$, if and only if Ψ is a norm.

If Ψ is a gauge function, we define the closed set (possibly empty) $\hat{N}[\Psi, \xi] = \{x \in \mathbb{R}^n : \Psi(x) \leq \xi\}$. On the other hand, the set $\hat{N}[\Psi, \xi]$ is a C-set for all $\xi > 0$. Any C-set $\mathcal{E}$ induces a gauge function $\Psi_{\mathcal{E}}(x)$ (Known as Minkowski function of $\mathcal{E}$), which is defined as $\Psi(x) \doteq \inf\{\mu > 0 : x \in \mu\mathcal{E}\}$. Therefore a C-set $\mathcal{E}$ can be thought of as the unit ball $\mathcal{E} = \hat{N}[\Psi, 1]$ of a gauge function Ψ and $x \in \mathcal{E} \Leftrightarrow \Psi(x) \leq 1$.

Lemma 6.1 (Blanchini 1994) If $\mathcal{E}$ is is SPDI (or PDI if $\lambda = 1$) set for system (6.1) with convergence index $\lambda \leq 1$, then $\mu\mathcal{E}$ is so for all $\mu \geq 1$.

Proof : Let $e \in \mu\mathcal{E}$, hence $\mu^{-1}e \in \mathcal{E}$, so $post(\mu^{-1}e, \mathcal{W}, \mathcal{D}) \subset \lambda\mathcal{E}$. Note $\mu^{-1}\mathcal{D} \subset \mathcal{D}$, so $post(e, \mathcal{W}, \mathcal{D}) = \mu post(\mu^{-1}e, \mathcal{W}, \mu^{-1}\mathcal{D}) \subset \mu post(\mu^{-1}e, \mathcal{W}, \mathcal{D}) \subset \lambda\mu\mathcal{E}$. □

Lemma 6.2 (Blanchini 1994) A C-set $\mathcal{E}$ is SPDI set for system (6.1) with convergence index $\lambda < 1$ *if and only if* there exists a gauge function $\Psi(e)$ such that the unit ball $\hat{N}[\Psi, 1] \subset \mathcal{E}$ and, if $e \notin int\{\hat{N}[\Psi, 1]\}$, then $\Psi(post(e, w, d)) \leq \lambda\Psi(e)$ for all $w \in \mathcal{W}$ and $d \in \mathcal{D}$ (or. equivalently, $\hat{N}[\Psi, \mu]$ is λ-contractive for all $\mu \geq 1$).

Although we only present the above two lemmas for the discrete-time case, their extensions to the continuous-time case are immediate by employing EAS. In view of the above two lemmas, we can derive the following theorem about the uniformly ultimate boundedness of the estimation error $e(t)$.

Theorem 6.1 The observation error $e(t)$ for the observer designed in Section 4 is uniformly ultimate bounded with convergence rate $0 < \lambda < 1$ (or. $\beta = \frac{1-\lambda}{\tau}$) in the given C-set $\mathcal{E}$, *if* *and only if* the inequalities (4.5) (or. the inequalities (4.12) respectively) are feasible. In addition,

$$x(t) \in \mathcal{X}(t) \oplus \mathcal{E} \tag{6.3}$$

for t large enough, where $\oplus$ stands for the Minkowski sum.

Proof : $\mathcal{E}$ is a C-set, and let $\psi(e) = \Psi_{\mathcal{E}}(e)$ be its Minkowski functional. For any $e \in \mathbb{R}^n$, we have $\psi(e(t+1)) \leq \lambda\psi(e(t))$ for all $e(t) \notin int\{\mathcal{E}\}$, because of linear inequalities (4.5) (or. the inequalities (4.12) respectively) and according to Lemma 5.1. Then $\psi(e)$ is a Lyapunov function for system (6.1) (or. for system (6.2) respectively), which is uniquely generated from the target set $\mathcal{E}$ for any fixed λ. Such a function has been named Set-induced Lyapunov Function (SILF). Then the existence of the Lyapunov function implies the exponential convergence of the estimation error to $\mathcal{E}$ according to Lemma 5.2. The exponential convergence is in the sense that $\psi(e(t+1)) \leq \lambda\psi(e(t))$ in discrete-time case or $\psi(e(t+\delta)) \leq e^{-\beta\delta}\psi(e(t))$ in continuous-time case (where $\beta = \frac{1-\lambda}{\tau}$). Also for any initial value of the estimation error $e(t_0)$, $\exists T \geq t_0$ such that for all $t \geq T$, $e(t) \in \mathcal{E}$ and $x(t)+e(t) \in \mathcal{X}(t)$. Therefore, $x(t) \in \mathcal{X}(t) \oplus \mathcal{E}$, where $\oplus$ stands for the Minkowski sum. $\square$

It is worthy to point out that for the linear constrained case the set-induced Lyapunov function $\psi(e) = \Psi_{\mathcal{E}}(e)$ can be derived explicitly. Let $\mathcal{E}$ be a polyhedral C-set for which the following plane description is given:

$$\mathcal{E} = \{e : \ f_i^T e \leq \theta_i, \ \theta_i > 0, \ i = 1, \cdots, s, \ \text{or} \ Fe \leq \theta\} \tag{6.4}$$

Then we can express the Minkowski function of $\mathcal{E}$ as:

$$\Psi_{\mathcal{E}}(e) = \max_{1 \leq i \leq s}\{f_i^T e\} \tag{6.5}$$

which is considered as the set-induced Lyapunov function $\psi(e) = \Psi_{\mathcal{E}}(e)$. The Lyapunov functions of the above form, which are usually called piecewise linear Lyapunov functions, have been used extensively for the analysis and synthesis of dynamical systems in the literature, see for example Michel, Nam and Vittal (1984), Blanchini (1990), Polanski (1995) and references therein.

7 Numerical Example

Consider the following continuous-time uncertain systems:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -0.1 & w \\ 1 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ w \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} d(t) \\ y(t) &= \begin{bmatrix} 1 & w \end{bmatrix} x(t) + n(t) \end{aligned}$$

We assume that the uncertain parameter w is subjected to the constraint $1 \leq w \leq 2$, the measurement noise $n(t) \in \mathcal{N} = \{n : -0.01 \leq n \leq 0.01\}$, and the continuous disturbance $d(t)$ is bounded by $d \in \mathcal{D} = \{d : -0.01 \leq d \leq 0.01\}$. We consider a full state observer (4.1):

$$\dot{\hat{x}}(t) = (\begin{bmatrix} -0.1 & w \\ 1 & -0.1 \end{bmatrix} - L \begin{bmatrix} 1 & w \end{bmatrix})\hat{x}(t) + \begin{bmatrix} 0 \\ w \end{bmatrix} u(t) + Ly(t)$$

Then, the estimation error $e(t)$ satisfies

$$\dot{e}(t) = (\begin{bmatrix} -0.1 & w \\ 1 & -0.1 \end{bmatrix} - L \begin{bmatrix} 1 & w \end{bmatrix})e(t) + Ln(t) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} d(t)$$

where we assume that the specified set $\mathcal{E} = \{e \in R^2 : \|e\|_\infty \leq 0.1\}$. Our problem is to design the matrix $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$, such that the estimation error $e(t)$ converges exponentially to $\mathcal{E}$.

Using (3.6) with $\tau = 1$, we obtain the EAS system for estimation error:

$$e(t+1) = (\begin{bmatrix} 0.9 & w \\ 1 & 0.9 \end{bmatrix} - L \begin{bmatrix} 1 & w \end{bmatrix})e(t) + Ln(t) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} d(t)$$

Then, using (4.5) for the above EAS system with $\lambda = 0.9$,

$$(\begin{bmatrix} 0.9 & w \\ 1 & 0.9 \end{bmatrix} - L \begin{bmatrix} 1 & w \end{bmatrix})e(t) + Ln(t) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} d(t) \in 0.9\mathcal{C}_j$$

where e_j is a vertex of $\mathcal{E}$, and $\mathcal{C}_j$ is a convex cone corresponding to the vertex v_j (see Section 3). Now

$$F_j(\begin{bmatrix} 0.9 & w \\ 1 & 0.9 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 0 & w \end{bmatrix})e_j + F_j \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} n_l \leq \begin{bmatrix} 0.8 \\ 0.8 \end{bmatrix}$$

where e_j corresponds to the four vertices of $\mathcal{E}$: $e_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}$, $e_3 = \begin{bmatrix} -0.1 \\ -0.1 \end{bmatrix}$, and $e_4 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$. F_j is the corresponding representation matrix for $\mathcal{C}_j$, that is $F_1 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, $F_2 = \begin{bmatrix} 10 & 0 \\ 0 & -10 \end{bmatrix}$, $F_3 = \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix}$, and $F_4 = \begin{bmatrix} -10 & 0 \\ 0 & 10 \end{bmatrix}$.

Solving the above inequalities with $w_1 = 1$ or $w_2 = 2$, and $n_1 = -0.01$ or $n_2 = 0.01$, we obtain the following conditions with respect to $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$.

$$0.7241 \leq l_1 \leq 1.7273, \quad 0.5789 \leq l_2 \leq 9$$

If we select $L = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then the set-valued observer is:

$$\dot{\hat{x}}(t) = \begin{bmatrix} -1.1 & 0 \\ 0 & -0.1 - w \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 0 \\ w \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} y(t)$$

Note that the evolution of the observer with given initial condition $\hat{x}(0) = \hat{x}_0$ and under the LTI uncertainty assumption can be explicitly expressed as:

$$\hat{x}(w,t) = \begin{bmatrix} e^{-1.1t} & 0 \\ 0 & e^{(-0.1-w)t} \end{bmatrix} \hat{x}_0 + \int_0^t (\begin{bmatrix} 0 \\ we^{(-0.1-w)(t-\tau)} \end{bmatrix} u(\tau) + \begin{bmatrix} e^{-1.1(t-\tau)} \\ e^{(-0.1-w)(t-\tau)} \end{bmatrix} y(\tau))d\tau$$

At each time instant t, the output of the observer, $\mathcal{X}(t)$, is given as the domain of mapping $\hat{x}(\cdot,t) : \mathcal{W} \to \mathbb{R}^n$. And $x(t) \in \mathcal{X}(t) \oplus \mathcal{E}$ for t large enough, where $\oplus$ stands for the Minkowski sum. The estimation error $e(t)$ satisfies

$$\dot{e}(t) = \begin{bmatrix} -1.1 & 0 \\ 0 & -0.1 - w \end{bmatrix} e(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} n(t) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} d(t)$$

Following Section 6, we take the set-induced Lyapunov function from $\mathcal{E} = \{e \in R^2 : \|e\|_\infty \leq 0.1\}$ as $\Psi(x) = \max_{1\leq i\leq s}\{f_i^T e\} = \|10 \cdot e\|_\infty$. We know $\mathcal{E}$ is λ-contractive ($\lambda = 0.9$) by the above design procedure, so by Theorem 6.1, the estimation is uniformly ultimately bounded in $\mathcal{E}$ with rate $\beta = \frac{1-\lambda}{\tau} = 0.1$.

Based on the results in Cugueró et al. (2002), we obtain the l^∞ norm of the estimation errors' upper bound, which is plotted in Figure 2 from several different initial conditions and under the assumption that $\mathcal{D} = \mathcal{N} = \{0\}$.

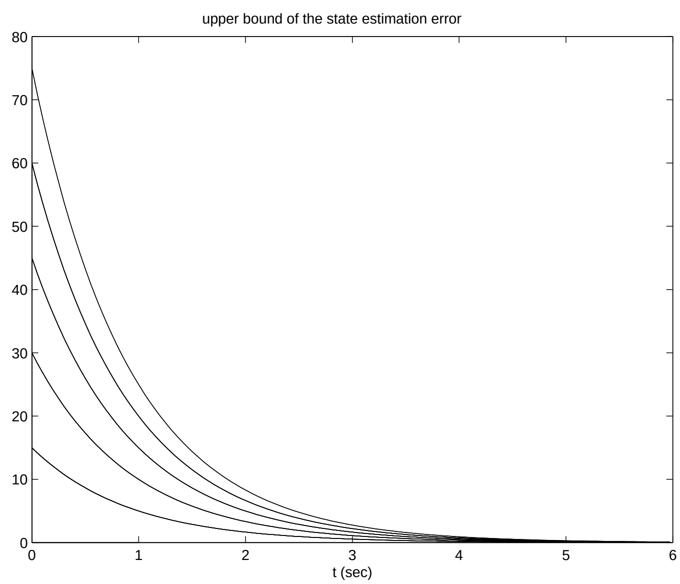

Figure 2: The simulation results for l^∞ norm of the estimation errors' upper bound.

8 Conclusions

In this paper, we developed a set-valued state observer for a class of uncertain linear systems affected by parameter variation, persistent disturbance and measurement noise. The design procedure proposed assures that the estimation error will be ultimately bounded within a given convex and compact set containing the origin with an assigned rate of convergence. The techniques were based on positive set invariance theory and set-induced Lyapunov functions. The advantage of the method comes from its simplicity. To design the observer one only needs to solve sets of linear inequalities, and to implement the observer one only needs to consider finite number of cases corresponding to the polytopic vertices.

The existence of the robust observer in form of (4.1), whose state estimation error is uniformly ultimately bounded in a specified region $\mathcal{E}$, is equivalent to the feasibility of the linear inequalities in (4.5) (or. the inequalities (4.12) respectively). However, the answer to the existence problem is not satisfactory, because the feasibility of these linear inequalities in (4.5) or (4.12), namely the condition that $\mathcal{E}$ is SPDI for some given $0 < \lambda < 1$ with respect to system (6.1) (or. system (6.2)), is not easy to check. Some algebraic conditions or computational procedures need to be obtained for the existence problem. In this paper, we only consider full order observer. A possible generalization is to the case of reduced order set-valued observer. Furthermore, instead of consider a constant matrix L, a parameterized matrix $L(\alpha)$ may give stronger results and offer more flexibility in design.

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References

Akpan, E. (2001). Robust observer for uncertain linear systems, *Proceedings of American Control Conference*, Vol. 6, pp. 4220–4221.

Bhattacharyya, S. P. (1976). The structure of robust observers, *IEEE Transactions on Automatic Control* **21**: 581–588.

Bitsoris, G. and Vassilaki, M. (1995). Constrained regulation of linear systems, *Automatica* **31**: 223–227.

Blanchini, F. (1990). Feedback control for linear time-invariant system with state and control bounds in the presence of disturbance, *IEEE Transactions on Automatic Control* **35**: 1231–1234.

Blanchini, F. (1994). Ultimate boundedness control for discrete-time uncertain system via set-induced lyapunov functions, *IEEE Transactions on Automatic Control* **39**: 428–433.

Blanchini, F. (1999). Set invariance in control, *Automatica* **35**: 1747–1767.

Collins, E. G. and Song, T. (2001). Robust l_1 estimation using the popov-tsypkin multiplier with applications to robust fault detection, *International Journal of Control* **74**: 303–313.

Cugueró, P., Puig, V., Saludes, J. and Escobet, T. (2002). A class of uncertain linear interval models for which a set based robust simulation can be reduced to few pointwise simulations, *Proceedings of the 41st Conference on Decision and Control*, pp. 1862–1863.

Haddad, W. M. and Berstein, D. S. (1995). Parameter dependent lyapunov functions and the popov criterion in robust analysis and synthesis, *IEEE Transactions on Automatic Control* **40**: 536–543.

Jaulin, L., Kieffer, M., Didrit, O. and Walter, E. (2001). *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control and Robotics*, Springer-Verlag, London.

Lin, H., Zhai, G. and Antsaklis, P. J. (2003). Set-valued observer for a class of discrete-time uncertain linear systems with persistent disturbance, *Proceedings of American Control Conference*.

Luenberger, D. G. (1966). Observers for multivariable systems, *IEEE Transactions on Automatic Control* **AC-11**: 190–197.

Michel, A. N., Nam, B. and Vittal, V. (1984). Computer generated lyapunov functions for interconnected systems: Improved results with applications to power systems, *IEEE Transactions on Circuits and Systems* **CAS-31**: 189–198.

Nagpal, K. M. and Khargonekar, P. P. (1991). Filtering and smoothing in an $\mathcal{H}^\infty$ setting, *IEEE Transactions on Automatic Control* **36**: 152–166.

O'Reilly, J. (1983). *Observers for Linear Systems*, Academic Press, London.

Polanski, A. (1995). On infinity norms as lyapunov functions for linear systems, *IEEE Transactions on Automatic Control* **40**: 1270–1273.

Shamma, J. S. and Tu, K.-Y. (1999). Set-valued observers and optimal disturbance rejection, *IEEE Transactions on Automatic Control* **44**: 253–264.

Voulgaris, P. (1995). On optimal l^∞ to l^∞ filtering, *Automatica* **31**: 489–495.