

Practical Stabilization of Integrator Switched Systems *

Xuping Xu[†]

Panos J. Antsaklis[‡]

Abstract

In this paper, practical stabilization problems for integrator switched systems are studied. In such class of switched systems, no subsystem has an equilibrium. However, the system can still exhibit interesting behaviors around a given point under appropriate switching laws. Such behaviors are similar to those of a conventional stable system near an equilibrium. We formally introduce some practical stability notions to define such behaviors. In particular, a necessary and sufficient condition for practical stabilizability of such systems is proved in the paper. Moreover, for practically stabilizable systems, we develop a minimum dwell time switching law which can easily be implemented. Finally, as an application, we apply the switching law to a batch process example.

1 Introduction

A switched system is a particular kind of hybrid system that consists of several subsystems and a switching law orchestrating the active subsystem at each time instant. Many results on stability analysis and stabilization of switched systems have been reported in the literature [1, 6]. Most of the available literature results consider switched systems whose subsystems share a common equilibrium. Methods based on single or multiple Lyapunov functions have been reported for the stability analysis and design of such systems. Methods based on geometric properties of the subsystem vector fields have also been reported [12].

In our recent research, we found that the assumption that all subsystems share a common equilibrium may not hold for all switched systems and may limit the applicability of switched systems stability results. In the case that such an assumption does not hold, in other words when subsystems have different equilibria or no equilibrium, we found that under appropriate switching laws the system can still exhibit interesting behaviors. Such behaviors are similar to those of a conventional stable system near an equilibrium point. In this paper, we formally introduce some practical stability notions to define such behaviors. Such practical stability notions for switched systems are extensions of the traditional literature concepts on practical stability [3, 4], which are concerned with bringing the system trajectories to be within a given bound.

In this paper, we focus on practical stabilization problems for a simple yet important class of switched systems — integrator switched systems. Many real-world processes including chemical processes [5, 10] can be modeled as such systems. After introducing some notions of practical stability, we propose and prove a necessary and sufficient condition for the practical stabilizability of such systems (Theorem 3.1). Additional feasible ways for checking the condition are also proposed. Moreover, for practically stabilizable systems, we develop a minimum dwell time switching law which can easily be implemented to achieve ϵ -practically

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[†]Department of Electrical and Computer Engineering, Penn State Erie, Erie, PA 16563 USA. Tel: 1-814-898-7169; Fax: 1-814-898-6125; E-mail: Xuping-Xu@psu.edu.

[‡]Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556 USA. Tel: 1-574-631-5792; Fax: 1-574-631-4393; E-mail: antsaklis.1@nd.edu.

asymptotic stability. The construction of the switching law is described in detail. The switching law is then applied to a three tank problem in chemical batch process to illustrate its effectiveness.

The structure of the paper is as follows. In Section 2, we propose some notions of practical stability for switched systems. In Section 3, we report some theoretical results and conditions for practical stabilizability, and propose a minimum dwell time switching law. In Section 4, we apply the stabilization results to a three tank problem. Section 5 concludes the paper. Examples are given throughout the paper.

2 Practical Stability Notions for Switched Systems

2.1 Switched Systems

A *switched system* is a dynamic system which consists of subsystems

$$\dot{x} = f_i(x), \quad f_i : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad i \in I = \{1, 2, \dots, M\}, \quad (2.1)$$

and a switching law orchestrating the active subsystem at each time instant. The state trajectory of a switched system is determined by the initial state and the timed sequence of active subsystems. A *switching sequence* in $t \in [t_0, t_f]$ regulates the timed sequence of active subsystems and is defined as follows.

Definition 2.1 (Switching Sequence) A *switching sequence* σ in $[t_0, t_f]$ is defined as

$$\sigma = ((t_0, i_0), (t_1, i_1), \dots, (t_K, i_K)) \quad (2.2)$$

where $0 \leq K < \infty$, $t_0 \leq t_1 \leq t_2 \leq \dots \leq t_K \leq t_f$, $i_k \in I$ for $k = 0, 1, \dots, K$.

We also define $\Sigma_{[t_0, t_f]} = \{\text{switching sequence } \sigma \text{'s in } [t_0, t_f]\}$. □

Note here (t_k, i_k) indicates that at instant t_k , the system switches from subsystem i_{k-1} to subsystem i_k ; during the time interval $[t_k, t_{k+1})$, subsystem i_k is active. For a switched system to be well-behaved, we only consider **nonZero** sequences which switch at most a finite number of times in any finite time interval $[t_0, t_f]$, though different sequences may have different numbers of switchings. Finally, we note that the feature distinguishing a switched system from a general hybrid system is that its continuous state does not exhibit jumps at switching instants.

Switching sequences as defined above are usually generated by switching laws which are defined below.

Definition 2.2 (Switching Law) For switched system (2.1), a *switching law* \mathcal{S} is defined to be a mapping $\mathcal{S} : \mathbb{R}^n \times \mathbb{R} \rightarrow \bigcup_{t_0} \Sigma_{[t_0, \infty)}$ which specifies a switching sequence $\sigma = \sigma(x_0, t_0) \in \Sigma_{[t_0, \infty)}$ for any initial point x_0 and any initial time t_0 . □

Remark 2.1 More often than not, the mapping \mathcal{S} is described by some rules or algorithms using words rather than explicit mathematical formulae. Such rules or algorithms describe how to generate a switching sequence given x_0 and t_0 . In this paper, we will specify our switching laws using such descriptions. □

Remark 2.2 Sometimes we are only interested in the behavior of a switched system in a finite time duration $[t_0, t_f]$. In such cases, we can use the same definition of switching law as the above one but only pay attention to the subsequences of $\sigma(x_0, t_0)$ in $[t_0, t_f]$. □

Remark 2.3 Once a switching law is given, the switched system may be described as

$$\dot{x}(t) = f_{i(t)}(x(t)) \quad (2.3)$$

$$i(t) = \varphi(x(t), i(t^-), t), \quad (2.4)$$

where $\varphi : \mathbb{R}^n \times I \times \mathbb{R} \rightarrow I$ determines the active subsystem at time t . Note that (2.3)-(2.4) are indeed used as the definition of switched systems in some literatures for stability analysis (e.g. [8]). Here we adopt (2.1) as the definition of switched systems rather than (2.3)-(2.4) because in stabilization design problems it is the designer's task to design a switching law and therefore φ in (2.4) is not given a priori. \square

2.2 Some Practical Stability Notions

Many literature results have appeared on stability analysis and stabilization of switched systems. In most of the results, it is assumed that a common equilibrium exists for all subsystems. However, this assumption may not be true for all switched systems and may limit the applicability of switched systems. In the case when subsystems have different equilibria or no equilibrium, the system can still exhibit interesting behaviors around a given point under appropriate switching laws. The behaviors are similar to those of a conventional stable system near an equilibrium. The following example illustrates such behaviors.

Example 2.1 Consider a switched system consisting of four subsystems: subsystem 1: $\dot{x} = [-3, 2.5]^T$; subsystem 2: $\dot{x} = [-2.5, -3]^T$; subsystem 3: $\dot{x} = [3, -2.5]^T$; subsystem 4: $\dot{x} = [2.5, 3]^T$. If we apply the switching rule which makes subsystem 1 active in quadrant I, subsystem 2 active in quadrant II, subsystem 3 active in quadrant III, and subsystem active 4 in quadrant IV, then the system will exhibit “convergent behaviors” around the origin. Figure 1 shows a sample trajectory starting from $x_0 = [2, 1]^T$ under this switching rule in a finite time duration. \square

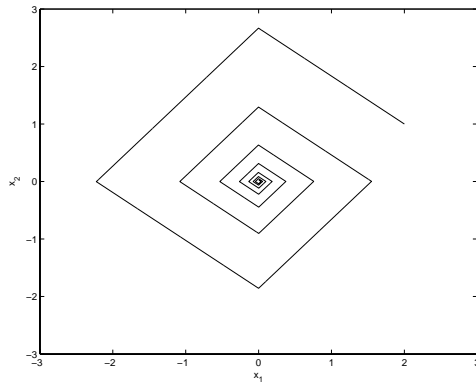


Figure 1: A sample trajectory starting from $x_0 = [2, 1]^T$ for Example 2.1.

In the above example, we see that under the given switching rule the origin exhibits behaviors similar to those of an asymptotically stable system. However, we note that as the trajectory becomes closer and closer to the origin, the system needs to switch faster and faster. This violates the nonZenoness requirement for valid switching sequences. In practice, a lower bound for the time between switchings should usually be imposed which prevents Zenoness. Such a lower bound is called the *minimum dwell time* [2] and its value may be different for different application objectives. If we incorporate a minimum dwell time into the switching rule in Example 2.1, system trajectories starting from any point in \mathbb{R}^2 will be attracted toward the origin and eventually oscillate near the origin within certain bound.

The concept of bringing the system trajectories to be within a given bound is quite useful in practice. For example, in temperature control systems, usually we are more interested in keeping the temperature within certain bounds, rather than in stabilizing the system asymptotically to a set-point. In fact, such concept has been formally termed practical stability in [3, 4] for ordinary differential equations. In the following, we adapt and expand some practical stability notions to switched systems and formally define

the notion of practical stabilizability of switched systems. Without loss of generality, we only discuss the case of the origin. For any point other than the origin, the definitions and results in the followings can similarly be obtained (we only need to apply some shift of points).

Definition 2.3 (ϵ -Practical Stability) *Assume a switching law \mathcal{S} is given for the switched system (2.1). Given $\epsilon > 0$, the switched system is said to be ϵ -practically stable around the origin under the switching law \mathcal{S} if there exists $\delta = \delta(\epsilon) > 0$ such that $x(t) < \epsilon$ whenever $x(0) = x_0$ satisfies $\|x_0\| < \delta$. \square*

Definition 2.4 (ϵ -Attractivity) *Assume a switching law \mathcal{S} is given for the switched system (2.1). Given $\epsilon > 0$, the origin is said to be ϵ -attractive if there exists $\eta = \eta(\epsilon) > 0$ such that for every $x(0) = x_0$ satisfying $\|x_0\| < \eta$, there exists $T = T(x_0) \geq 0$ such that $\|x(t)\| < \epsilon$ for any $t \geq T$. Moreover, the origin is said to be globally ϵ -attractive if η can be chosen to be ∞ . \square*

Definition 2.5 (ϵ -Practically Asymptotic Stability) *Assume a switching law \mathcal{S} is given for the switched system (2.1). Given $\epsilon > 0$, the switched system is said to be ϵ -practically asymptotically stable around the origin under the switching law \mathcal{S} if it is ϵ -practically stable and the origin is ϵ -attractive. Moreover, the system is said to be globally ϵ -practically asymptotically stable if it is ϵ -practically stable and the origin is globally ϵ -attractive. \square*

Remark 2.4 Note that in the definition of ϵ -practically asymptotic stability, we require not only ϵ -attractivity but also ϵ -practical stability. ϵ -attractivity does not imply ϵ -practical stability. This is because it is possible that for any $\delta < \epsilon$, a trajectory exists that starts at $x(0)$ with $\|x(0)\| < \delta$ and violates $\|x(t)\| < \epsilon$ for some time and finally settles down with $\|x(t)\| < \epsilon$. This still satisfies ϵ -attractivity; however, ϵ -practical stability is not satisfied. \square

Definition 2.6 (Practical Stabilizability) *The switched system (2.1) is said to be practically stabilizable if for any $\epsilon > 0$, there exists a switching law $\mathcal{S} = \mathcal{S}(\epsilon)$ such that the system is ϵ -practically asymptotically stable around the origin under \mathcal{S} . Moreover, it is said to be globally practically stabilizable if for any $\epsilon > 0$, $\mathcal{S} = \mathcal{S}(\epsilon)$ exists such that the system is globally ϵ -practically asymptotically stable around the origin under \mathcal{S} . \square*

Remark 2.5 Note that in the definition of practical stabilizability, the ϵ can be varied as opposed to the fixed ϵ in the previous several definitions. Hence a practically stabilizable system has the property that, for any given bound, a valid switching law can be found which brings the system trajectory into the bound and keeps it within the bound. \square

3 Practical Stabilization Results for Integrator Switched Systems

In the sequel, we will focus on a special class of switched systems — integrator switched systems, which consist of subsystems

$$\dot{x} = a_i, \quad i \in I = \{1, 2, \dots, M\} \quad (3.1)$$

where $a_i \in \mathbb{R}^n$ ($a_i \neq 0$), $i \in I$ are constant vectors and $x \in \mathbb{R}^n$ is the continuous state. Such class of systems receives particular attention due to the following reasons. First, such systems can model many real world processes, such as chemical batch processes [5, 10]. Second, the simple structure of such systems makes possible rigorous analysis which leads to nice theoretical and practical results. Third, the complete exploration of such systems is the first step toward the study of practical stability properties of general nonlinear switched systems, because a nonlinear switched system may be approximated locally by an integrator switched systems around a given point when this point is not an equilibrium for any subsystem.

3.1 Results on Practical Stabilizability

For an integrator switched system, practical stabilizability is equivalent to globally practical stabilizability. In the following, with the help of some convex analysis notions and results (see Appendix A), we propose and prove some necessary and sufficient conditions for globally practical stabilizability of system (3.1). Theorem 3.1 provides a necessary and sufficient condition for practical stabilizability. Lemma 3.1 provides a feasible way of verifying the condition in Theorem 3.1. Then Theorem 3.2 and three corollaries are proposed which illustrate some implications of the necessary and sufficient condition and emphasize more on systems with $n + 1$ subsystems in \mathbb{R}^n . In order to make the main results stand out, we have put all the proofs in Appendix B.

Theorem 3.1 (Necessary and Sufficient Condition) *An integrator switched system (3.1) in \mathbb{R}^n is globally practically stabilizable if and only if $C = \mathbb{R}^n$, where C is the convex cone $C = \{\sum_{i=1}^M \lambda_i a_i | \lambda_1 \geq 0, \dots, \lambda_M \geq 0\}$.*

Proof: See Appendix B. □

In order to apply Theorem 3.1, we need to verify the validity of the condition $C = \mathbb{R}^n$. Exhaustively checking whether $x \in C$ for any $x \in \mathbb{R}^n$ is not an option due to the infinite number of x to check. The following lemma provides a necessary and sufficient condition which is equivalent to $C = \mathbb{R}^n$ and computational feasible to verify.

Lemma 3.1 *$C = \mathbb{R}^n$ if and only if there exists a subset $\{a_{i_1}, \dots, a_{i_l}\}$ of $\{a_1, \dots, a_M\}$ which satisfies the following conditions:*

- (a). $\text{span}\{a_{i_1}, \dots, a_{i_l}\} = \mathbb{R}^n$ and
- (b). there exist $\lambda_j > 0$, $j = 1, \dots, l$, such that $\sum_{j=1}^l \lambda_j a_{i_j} = 0$.

Proof: See Appendix B. □

Remark 3.1 Lemma 3.1 provides us with a feasible way of checking whether $C = \mathbb{R}^n$ or not. By exhaustively checking all possible subsets of $\{a_1, \dots, a_M\}$ for the validity of conditions (a) and (b), we can determine whether a given system is practically stabilizable or not. Because there are at most 2^M subsets and we only need to check condition (b) for the 0 point, the computation can finish in finite time and therefore is feasible. □

Furthermore, from Lemma 3.1, we can immediately conclude that the number of subsystems in a globally practically stabilizable system should be great than or equal to $n + 1$. The following theorem confirms it.

Theorem 3.2

- (a). *If an integrator switched system (3.1) in \mathbb{R}^n is globally practically stabilizable, then there are at least $n + 1$ subsystems.*
- (b). *Moreover, there exists an integrator switched system consisting of $n + 1$ subsystems which is globally practically stabilizable.*

Proof: See Appendix B. □

The case of $n + 1$ subsystems that form a globally practically stabilizable system is important, because in many stabilizable systems such $n + 1$ subsystems do exist. The following three corollaries related to such case can be inferred from the above theorems and lemma.

Corollary 3.1 *An integrator switched system (3.1) in \mathbb{R}^n consisting $n + 1$ subsystems with vector fields a_1, \dots, a_{n+1} is globally practically stabilizable if and only if $\text{span}\{a_1, \dots, a_{n+1}\} = \mathbb{R}^n$ and there exist $\lambda_i > 0$, $i = 1, \dots, n + 1$ such that $\sum_{i=1}^{n+1} \lambda_i a_i = 0$.*

Proof: See Appendix B. □

Corollary 3.2 *An integrator switched system (3.1) in \mathbb{R}^n with $n + 1$ subsystems with vector fields a_1, \dots, a_{n+1} is globally practically stabilizable if and only if any n vectors in the set $\{a_1, \dots, a_{n+1}\}$ are linearly independent and there exist $\lambda_i > 0$, $i = 1, \dots, n + 1$ not all zero such that $\sum_{i=1}^{n+1} \lambda_i a_i = 0$.*

Proof: See Appendix B. □

In many cases, even though a system has many subsystems, yet we can find $n + 1$ subsystems which can be used for determining the practical stabilizability of the system. The following corollary provides a sufficient condition for doing so.

Corollary 3.3 (A Sufficient Condition) *An integrator switched system (3.1) in \mathbb{R}^n with M ($M \geq n + 1$) subsystems is globally practically stabilizable if there exists a subset of $n + 1$ subsystems which, if regarded as a switched system with $n + 1$ subsystems, is globally practically stabilizable.*

Proof: See Appendix B. □

3.2 A Minimum Dwell Time Switching Law

Now we will construct a switching law that makes the system globally ϵ -practically asymptotically stable if the system is determined to be globally practically stabilizable by the conditions proposed in Section 3.1. Note that in the proof of the “If” part of Theorem 3.1 (see Appendix B), a valid switching law is constructed. However, it is very difficult to implement in practice due to the need to solve for the convex combination at each x (see equation (B.3) in Appendix B).

In this subsection, we focus on integrator switched systems in \mathbb{R}^n with $n + 1$ subsystems which are globally practically stabilizable and propose a valid minimum dwell time switching law. As mentioned in Section 3.1, the case of $n + 1$ subsystems is important because in many stabilizable systems such $n + 1$ subsystems do exist. Hence the switching law proposed here can actually be applied to many switched systems with more than $n + 1$ subsystems.

Let us first illustrate the idea of our switching law by the following example.

Example 3.1 Consider a switched system in \mathbb{R}^2 consisting of three subsystems: subsystem 1: $\dot{x} = a_1 = [1, 0.5]^T$; subsystem 2: $\dot{x} = a_2 = [-1, 1.5]^T$; subsystem 3: $\dot{x} = a_3 = [-0.5, -1]^T$ (see figure 2(a)). It can be verified by Corollary 3.2 that the system is globally practically stabilizable.

For a given $\epsilon > 0$, we now propose a valid switching law to achieve ϵ -practically asymptotic stability. To do so, we first denote by C_1 the convex cone generated by the vectors $-a_2, -a_3$; denote by C_2 the convex cone generated by $-a_1, -a_3$; denote by C_3 the convex cone generated by $-a_1, -a_2$ (see figure 2(b)). Note that C_1, C_2 , and C_3 have mutually disjoint interiors and $C_1 \cup C_2 \cup C_3 = \mathbb{R}^2$. The switching law is now stated as follows.

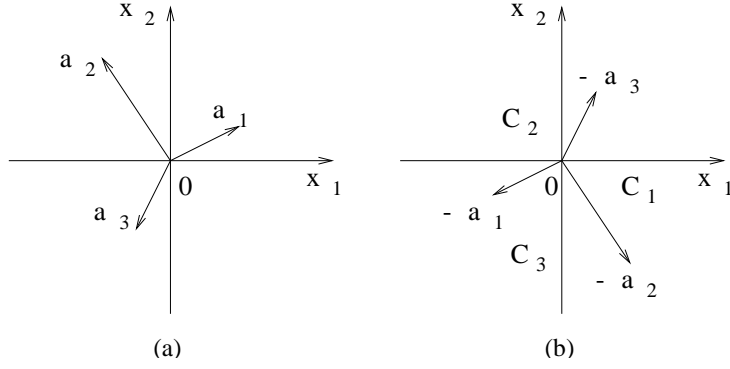


Figure 2: Example 3.1: (a) Vector fields a_1, a_2, a_3 . (b) Convex cones C_1, C_2, C_3 .

A minimum dwell time switching law: Let subsystem 2 be active in $\text{Int}(C_1)$, subsystem 3 be active in $\text{Int}(C_2)$, subsystem 1 be active in $\text{Int}(C_3)$. When the state is on the common boundary of any two convex cones, we choose the active subsystem to be the one corresponding to the convex cone that the trajectory has the potential to enter next, if the system still evolves according to the current active subsystem. For example, if x evolves in C_1 (following subsystem 2) and intersects the ray in the same direction as $-a_3$, then subsystem 3 will become active. Moreover, in order to eliminate the Zenoness phenomenon near the origin, besides the above rules, we also impose a minimum dwell time τ (i.e., the minimum time duration that any subsystem must be active before the system can switch again).

The choice of a minimum dwell time τ : In general, the smaller the τ is, the smaller the ϵ can be, so that the system can be made ϵ -practically asymptotically stable. As $\tau \rightarrow 0$, we find that ϵ can also go to 0. However when $\tau = 0$, Zenoness problem will occur, therefore τ cannot be infinitely small either. For this example in \mathbb{R}^2 , some geometric observations suggest that we can choose a τ satisfying the following inequality

$$\tau \leq \min \left\{ \frac{1}{\|a_1\|} \left(\epsilon - \frac{\delta}{\sin \theta_{12}} \right), \frac{1}{\|a_2\|} \left(\epsilon - \frac{\delta}{\sin \theta_{23}} \right), \frac{1}{\|a_3\|} \left(\epsilon - \frac{\delta}{\sin \theta_{31}} \right), \frac{\delta}{\|a_1\|}, \frac{\delta}{\|a_2\|}, \frac{\delta}{\|a_3\|} \right\}, \quad (3.2)$$

where θ_{12} is the angle extended by l_1 and l_2 ($0 < \theta_{12} < \pi$). Similar definitions apply for θ_{23} and θ_{31} . The δ in (3.2) corresponds to the δ in Definition 2.3 and can be chosen to be a value that satisfies

$$\delta < \min \{ \epsilon \sin \theta_{12}, \epsilon \sin \theta_{23}, \epsilon \sin \theta_{31} \}. \quad (3.3)$$

Note that the details of the derivation of (3.2) and (3.3) are given in Appendix C.

Equipped with the switching law and (3.2), we return to our example. We choose $\epsilon = 0.3$, and $\delta = 0.1$ which satisfies (3.3) for this example, it can then be determined from (3.2) that $\tau \leq 0.0555$ will lead to a valid switching law which makes the system ϵ -practically asymptotically stable. Figure 3 shows a trajectory starting from $[1, 1]^T$ with $\tau = 0.05$. Figure 4 shows $x_1(t)$ and $x_2(t)$. Note that when time becomes large, the maximum deviation from 0 is -0.1165 for x_1 , and 0.075 for x_2 . So the state is actually within a ball with radius 0.1386 which is smaller than 0.3. The requirement is therefore satisfied. \square

The switching law proposed in Example 3.1 can be extended to the case of practically stabilizable systems in \mathbb{R}^n with $n + 1$ subsystems. First of all, we denote by C_k the convex cone generated by the vectors $-a_1, \dots, -a_{k-1}, -a_{k+1}, \dots, -a_{n+1}$ (if $k = n + 1$ then regard subsystem 1 as subsystem $k + 1$) for all $1 \leq k \leq n + 1$. It can be shown that C_1, \dots, C_{n+1} have mutually disjoint interiors and $C_1 \cup \dots \cup C_{n+1} = \mathbb{R}^n$ (due to the space limitation, we do not prove it here). Then we propose a switching law with a minimum dwell time τ as follows.

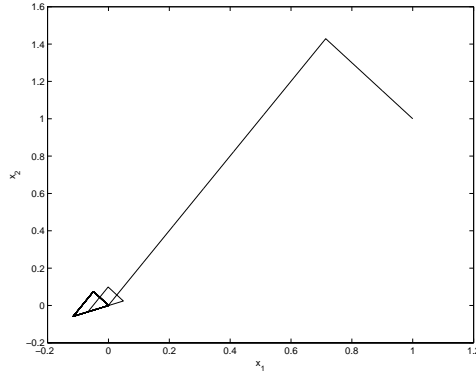


Figure 3: Example 3.1: A trajectory starting from $[1, 1]^T$.

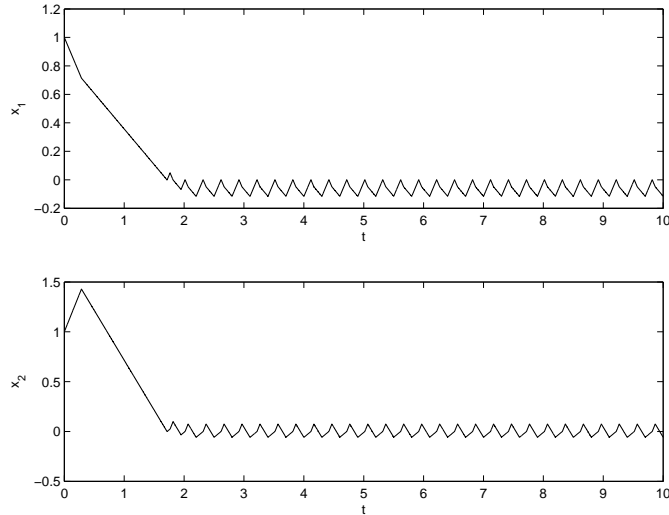


Figure 4: Example 3.1: $x_1(t)$ and $x_2(t)$.

A minimum dwell time switching law: Let subsystem $k + 1$ be active whenever the system state is in $\text{Int}(C_k)$, $1 \leq k \leq n + 1$. When the state is on the common boundary of convex cones, we choose the active subsystem to be the one corresponding to the convex cone that the trajectory has the potential to enter next, if the system still evolves according to the current active subsystem. Moreover, in order to eliminate the Zenoness phenomenon near the origin, besides the above rules, we also impose a minimum dwell time τ .

The choice of a minimum dwell time τ : For systems in \mathbb{R}^n , geometric observations as those in Example 3.1 are currently still under our research, because direct extensions of results \mathbb{R}^2 into higher dimensional space are not readily available. However, we note that given any $\epsilon > 0$, the above switching law will behave as ϵ -practically asymptotically stable if we choose τ small enough. In practice, we usually specify an ϵ and then reduce the value of τ and test the resulting trajectory until the ϵ -practically asymptotic stability is achieved.

Remark 3.2 Note that there are altogether $n + 1$ convex cones which cover \mathbb{R}^n and have nonintersecting interiors. In implementing the above switching law, we need to check to see which cone a given point x belongs to. For a practically stabilizable system with $n + 1$ subsystems, Corollary 3.3 tells us that any n vectors are linearly independent. Therefore the equation

$$[-a_1, \dots, -a_{k-1}, -a_{k+1}, \dots, -a_{n+1}]y_k = x \quad (3.4)$$

will have a unique solution y_k . If x is in the interior of a set C_{k_1} , then all the elements of y_{k_1} will be positive. However, for any $k \neq k_1$, the y_k will not have all positive elements. This fact can be used to determine which C_k the x is in. To do this, we only need to solve at most $n + 1$ linear equations (3.4) for different k 's. If x is on the boundary of two C_k 's, then the corresponding two y_k 's will both have all elements nonnegative, but not all positive. \square

4 A Three Tank Example

Now we apply the practical stabilization results developed in Section 3 to a chemical batch process example.

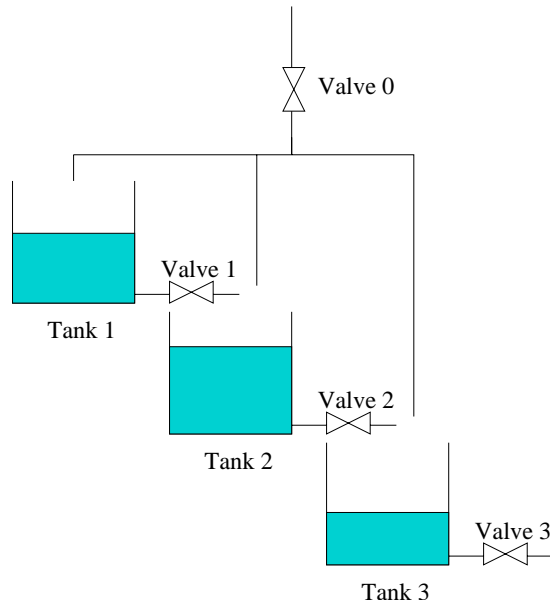


Figure 5: Example 4.1: The three tanks system.

Example 4.1 (A Three Tank Example) Consider the three tanks system in figure 5. All three tanks are identical and all flows cause the tank-levels to rise or decrease by 0.1 unit/sec. There are four allowable operating modes: Mode 1: Valve 0 is on, Valves 1,2,3 are off, the corresponding dynamics $\dot{x} = a_1 = [0.1, 0.1, 0.1]^T$; Mode 2: Valve 1 is on, Valves 0,2,3 are off, the corresponding dynamics $\dot{x} = a_2 = [-0.1, 0.1, 0]^T$; Mode 3: Valve 2 is on, Valves 0,1,3 are off, the corresponding dynamics $\dot{x} = a_3 = [0, -0.1, 0.1]^T$; Mode 4: Valve 3 is on, Valves 0,1,2 are off, the corresponding dynamics $\dot{x} = a_4 = [0, 0, -0.1]^T$. We want to develop a switching law such that the water levels in the tanks are driven toward the desired value $[80, 50, 70]^T$ and each tank level is then kept within $[-2, +2]$ range around the desired level.

Using Corollary 3.2, we can show this system with 4 subsystems in \mathbb{R}^3 is practically stabilizable. For this problem, we can choose $\epsilon = 2$ and apply the switching law proposed in Section 3.2 to make the system ϵ -practically asymptotically stable around the point $[80, 50, 70]^T$ (although the point is not the origin, but with state shift, the stabilization result can be applied). We choose $\tau = 5$ sec. Figure 6 shows the three tank levels starting from $[90, 45, 75]^T$. Note that when time becomes large, the maximum deviations from the desired point are 0.4999 for x_1 , 0.9999 for x_2 , and 1.4999 for x_3 . They satisfy the requirements. \square

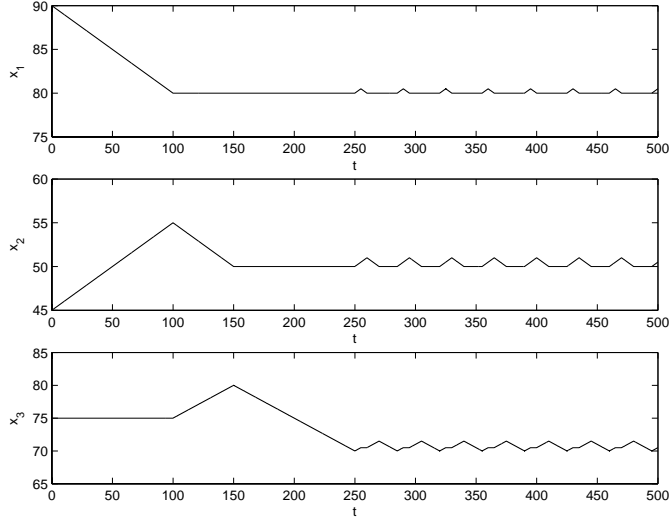


Figure 6: Example 4.1: The three tank levels starting from $[90, 45, 75]^T$.

5 Conclusion

This paper reports some results for practical stabilization problems of integrator switched systems. Some practical stability notions were introduced, and a necessary and sufficient condition for practical stabilizability of integrator switched systems was then given. Moreover, a minimum dwell time switching law for practically stabilizable systems in \mathbb{R}^n consisting of $n + 1$ subsystems was proposed which can be used to achieve ϵ -practically asymptotic stability. The research in this paper is a first step toward the studies of general nonlinear subsystems. Future research includes the estimation of bound for minimum dwell time for systems in \mathbb{R}^n , and extensions of the results to the studies of local behaviors of switched systems with nonintegrator subsystems.

Appendix A: Some Notions and Results from Convex Analysis

The following notions and results from convex analysis are needed in the proofs of the results in this paper. They can be found in [7, 9, 11].

First we introduce some topological concepts. Let $a \in \mathbb{R}^n$ and $r > 0$, the set $B(a; r) = \{x \in \mathbb{R}^n \mid \|x - a\| < r\}$ is said to be an open ball and the set $B[a; r] = \{x \in \mathbb{R}^n \mid \|x - a\| \leq r\}$ is said to be a closed ball. A point $x \in A$ is said to be an interior point of a set A if $B(x; r) \subseteq A$ for some $r > 0$. The set of all interior points of A is called the interior of A and is denoted by $\text{Int}(A)$.

Definition A.1 (Hyperplane) A hyperplane $H \subseteq \mathbb{R}^n$ is a set of the form $H = \{x \in \mathbb{R}^n \mid \nu^T x = d\}$, $\nu \in \mathbb{R}^n$, $\nu \neq 0$, $d \in \mathbb{R}$. \square

A hyperplane divides \mathbb{R}^n into two closed halfspaces $H^+ = \{x \in \mathbb{R}^n \mid \nu^T x \geq d\}$ and $H^- = \{x \in \mathbb{R}^n \mid \nu^T x \leq d\}$. A hyperplane H is said to *separate* two sets A and B if A lies in one of the closed halfspaces determined by H , and B lies in the other. If A and B lie in opposite open halfspaces $\text{Int}(H^+)$ and $\text{Int}(H^-)$ determined by H , then H is said to *separate* A and B *strictly*.

Definition A.2 (Convex Set) A set $A \subseteq \mathbb{R}^n$ is said to be convex if $x \in A$, $y \in A$ implies that $\lambda x + (1 - \lambda)y \in A$ for any $0 \leq \lambda \leq 1$. \square

Definition A.3 (Cone) A nonempty set $A \subset \mathbb{R}^n$ is called a cone if $\lambda a \in A$ whenever $a \in A$ and $\lambda \geq 0$. \square

Given a finite number of nonzero vectors $a_1, a_2, \dots, a_M \in \mathbb{R}^n$, the set $C = \{\sum_{i=1}^M \lambda_i a_i \mid \lambda_1 \geq 0, \dots, \lambda_M \geq 0\}$ is a cone and is also convex. C is often mentioned as the *convex cone* generated by $\{a_1, a_2, \dots, a_M\}$. It can also be shown C is closed.

The following lemma is one of the most useful results in convex analysis.

Lemma A.1 (Separation Lemma) *Let A and B be two disjoint convex sets in \mathbb{R}^n with A closed and B compact. Then A and B can be strictly separated by a hyperplane in \mathbb{R}^n .* \square

From Lemma A.1, we have the following corollary which is used in the proofs of our main results in Section 3.1.

Corollary A.1 *In \mathbb{R}^n let A be a closed convex set and let b be a point not lying in A . Then A and $\{b\}$ can be strictly separated by a hyperplane in \mathbb{R}^n .* \square

Appendix B: Some Proofs for Section 3

Proof of Theorem 3.1: “Only if” part: Assume that the integrator switched system is globally practically stabilizable, but $C \neq \mathbb{R}^n$. Then there exists a nonzero vector b such that $b \notin C$. Since $b \notin C$, it must be true that $0 = -b + b \notin -b + C$.

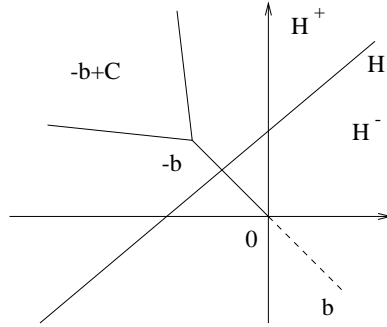


Figure 7: There exists a hyperplane H which strictly separates $-b + C$ and $\{0\}$.

Note that the set $-b + C$ is a translation of the set C . So $-b + C$ is a closed convex set because C is so as indicated in Appendix A. And 0 is a point not lying in $-b + C$. Then by Corollary 3.1, there exists a hyperplane H which strictly separates $-b + C$ and $\{0\}$ (see figure 7). It can be seen that under any switching sequence, the trajectory starting from $x_0 = -b$ must be within the set $-b + C$. Therefore the trajectory cannot enter the open halfspace $\text{Int}(H^-)$ where 0 is in. Hence there exists $\epsilon > 0$ small enough such that $B(0, \epsilon)$ is all in the open halfspace $\text{Int}(H^-)$. Consequently, the trajectory starting from $x_0 = -b$ cannot enter $B(0, \epsilon)$ under any valid switching law. This leads to a contradiction because the origin should be globally ϵ -attractive under some switching control law due to the globally practical stabilizability assumption.

“If” part: We prove this by actually constructing a valid switching law $\mathcal{S} = \mathcal{S}(\epsilon)$ that renders the system ϵ -practically stable and the origin globally ϵ -attractive, given any $\epsilon > 0$.

Assume we are given an $\epsilon > 0$. Let us first consider how to find a switching law for the ϵ -practical stability of the system. In order to do so, we first claim that we can find a switching law such that there exists a $G > 0$ and for any initial point in the closed unit ball $B[0; 1]$, the trajectory satisfies $\|x(t)\| < G$ for any $t \geq 0$. This switching law is constructed as follows. First let us consider the unit vectors e_1, e_2, \dots, e_n in \mathbb{R}^n (e_j is a column vector with all 0's except for the j -th element being 1) and their negatives $-e_1, -e_2, \dots, -e_n$. We denote them as $\hat{e}_1 = e_1, \hat{e}_2 = e_2, \dots, \hat{e}_n = e_n, \hat{e}_{n+1} = -e_1, \hat{e}_{n+2} = -e_2, \dots, \hat{e}_{2n} = -e_n$. Since $C = \mathbb{R}^n$, they have the representations

$$\hat{e}_1 = \sum_{i=1}^M \lambda_{1,i} a_i, \dots, \hat{e}_{2n} = \sum_{i=1}^M \lambda_{2n,i} a_i, \quad (\text{B.1})$$

with $\lambda_{k,i} \geq 0$. Furthermore, we note that every vector x in the $B[0, 1]$ can be represented as $x = \sum_{k=1}^n \alpha_k e_k$ where $\alpha_k \in \mathbb{R}$, $\sum_{k=1}^n \alpha_k^2 \leq 1$. By using the \hat{e}_k 's, x can be represented as

$$x = \sum_{k=1}^{2n} \beta_k \hat{e}_k, \quad (\text{B.2})$$

where, for $1 \leq k \leq n$

$$\beta_k = \begin{cases} \alpha_k, & \text{if } \alpha_k \leq 0, \\ 0, & \text{if } \alpha_k > 0, \end{cases}$$

and for $n + 1 \leq k \leq 2n$

$$\beta_k = \begin{cases} -\alpha_{k-n}, & \text{if } \alpha_{k-n} > 0, \\ 0, & \text{if } \alpha_{k-n} \leq 0. \end{cases}$$

Note that $\beta_k \leq 0$ for any $1 \leq k \leq 2n$, and $\sum_{k=1}^{2n} \beta_k^2 = \sum_{k=1}^n \alpha_k^2 \leq 1$.

Substituting (B.1) into (B.2), we can write x as

$$x = \sum_{k=1}^{2n} \beta_k \hat{e}_k = \sum_{k=1}^{2n} \beta_k \left(\sum_{i=1}^M \lambda_{k,i} a_i \right) = \sum_{i=1}^M \left(\sum_{k=1}^{2n} \beta_k \lambda_{k,i} \right) a_i = \sum_{i=1}^M \gamma_i a_i \quad (\text{B.3})$$

where $\gamma_i = \sum_{k=1}^{2n} \beta_k \lambda_{k,i} \leq 0$ for any $1 \leq i \leq M$.

Based on (B.3), we can construct the following switching law:

Switching Law A (for $x \in B[0; 1]$):

- (1). Assume that the system trajectory starts from $x \in B[0; 1]$ at time 0. Set the current state $x_{current} = x$ and the current time $t_{current} = 0$.
- (2). Obtain the expression for the current state $x_{current} = \sum_{i=1}^M \gamma_i a_i$. First switch to subsystem 1 and stay for time $|\gamma_1|$, then switch to subsystem 2 and stay for time $|\gamma_2|$ and so on. In other words, we obtain a switching sequence

$$((t_{current}, 1), (t_{current} + |\gamma_1|, 2), (t_{current} + |\gamma_1| + |\gamma_2|, 3), \dots, (t_{current} + |\gamma_1| + \dots + |\gamma_{M-1}|, M))$$

from time $t_{current}$ to $t_{current} + \sum_{i=1}^M |\gamma_i|$.

- (3). At time $t_{current} + \sum_{i=1}^M |\gamma_i|$, the trajectory reaches the origin, then we can let the system stay at subsystem M for time $\frac{1}{\|a_M\|}$ until it intersects the unit sphere.
- (4). Update $x_{current}$ to be the intersecting point and $t_{current}$ to be the time instant at intersection. Repeat steps (2) and (3).

Using Switching Law A, we obtain nonZero switching sequences for initial $x \in B[0; 1]$. The switching sequences are valid because $\frac{1}{\|a_M\|} > 0$ and every repetition of steps (2) and (3) will require at most M switchings in a time duration $\sum_{i=1}^M |\gamma_i| + \frac{1}{\|a_M\|}$.

Switching Law A also generates bounded trajectories for initial $x \in B[0; 1]$. Note that a trajectory starting from x will take time $\sum_{i=1}^M |\gamma_i|$ to reach the origin in step (2). For any $0 \leq t \leq \sum_{i=1}^M |\gamma_i|$ during this time period, we must have

$$\begin{aligned} \|x(t)\| &\leq \|x\| + \sum_{i=1}^M |\gamma_i| \cdot \max_{1 \leq i \leq M} (\|a_i\|) \\ &\leq 1 + \sum_{i=1}^M \left(\left(\sum_{k=1}^{2n} \beta_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{2n} \lambda_{k,i}^2 \right)^{\frac{1}{2}} \right) \cdot \max_{1 \leq i \leq M} (\|a_i\|), \text{ by using Cauchy-Schwarz Inequality} \\ &\leq 1 + M \max_{1 \leq i \leq M} \left(\sum_{k=1}^{2n} \lambda_{k,i}^2 \right)^{\frac{1}{2}} \cdot \max_{1 \leq i \leq M} (\|a_i\|) \end{aligned}$$

Define $G \triangleq 2 + M \max_{1 \leq i \leq M} \left(\sum_{k=1}^{2n} \lambda_{k,i}^2 \right)^{\frac{1}{2}} \cdot \max_{1 \leq i \leq M} (\|a_i\|)$, we will have $\|x(t)\| < G$ for any $t \geq 0$. It follows that for any $\epsilon > 0$, if we choose $\delta = \frac{\epsilon}{G}$, we can design a switching law similar to Switching Law A except for the scaling to the points starting in $B[0; \delta]$. We call such a switching law **Switching Law B**. Under Switching Law B, ϵ -practical stability of the system can be achieved.

Next let us consider the global ϵ -attractiveness of the origin. Starting from any initial point $x \in \mathbb{R}^n$, since $C = \mathbb{R}^n$, we have $-x = \sum_{i=1}^M \lambda_i a_i$, $\lambda_i \geq 0$. Hence $x = \sum_{i=1}^M (-\lambda_i) a_i$. We modify Switching Law B as follows. If $x \in B[0; \delta]$, apply Switching Law B for $x \in B[0; \delta]$ as mentioned above. If $x \notin B[0; \delta]$, we can first choose the switching sequence as

$$((0, 1), (\lambda_1, 2), (\lambda_1 + \lambda_2, 3), \dots, (\lambda_1 + \dots + \lambda_{M-1}, M))$$

until the trajectory reaches 0 at time $\sum_{i=1}^M \lambda_i$ and then follow Switching Law B. Hence the trajectory will always be inside the ball $B(0; \epsilon)$ after $T = \sum_{i=1}^M \lambda_i$. Such a modification provides us with a new switching law. We call it **Switching Law C**.

Clearly from the above constructions, if we choose $\mathcal{S} = \mathcal{S}(\epsilon)$ to be Switching Law C. \mathcal{S} renders the system ϵ -practical stable and the origin globally ϵ -attractive. Therefore the switched system is globally practically stabilizable. \square

Proof of Lemma 3.1: “If” part: Assume that a subset $\{a_{i_1}, \dots, a_{i_l}\}$ exists such that conditions (a) and (b) are satisfied. Without loss of generality, let us assume that it is $\{a_1, \dots, a_l\}$.

For any $x \in \mathbb{R}^n$, from (a), there exist α_j 's, $j = 1, \dots, l$ such that

$$x = \sum_{j=1}^l \alpha_j a_j. \quad (\text{B.4})$$

From (b), we have

$$\sum_{j=1}^l \lambda_j a_{i_j} = 0, \lambda_j > 0, j = 1, \dots, l. \quad (\text{B.5})$$

Multiplying (B.5) by a constant $c > 0$ and then adding to (B.4), we obtain

$$x = \sum_{j=1}^l (\alpha_j + c\lambda_j) a_j. \quad (\text{B.6})$$

Since $\lambda_j > 0$, we can have $\alpha_j + c\lambda_j \geq 0$ for all $1 \leq j \leq l$ if we choose c to be large enough. Now define

$$\hat{\lambda}_i = \begin{cases} \alpha_i + c\lambda_i, & i = 1, \dots, l, \\ 0, & i = l+1, \dots, M. \end{cases} \quad (\text{B.7})$$

x can then be expressed as $\sum_{i=1}^M \hat{\lambda}_i a_i$. Consequently, this shows $x \in C$ for any $x \in \mathbb{R}^n$ which is in C . Hence $C = \mathbb{R}^n$.

“Only if” part: Consider the unit vectors $\hat{e}_1, \dots, \hat{e}_{2n}$ defined in the proof of the ‘If’ part of Theorem 3.1 and their representations (B.1). Define A_1 to be the set of all a_i ’s for which the corresponding $\lambda_{1,i} > 0$ in the expression $\hat{e}_1 = \sum_{i=1}^M \lambda_{1,i} a_i$. Similarly define A_2 to be the set of all a_i ’s for which the corresponding $\lambda_{2,i} > 0$ in the expression $\hat{e}_2 = \sum_{i=1}^M \lambda_{2,i} a_i$, and so on. In this way, we can define the subsets A_1, A_2, \dots, A_{2n} corresponding to the expressions of $\hat{e}_1, \hat{e}_2, \dots, \hat{e}_{2n}$. Now if we define the subset $A = \cup_{k=1}^{2n} A_k$, we claim that A satisfies conditions (a) and (b). The reasons are as follows.

For (a), since any $x \in \mathbb{R}^n$ can be represented as a linear combination of $\hat{e}_1, \dots, \hat{e}_{2n}$ and every \hat{e}_k can be represented as a linear combination of the vectors in A , we conclude that x can be represented as a linear combination of the vectors in A . Hence (a) holds true.

For (b), assume that $A = \{a_{i_1}, \dots, a_{i_l}\}$. Now consider $\sum_{k=1}^{2n} \hat{e}_k$, by substituting the expressions $\hat{e}_1 = \sum_{i=1}^M \lambda_{1,i} a_i, \dots, \hat{e}_{2n} = \sum_{i=1}^M \lambda_{2n,i} a_i$ into it, we conclude that $\sum_{k=1}^{2n} \hat{e}_k = \sum_{j=1}^l \hat{\lambda}_j a_{i_j}$. Note $\hat{\lambda}_j > 0, j = 1, \dots, l$, because for each a_{i_j} there must be at least one \hat{e}_k in the expression of which $\lambda_{k,i_j} > 0$. On the other hand, we note that $\sum_{k=1}^{2n} \hat{e}_k = \sum_{k=1}^n e_k - \sum_{k=1}^n e_k = 0$. Therefore, (b) holds true. \square

Proof of Theorem 3.2: (a). If system (3.1) is globally practically stabilizable, then by Theorem 3.1, we have $C = \mathbb{R}^n$. By Lemma 3.1, there exists a subset $\{a_{i_1}, \dots, a_{i_l}\}$ of $\{a_1, \dots, a_M\}$ whose span is \mathbb{R}^n , and therefore $l \geq n$. However, if $l = n$, the only solution to $\sum_{j=1}^l \lambda_j a_{i_j} = 0$ is $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$ since a_{i_j} ’s must be linearly independent in this case. This is a contradiction to condition (b) in Lemma 3.1. Hence $l > n$ and consequently we conclude that $M \geq l \geq n+1$.

(b). In the following, we construct an integrator switched system consisting of $n+1$ subsystems which is globally practically stabilizable. Assume that a_1, \dots, a_n are linearly independent vectors in \mathbb{R}^n , let

$$a_{n+1} = -\sum_{i=1}^n \lambda_i a_i, \lambda_i > 0. \quad (\text{B.8})$$

Now let us show that the convex cone formed by $\{a_1, a_2, \dots, a_n, a_{n+1}\}$, i.e., $C = \{\sum_{i=1}^{n+1} \lambda_i a_i | \lambda_i \geq 0\}$ satisfies $C = \mathbb{R}^n$.

First of all, for any $x \in \mathbb{R}^n$, there exist $\alpha_1, \dots, \alpha_n \in \mathbb{R}^n$ (not necessarily nonnegative) such that

$$x = \sum_{i=1}^n \alpha_i a_i. \quad (\text{B.9})$$

Next let us discuss the two possible cases of α_i ’s as follows.

Case 1. If all $\alpha_i \geq 0$, then it follows that $x \in C$.

Case 2. If there exists an $\alpha_i < 0$, then notice from (B.8) we have

$$0 = a_{n+1} + \sum_{i=1}^n \lambda_i a_i. \quad (\text{B.10})$$

Let $c = \max_{\{i | \alpha_i < 0\}} \{-\frac{\alpha_i}{\lambda_i}\}$. Multiplying (B.10) by c and adding to (B.9), we obtain

$$x = ca_{n+1} + (\alpha_i + c\lambda_i) a_i.$$

Since $c > 0$ and $\alpha_i + c\lambda_i \geq 0$, $x \in C$ is true in this case.

From the above arguments, we conclude that any $x \in \mathbb{R}^n$ can be expressed as a nonnegative linear combination of a_1, \dots, a_{n+1} . Consequently, we conclude $b \in C$ and hence $\mathbb{R}^n \subseteq C$. Since C is also a subset of \mathbb{R}^n , $C = \mathbb{R}^n$ is proved. \square

Proof of Corollary 3.1: From the proof of part (a) of Theorem 3.2, we conclude that the only subset that satisfies the conditions of Lemma 3.1 is the set $\{a_1, \dots, a_{n+1}\}$ in this case. This completes the proof. \square

Proof of Corollary 3.2: We only need to prove that the condition in Corollary 3.2 is equivalent to the condition in Corollary 3.1.

First of all, assume the condition “ $\text{span}\{a_1, \dots, a_{n+1}\} = \mathbb{R}^n$ and there exist $\lambda_i > 0, i = 1, \dots, n+1$ such that $\sum_{i=1}^{n+1} \lambda_i a_i = 0$ ” in Corollary 3.1 is true. However, assume the condition in Corollary 3.2 does not hold. Without loss of generality, assume a_1, \dots, a_n are linearly dependent. Then the dimensionality of $\text{span}\{a_1, \dots, a_n\}$ is less than n . Since $\sum_{i=1}^{n+1} \lambda_i a_i = 0, \lambda_i > 0, i = 1, \dots, n+1$, we conclude that a_{n+1} can be represented as a linear combination of a_1, \dots, a_n . Hence the dimensionality of $\text{span}\{a_1, \dots, a_{n+1}\}$ is the same as that of $\text{span}\{a_1, \dots, a_n\}$ which is less than n . This leads to a contradiction to $\text{span}\{a_1, \dots, a_{n+1}\} = \mathbb{R}^n$. Hence the condition in Corollary 3.2 holds true.

On the other hand, it is clear that the condition in Corollary 3.2 implies the condition in Corollary 3.1. Therefore the proof is completed. \square

Proof of Corollary 3.3: Assume without loss of generality that the subset is $\{a_1, \dots, a_{n+1}\}$, then we only need to switch among these $n+1$ subsystems in order to practically stabilize the system. The practical stabilizability of the system consisting of these $n+1$ subsystems can therefore lead to the practical stabilizability of the original system. \square

Appendix C: Some Geometric Observations for Choosing τ in Example 3.1

Given an $\epsilon > 0$, we can choose τ based on the following reasonings. Figure 8 helps our reasonings below.

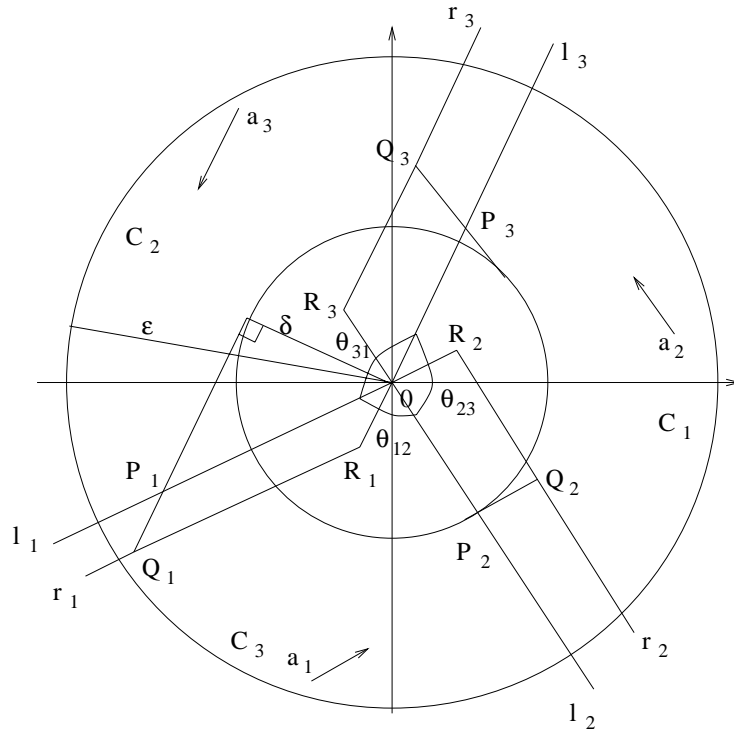


Figure 8: Geometric observations for choosing τ for Example 3.1.

First of all, let us consider ϵ -practical stability. From Definition 2.3, we need to have a ball $B[0; \delta]$ (see Appendix A for the definition of balls) such that any trajectory starting in this ball will remain in $B(0; \epsilon)$. Figure 8 depicts the two balls. l_i 's are the rays corresponding to $-a_i$'s. Assume that x is in $B[0; \delta] \cap C_2$ and subsystem 3 is active. Also assume that the points P_1, Q_1 are on the line which is tangent to $B[0; \delta]$ and parallel to l_3 , P_1 is on l_1 , and $|P_1Q_1| = \|a_3\|\tau$. Moreover, assume the points O, P_1, Q_1 , and R_1 form a parallelogram. Now we note that for any point in $B[0; \delta] \cap C_2$, when subsystem 3 is active and the system follows the minimum dwell time switching law proposed in Example 3.1, the trajectory will either intersect l_1 and switch to subsystem 1 immediately (when the time elapsed is no less than τ), or it will enter $OP_1Q_1R_1$ and then switch to subsystem 1 (when time elapsed is equal to τ). The importance of $OP_1Q_1R_1$ lies in the fact that for trajectories starting from $B[0; \delta] \cap C_2$ and following subsystem 3, all trajectories will switch to subsystem 1 in $OP_1Q_1R_1$. As long as the line segment OR_1 is in $B[0; \delta]$, by following subsystem 1, the trajectory will eventually intersect OR_1 and hence be in $B[0; \delta] \cap C_3$. Similar

arguments can be applied to show that the trajectories starting in $B[0; \delta] \cap C_3$ will switch in the parallelogram $0P_2Q_2R_2$ and then enter into $B[0; \delta] \cap C_1$; and the trajectories starting in $B[0; \delta] \cap C_1$ will switch in the parallelogram $0P_3Q_3R_3$ and then enter into $B[0; \delta] \cap C_2$. Now in order to achieve ϵ -practical stability, a sufficient condition is to require that the farthest point Q_1 of the parallelogram $0P_1Q_1R_1$ be inside $B(0; \epsilon)$. A sufficient condition for this is $|0P_1| + |P_1Q_1| \leq \epsilon$ which can also be written as

$$\|a_3\|\tau + \frac{\delta}{\sin \theta_{31}} \leq \epsilon, \quad (\text{C.1})$$

where θ_{31} is the angle extended by l_3 and l_1 ($0 < \theta_{31} < \pi$). Also note from our above discussion, we require that $0R_1$ be in $B[0; \delta]$, which is equivalent to

$$\|a_3\|\tau \leq \delta. \quad (\text{C.2})$$

Similarly, we can obtain the inequalities

$$\|a_1\|\tau + \frac{\delta}{\sin \theta_{12}} \leq \epsilon, \quad (\text{C.3})$$

$$\|a_1\|\tau \leq \delta, \quad (\text{C.4})$$

$$\|a_2\|\tau + \frac{\delta}{\sin \theta_{23}} \leq \epsilon, \quad (\text{C.5})$$

$$\|a_2\|\tau \leq \delta. \quad (\text{C.6})$$

From (C.1)-(C.6), we find that if we choose

$$\tau \leq \min \left\{ \frac{1}{\|a_1\|} \left(\epsilon - \frac{\delta}{\sin \theta_{12}} \right), \frac{1}{\|a_2\|} \left(\epsilon - \frac{\delta}{\sin \theta_{23}} \right), \frac{1}{\|a_3\|} \left(\epsilon - \frac{\delta}{\sin \theta_{31}} \right), \frac{\delta}{\|a_1\|}, \frac{\delta}{\|a_2\|}, \frac{\delta}{\|a_3\|} \right\}, \quad (\text{C.7})$$

then the switching law with τ satisfying (C.7) will lead to ϵ -practical stability. Note that the δ in (C.7) corresponds to the δ in Definition 2.3 and can be chosen by the designer, however it must satisfy the following condition

$$\delta < \min \{ \epsilon \sin \theta_{12}, \epsilon \sin \theta_{23}, \epsilon \sin \theta_{31} \}, \quad (\text{C.8})$$

so that the τ in (C.1), (C.3), (C.5) can take positive value. Besides the constraint (C.8), we can freely choose δ to achieve different bounds for τ .

We claim that the switching law in Example 3.1 with τ satisfying (C.7) also achieves ϵ -attractiveness. This is because any trajectory starting in C_2 following subsystem 3 will enter into the band formed by l_1 , $0R_1$, and the ray r_1 which emits from R_1 and is in the direction of R_1Q_1 (see figure 8) and then switch to subsystem 1. Therefore, after one switching from subsystem 3 to 1, all trajectories starting in C_2 can then intersect $0R_1$, which is in $B[0; \delta]$. Then by the above arguments for ϵ -practical stability, the trajectory will always be in $B(0, \epsilon)$.

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