Optimal Control of Hybrid Autonomous Systems with State Jumps *

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Abstract

In this paper, optimal control problems for hybrid autonomous systems with state jumps are studied. In particular, we focus on problems in which a prespecified sequence of active subsystems is given and propose an approach to find the optimal switching instants. Specifically, the derivatives of the cost with respect to the switching instants are derived and nonlinear optimization techniques are used to locate the optimal switching instants. The approach is then applied to general quadratic problems for hybrid autonomous systems with linear subsystems and linear state jumps, where it is shown that the special structure of the problem leads to reduced computational effort. As an application, we use our optimal control results to address important reachability problems. Examples illustrate the results.

1 Introduction

A hybrid system is a dynamic system that involves both continuous and discrete event dynamics. The subsystem continuous dynamics are usually described by differential/difference equations and the discrete event dynamics are described by switching laws. Discontinuous jumps of continuous states may occur when the system switches from one subsystem to another. Examples of hybrid systems can be found in chemical processes, automotive systems, and electrical circuit systems.

Recently, many results for optimal control of hybrid systems have appeared in the literature. [4, 5, 9, 13, 14, 15, 17] report some theoretical results, for example, extensions of the classical maximum principle and/or the dynamic programming to such problems. However, due to the lack of efficient constructive methodologies in these papers, it is difficult to apply such theoretical results to locate the optimal solutions. On the other hand, several practical approaches have been proposed for finding numerical solutions to various classes of hybrid systems optimal control problems (see, e.g., [2, 7, 8, 10, 11, 12, 16, 22, 23]). Many of these papers find approximations to local optimal solutions.

In this paper, we focus on optimal control problems for a class of hybrid systems where each subsystem is autonomous (i.e., with no continuous input) and state jumps are present at the switching instants. For such problems, we develop an effective approach for finding accurate numerical values of local optimal solutions. In particular, we focus on problems in which a prespecified sequence of active subsystems is given. Such problems arise naturally in multimodal control and in logic-based control systems whose controllers are switched among several given controllers. Nonlinear autonomous subsystems and performance costs which are not necessarily quadratic are considered in the paper. We note that the cost is actually a function of the switching instants for such problems and use constrained nonlinear optimization techniques to locate

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the optimal switching instants. To apply nonlinear optimization techniques, we need first to determine the values of the derivatives of the cost with respect to the switching instants. An approach is proposed for their derivations and is presented in detail. One of the main results of the paper is Theorem 3.1 which gives us the expressions of the derivatives and makes possible the calculation of accurate values of the derivatives. Then the approach is applied to general quadratic problems for hybrid autonomous systems with linear subsystems and linear state jumps. The computation of the derivatives can further be simplified by utilizing the special structure of such problems. Finally, we apply the optimal control approach to reachability problems. Using the approach, the reachability switching instants can be determined if a final state is determined to be reachable from an initial state.

In our earlier papers [19, 22, 23], we studied switched systems optimal control problems which require the solutions of optimal switching instants and optimal continuous inputs. Two approaches were proposed for finding the values of the derivatives of the optimal cost with respect to the switching instants; [23] reports an approach that finds the approximations of the derivatives based on direct differentiations of the value functions, and [19, 22] report an other approach that finds accurate numerical values of the derivatives based on solutions of two point boundary differential algebraic equations. As opposed to switched systems without jumps in [19, 22, 23], in this paper we focus on hybrid autonomous systems with state jumps which are an important class of hybrid systems. Here we extend the results in [23] to such hybrid systems and, moreover, by taking advantage of the autonomous subsystems we develop an approach to obtain accurate derivative values as opposed to the approximations in [23]. Our earlier results of the research in this paper were reported in [20, 21] for switched autonomous systems. Another earlier result for a special class of systems with state jumps was reported in [18].

It is worth noting that most of the available literature results on numerical solutions of hybrid systems optimal control problems are for discrete-time hybrid systems [2, 11, 12], or based on the discretizations of time and/or state spaces [10, 16]. However, the discretization approaches may lead to combinatoric explosion and the solutions obtained may not be accurate enough. Unlike these results, the problem we consider in this paper is for continuous-time systems and the approach here is not based on discretization; hence our approach can provide us with accurate values of local minima. The closest literature results to our paper, as far as we are aware of, are [7, 8] which present closed-loop solutions to a special class of problems, i.e., infinite horizon problems for switched linear autonomous systems. However, we should indicate that our approach has the following advantages. First, our approach can deal with finite horizon problems with nonlinear subsystems, and with costs which are not necessarily quadratic, as opposed to infinite horizon problems with linear subsystems and quadratic costs in [7, 8]. Moreover, our approach can be applied to reachability problems, while the approach in [7, 8] fits better for stability problems. In view of the above, we believe our results are new and contribute to the understanding and the solution of optimal control problems of hybrid systems.

The structure of the paper is as follows. In Section 2, we formulate the optimal control problem and propose an algorithm for solving it. In Section 3, detailed derivations are presented to show how to obtain the derivatives of the cost with respect to the switching instants. In Section 4, these results are applied to general quadratic problems for hybrid autonomous systems with linear subsystems and linear state jumps. In Section 5, we show how to apply the optimal control result to reachability problems. Examples are given in Section 6. Section 7 concludes the paper.

2 Problem Formulation

In this paper, we consider *hybrid autonomous systems with state jumps* defined as follows. The hybrid system consists of autonomous subsystems (i.e., without continuous input)

$$\dot{x} = f_i(x), \quad f_i : \mathbb{R}^n \to \mathbb{R}^n, \quad i \in I = \{1, 2, \cdots, M\}. \tag{2.1}$$

and whenever the system dynamics switches from subsystem i_k to subsystem i_{k+1}, a discontinuous jump of the state x will occur, which are described by a function

$$x(t_k^+) = \gamma^{i_k, i_{k+1}}\big(x(t_k^-)\big) \tag{2.2}$$

where $x(t_k^+)$ and $x(t_k^-)$ are the righthand limit and lefthand limit of the state x at t_k, respectively.

For such a hybrid system, one can control its state trajectory evolution by choosing appropriate switching sequences. Here a *switching sequence* σ in $[t_0, t_f]$ is defined as

$$\sigma = ((t_0, i_0), (t_1, i_1), (t_2, i_2), \cdots, (t_K, i_K)), \tag{2.3}$$

with $0 \le K < \infty$, $t_0 \le t_1 \le t_2 \le \cdots \le t_K \le t_f$, and $i_k \in I$, $k = 0, 1, 2, \cdots, K$. σ tells us that the switched system switches to subsystem i_k at time instant t_k.

In the following, we assume that a prespecified sequence of active subsystems is given (i.e., the untimed sequence $(i_0, i_1, \cdots, i_K)$ is given). Furthermore, we assume without loss of generality that the untimed sequence is $(1, 2, \cdots, K, K+1)$, i.e., subsystem k is active in $[t_{k-1}, t_k)$. Note that we can always do this by relabeling the subsystem indices and even expanding the collection of subsystems (i.e., two subsystems may actually refer to the same actual subsystem). Under such assumptions, we can simply denote the state jump function at the k-th switching as γ^k. We consider the following optimal control problem.

Problem 2.1 (Optimal Control Problem) *Consider a hybrid autonomous system with state jumps, which consists of subsystems $f_i(x)$, $i \in I$. Assume that a prespecified sequence of active subsystems $(1, 2, \cdots, K, K+1)$ is given. Find optimal switching instants $t_1, \cdots, t_K (t_0 \le t_1 \le \cdots \le t_K \le t_f)$ such that the corresponding continuous state trajectory x departs from a given initial state $x(t_0) = x_0$ and the cost*

$$J(t_1, \cdots, t_K) = \psi\big(x(t_f)\big) + \int_{t_0}^{t_f} L(x)\ dt + \sum_{k=1}^{K} \psi^k\big(x(t_k^-)\big) \tag{2.4}$$

is minimized. Here t_0, t_f are given. □

Problem 2.1 is an optimal control problem in Bolza form. Unlike conventional optimal control problems, here the cost J includes the costs ψ^k's for discontinuous jumps at t_k's. As in the usual practice of formulating optimal control problems (see [1]), in the sequel, we assume that f_k's, L are continuous and have continuous partial derivatives; ψ, ψ^k's, and γ^k's are assumed to have twice continuous derivatives.

Remark 2.1 Due to the smoothness assumptions for f_i's, L, ψ, ψ^k's, and γ^k's, we can observe that a small disturbance of $(t_1, \cdots, t_K)$ will only cause a small disturbance of the J value. Furthermore, it is not difficult to show that the cost J is a continuously differentiable function of $(t_1, \cdots, t_K)$. □

2.1 An Algorithm

Note that Problem 2.1 is actually a constrained multivariable optimization problem

$$\begin{array}{c} \min_{\hat{t}} J(\hat{t}) \\ \text{subject to } \hat{t} \in T \end{array} \tag{2.5}$$

where $T \triangleq \{\hat{t} = (t_1, t_2, \cdots, t_K)^T | t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_K \leq t_f\}$. The following algorithm can be adopted to solve such a nonlinear optimization problem.

Algorithm 2.1

(1). Set the iteration index $j = 0$. Choose an initial $\hat{t}^j$.

(2). Find $J(\hat{t}^j)$, $\frac{\partial J}{\partial \hat{t}}(\hat{t}^j)$ and $\frac{\partial^2 J}{\partial \hat{t}^2}(\hat{t}^j)$.

(3). Use some first-order or second-order feasible direction method (e.g., the gradient projection method or the constrained Newton's method [3]) to update $\hat{t}^j$ to be $\hat{t}^{j+1} = \hat{t}^j + \alpha^j d\hat{t}^j$ (here $d\hat{t}^j = -\big(\frac{\partial^2 J}{\partial \hat{t}^2}(\hat{t}^j)\big)^{-1}\big(\frac{\partial J}{\partial \hat{t}}(\hat{t}^j)\big)^T$ and the stepsize α^j can be chosen using, e.g., the Armijo's rule [3]). Set the iteration index $j = j + 1$.

(4). Repeat Steps (2), (3) and (4), until a prespecified termination condition is satisfied (e.g. $\|\frac{\partial J}{\partial \hat{t}}(\hat{t}^j)\|_2 < \epsilon$ where ϵ is a given small number). □

In order to apply the above algorithm, one needs to find the values of the derivatives $\frac{\partial J}{\partial \hat{t}}$ and $\frac{\partial^2 J}{\partial \hat{t}^2}$ (step (2)). Let us elaborate more on step (2) in the sequel.

3 Differentiations of the Cost Function

In this section, we propose an approach based on the direct differentiations of the cost function to finding the values of the derivatives $\frac{\partial J}{\partial \hat{t}}$ and $\frac{\partial^2 J}{\partial \hat{t}^2}$. This is extends the results in [20, 21, 23].

Assume that we have a nominal $\hat{t} = (t_1, \cdots, t_K)^T$ and the corresponding nominal state trajectory $x(t)$. For such nominal values, the cost is

$$J(t_1, \cdots, t_K) = \psi\big(x(t_f)\big) + \int_{t_0}^{t_1^-} L(x)\ dt + \int_{t_1^+}^{t_2^-} L(x)\ dt + \cdots + \int_{t_K^+}^{t_f} L(x)\ dt + \sum_{j=1}^{K} \psi^j\big(x(t_j^-)\big). \tag{3.1}$$

Since x_0 and t_0 are given in Problem 2.1, J will not be a function of them. Next we define the value function at the k-th switching instant to be

$$J^k\big(x(t_k^+), t_k, \cdots, t_K\big) \triangleq \psi\big(x(t_f)\big) + \int_{t_k^+}^{t_{k+1}^-} L(x)\ dt + \cdots + \int_{t_K^+}^{t_f} L(x)\ dt + \sum_{j=k+1}^{K} \psi^j\big(x(t_j^-)\big). \tag{3.2}$$

Note that, unlike J, J^k for $k = 1, \cdots, K$ will be a function of t_k and of the initial state $x(t_k^+)$ which depends on the trajectory before t_k. Also note that J^K does not have the state jump cost and it is

$$J^K\big(x(t_K^+), t_K\big) \triangleq \psi\big(x(t_f)\big) + \int_{t_K^+}^{t_f} L(x)\ dt. \tag{3.3}$$

The relationship between J^k and J^{k+1} is

$$J^k\big(x(t_k^+), t_k, \cdots, t_K\big) = \int_{t_k^+}^{t_{k+1}^-} L(x)\ dt + \psi^{k+1}\big(x(t_{k+1}^-)\big) + J^{k+1}\big(x(t_{k+1}^+), t_{k+1}, \cdots, t_K\big) \tag{3.4}$$

for $k = 1, 2, ..., K-1$. In order to make our presentation clear, in the sequel, we denote $\frac{\partial J^k}{\partial x}$ for the function J^k as a row vector J_x^k, $\frac{\partial^2 J^k}{\partial x^2}$ as an $n \times n$ matrix J_{xx}^k and so on.

3.1 Single Switching

Let us first consider the case of a single switching. Assume that we are given a nominal t_1 and the corresponding nominal state trajectory $x(t)$, we denote by $\hat{x}(t)$ the state trajectory after a variation dt_1 has taken place. In the sequel, we adopt the following notational convention. We write f and f_x with a superscript $1-$ (resp. $1+$) whenever the corresponding active vector field at t_1- (resp. t_1+) is used for evaluation at $x(t_1^-)$ (resp. $x(t_1^+)$. Examples of this convention are $f^{1-} \triangleq f_1\big(x(t_1^-)\big)$, $f^{1+} \triangleq f_2\big(x(t_1^+)\big)$, $f_x^{1-} \triangleq \frac{\partial f_1}{\partial x}\big(x(t_1^-)\big)$, $f_x^{1+} \triangleq \frac{\partial f_2}{\partial x}\big(x(t_1^+)\big)$. Also, we simply write a function's name with a superscript $1-$ (resp. $1+$) whenever the corresponding function is evaluated at $x(t_1^-)$ (resp. $x(t_1^+)$). Examples are $J^{1+} \triangleq J^1\big(x(t_1^+), t_1\big)$, $J_x^{1+} \triangleq \frac{\partial J^1}{\partial x}\big(x(t_1^+), t_1\big)$, $L^{1-} \triangleq L\big(x(t_1^-)\big)$, $L^{1+} \triangleq L\big(x(t_1^+)\big)$, $L_x^{1-} \triangleq \frac{\partial L}{\partial x}\big(x(t_1^-)\big)$, $\psi^{1-} \triangleq \psi^1\big(x(t_1^-)\big)$,$\cdots$ (be careful to distinguish the values J^{1+}, J_x^{1+}, L^{1-}, L_x^{1-},$\cdots$ from the functions $J^1\big(x(t_1^+), t_1\big)$, $J_x^1\big(x(t_1^+), t_1\big)$, $L(x)$, $L_x(x)$,$\cdots$). We also simply denote the lefthand (resp. righthand) limit of $(t_1 + dt_1)$ as $t_1 + dt_1^-$ (resp. $t_1 + dt_1^+$) instead of the longer notation $(t_1 + dt_1)-$ (resp. $(t_1 + dt_1)+$).

Now consider $J(t_1)$ which can be expressed as

$$J(t_1) = \int_{t_0}^{t_1^-} L(x)\ dt + \psi^1\big(x(t_1^-)\big) + J^1\big(x(t_1^+), t_1\big). \tag{3.5}$$

For a small variation dt_1 of t_1, we have

$$J(t_1 + dt_1) = \int_{t_0}^{t_1 + dt_1^-} L(\hat{x})\ dt + \psi^1\big(\hat{x}(t_1 + dt_1^-)\big) + J^1\big(\hat{x}(t_1 + dt_1^+), t_1 + dt_1\big). \tag{3.6}$$

There are three terms in (3.6). Let us consider the second order Taylor expansion of each term. In the following derivations we denote $dx(t_1^-) \triangleq \hat{x}(t_1 + dt_1^-) - x(t_1^-)$ and $dx(t_1^+) \triangleq \hat{x}(t_1 + dt_1^+) - x(t_1^+)$.

Consider the first term $\int_{t_0}^{t_1 + dt_1^-} L(\hat{x})\ dt$ in (3.6), if $dt_1 \geq 0$, we have

$$\begin{aligned} \int_{t_0}^{t_1 + dt_1^-} L(\hat{x})\ dt &= \int_{t_0}^{t_1^-} L(x)\ dt + \int_{t_1^-}^{t_1 + dt_1^-} L(\hat{x})\ dt \\ &= \int_{t_0}^{t_1^-} L(x)\ dt + L^{1-} dt_1 + \frac{1}{2} dt_1 L_x^{1-} dx(t_1^-) + \text{H.O.T.} \end{aligned} \tag{3.7}$$

where H.O.T. stands for Higher Order Terms. Note that in deriving (3.6), we have used the relationship $\hat{x}(t_1^-) = x(t_1^-)$. If $dt_1 < 0$, we have

$$\begin{aligned} \int_{t_0}^{t_1 + dt_1^-} L(\hat{x})\ dt &= \int_{t_0}^{t_1^-} L(x)\ dt + \int_{t_1^-}^{t_1 + dt_1^-} L(x)\ dt \\ &= \int_{t_0}^{t_1^-} L(x)\ dt + L^{1-} dt_1 + \frac{1}{2} dt_1 L_x^{1-} dx(t_1^-) + \text{H.O.T.} \end{aligned} \tag{3.8}$$

which has the same expression as (3.7) although the derivation is slightly different.

For the second term in (3.6), we have

$$\begin{aligned} \psi^1\big(\hat{x}(t_1 + dt_1^-)\big) &= \psi^1\big(x(t_1^-) + dx(t_1^-)\big) \\ &= \psi^{1-} + \psi_x^{1-} dx(t_1^-) + \frac{1}{2}\big(dx(t_1^-)\big)^T \psi_{xx}^{1-} dx(t_1^-) + \text{H.O.T.} \end{aligned} \tag{3.9}$$

For the third term in (3.6), we have the second order expansion

$$\begin{aligned} J^1\big(\hat{x}(t_1 + dt_1^+), t_1 + dt_1\big) &= J^{1+} + J_x^{1+} dx(t_1^+) + J_{t_1}^{1+} dt_1 + \frac{1}{2}\big(dx(t_1^+)\big)^T J_{xx}^{1+} dx(t_1^+) + \frac{1}{2} J_{t_1 t_1}^{1+} dt_1^2 \\ &\quad + dt_1 J_{t_1 x}^{1+} dx(t_1^+) + \text{H.O.T.} \end{aligned} \tag{3.10}$$

In order to express (3.6) into second order expansions with respect to dt_1, we need to find the second order expansions of $dx(t_1^-)$, $dx(t_1^+)$ in terms of dt_1. First note that

$$dx(t_1^-) \stackrel{\triangle}{=} \hat{x}(t_1+dt_1^-) - x(t_1^-) = f^{1-}dt_1 + \frac{1}{2}f_x^{1-}f^{1-}dt_1^2 + o(dt_1^2). \tag{3.11}$$

Note that in (3.11), $o(dt_1^2)$ refers to a column vector with each element being $o(dt_1^2)$. We will not explicitly mention this later in the paper since it will be clear from the context. Next we have

$$\begin{aligned} dx(t_1^+) &\stackrel{\triangle}{=} \hat{x}(t_1+dt_1^+) - x(t_1^+) = \gamma^1\big(\hat{x}(t_1+dt_1^-)\big) - \gamma^1\big(x(t_1^-)\big) \\ &= \gamma_x^{1-}dx(t_1^-) + \frac{1}{2}\begin{bmatrix} \big(dx(t_1^-)\big)^T \frac{\partial^2 \gamma^1_{(1)}\left(x(t_1^-)\right)}{\partial x^2} dx(t_1^-) \\ \vdots \\ \big(dx(t_1^-)\big)^T \frac{\partial^2 \gamma^1_{(n)}\left(x(t_1^-)\right)}{\partial x^2} dx(t_1^-) \end{bmatrix} + \text{H.O.T.} \end{aligned} \tag{3.12}$$

where $\gamma^1_{(j)}$ refers to the j-th element of the vector-valued function γ^1. Note that

$$\begin{bmatrix} \big(dx(t_1^-)\big)^T \frac{\partial^2 \gamma^1_{(1)}\left(x(t_1^-)\right)}{\partial x^2} dx(t_1^-) \\ \vdots \\ \big(dx(t_1^-)\big)^T \frac{\partial^2 \gamma^1_{(n)}\left(x(t_1^-)\right)}{\partial x^2} dx(t_1^-) \end{bmatrix} = \begin{bmatrix} (f^{1-})^T \frac{\partial^2 \gamma^1_{(1)}\left(x(t_1^-)\right)}{\partial x^2} \\ \vdots \\ (f^{1-})^T \frac{\partial^2 \gamma^1_{(n)}\left(x(t_1^-)\right)}{\partial x^2} \end{bmatrix} f^{1-}dt_1^2 + o(dt_1^2). \tag{3.13}$$

If we define

$$\xi^{1-} \stackrel{\triangle}{=} \begin{bmatrix} (f^{1-})^T \frac{\partial^2 \gamma^1_{(1)}\left(x(t_1^-)\right)}{\partial x^2} \\ \vdots \\ (f^{1-})^T \frac{\partial^2 \gamma^1_{(n)}\left(x(t_1^-)\right)}{\partial x^2} \end{bmatrix} \tag{3.14}$$

and substitute (3.13) into (3.12), we obtain

$$dx(t_1^+) = \gamma_x^{1-}f^{1-}dt_1 + \frac{1}{2}(\gamma_x^{1-}f_x^{1-} + \xi^{1-})f^{1-}dt_1^2 + o(dt_1^2) \tag{3.15}$$

Substituting (3.11) and (3.15) into (3.7), (3.9) and (3.10) and summing them, we obtain

$$\begin{aligned} J(t_1+dt_1) &= J(t_1) + L^{1-}dt_1 + \frac{1}{2}dt_1 L_x^{1-}dx(t_1^-) + \psi_x^{1-}dx(t_1^-) + \frac{1}{2}\big(dx(t_1^-)\big)^T \psi_{xx}^{1-}dx(t_1^-) \\ &\quad + J_x^{1+}dx(t_1^+) + J_{t_1}^{1+}dt_1 + \frac{1}{2}\big(dx(t_1^+)\big)^T J_{xx}^{1+}dx(t_1^+) + \frac{1}{2}J_{t_1t_1}^{1+}dt_1^2 \\ &\quad + dt_1 J_{t_1x}^{1+}dx(t_1^+) + \text{H.O.T.} \\ &= J(t_1) + (L^{1-} + \psi_x^{1-}f^{1-} + J_x^{1+}\gamma_x^{1-}f^{1-} + J_{t_1}^{1+})dt_1 \\ &\quad + \frac{1}{2}\Big(L_x^{1-}f^{1-} + \psi_x^{1-}f_x^{1-}f^{1-} + (f^{1-})^T\psi_{xx}^{1-}f^{1-} + J_x^{1+}(\gamma_x^{1-}f_x^{1-} + \xi^{1-})f^{1-} \\ &\quad + (f^{1-})^T(\gamma_x^{1-})^T J_{xx}^{1+}\gamma_x^{1-}f^{1-} + J_{t_1t_1}^{1+} + 2J_{t_1x}^{1+}\gamma_x^{1-}f^{1-}\Big)dt_1^2 + o(dt_1^2) \\ &\stackrel{\triangle}{=} J(t_1) + J_{t_1}dt_1 + \frac{1}{2}J_{t_1t_1}dt_1^2 + o(dt_1^2) \end{aligned} \tag{3.16}$$

Now let us consider $J^1\big(x(t_1^+), t_1\big)$ which is the value function for the given nominal $x(t_1^+)$ and t_1. The following dynamic programming equation holds for it

$$J_{t_1}^{1+} = -J_x^{1+}f^{1+} - L^{1+} \tag{3.17}$$

Note that (3.17) can be derived similarly to the HJB equation. However, the difference between it and the HJB equation is that (3.17) holds for any trajectory that is not necessarily optimal (for more details see [6]).

By differentiating (3.17), we obtain

$$\begin{aligned} J^{1+}_{t_1 x} &= -(f^{1+})^T J^{1+}_{xx} - J^{1+}_x f^{1+}_x - L^{1+}_x & (3.18) \\ J^{1+}_{t_1 t_1} &= -J^{1+}_{t_1 x} f^{1+} = (f^{1+})^T J^{1+}_{xx} f^{1+} + (J^{1+}_x f^{1+}_x + L^{1+}_x) f^{1+} & (3.19) \end{aligned}$$

Substituting these into (3.16) we have

$$\begin{aligned} J_{t_1} &= L^{1-} - L^{1+} + \psi^{1-}_x f^{1-} + J^{1+}_x (\gamma^{1-}_x f^{1-} - f^{1+}) & (3.20) \\ J_{t_1 t_1} &= (L^{1-}_x - L^{1+}_x \gamma^{1-}_x) f^{1-} + \psi^{1-}_x f^{1-}_x f^{1-} + (f^{1-})^T \psi^{1-}_{xx} f^{1-} \\ &\quad + J^{1+}_x (\gamma^{1-}_x f^{1-}_x + \xi^{1-} - f^{1+}_x \gamma^{1-}_x) f^{1-} - (J^{1+}_x f^{1+}_x + L^{1+}_x)(\gamma^{1-}_x f^{1-} - f^{1+}) \\ &\quad + (\gamma^{1-}_x f^{1-} - f^{1+})^T J^{1+}_{xx} (\gamma^{1-}_x f^{1-} - f^{1+}) & (3.21) \end{aligned}$$

3.2 Two or More Switchings

In order to construct a second-order optimization algorithm for hybrid systems with two or more switchings, we need more information to derive the derivatives of J with respect to the t_k's. Let us first consider the case of two switchings. Assume that a system switches from subsystem 1 to 2 at t_1 and from subsystem 2 to 3 at t_2 ($t_0 \le t_1 \le t_2 \le t_f$). The cost then is

$$\begin{aligned} J(t_1, t_2) &= \int_{t_0}^{t_1^-} L(x)\,dt + \psi^1\big(x(t_1^-)\big) + J^1\big(x(t_1^+), t_1, t_2\big) & (3.22) \\ &= \int_{t_0}^{t_1^-} L(x)\,dt + \psi^1\big(x(t_1^-)\big) + \int_{t_1^+}^{t_2^-} L(x)\,dt + \psi^2\big(x(t_2^-)\big) + J^2\big(x(t_2^+), t_2\big). & (3.23) \end{aligned}$$

Using (3.22), by holding t_2 fixed, J_{t_1}, $J_{t_1 t_1}$ can be derived similarly to that in subsection 3.1. On the other hand, if t_1 is held fixed, the first two terms in (3.23) will not contribute to the coefficients J_{t_2}, $J_{t_2 t_2}$. J_{t_2}, $J_{t_2 t_2}$ can then be derived using the expansion of the last three terms in (3.23) with respect to dt_2 similarly to that in subsection 3.1. However, we need additional information to derive $J_{t_1 t_2}$. Arguments from the calculus of variations will be used in the followings to derive it. Let us first define the important notion of incremental change which will be used in the sequel.

Definition 3.1 (Incremental Change) *Given any variations dt_1 and dt_2, we define $\delta x(t), \min\{t_1^+, t_1 + dt_1^+\} \le t \le \max\{t_2^-, t_2 + dt_2^-\}$ to be* the incremental change *of the state due to dt_1 and dt_2. In detail, it is defined as follows (see figure 1).*

Case 1: $dt_1 \ge 0, dt_2 \ge 0$ *(see figure 1(a))*

In this case, $\delta x(t)$ is defined to be

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), & t \in [t_1 + dt_1^+, t_2^-] \\ y_1(t) - x(t), & t \in [t_1^+, t_1 + dt_1^+] \\ \hat{x}(t) - z_1(t), & t \in [t_2^-, t_2 + dt_2^-] \end{cases} \qquad (3.24)$$

where $y_1(t)$ is the solution of

$$\begin{cases} \dot{y}_1(t) = f_2\big(y_1(t)\big), & t \in [t_1^+, t_1 + dt_1^+] \\ y_1(t_1 + dt_1^+) = \hat{x}(t_1 + dt_1^+) \end{cases} \qquad (3.25)$$

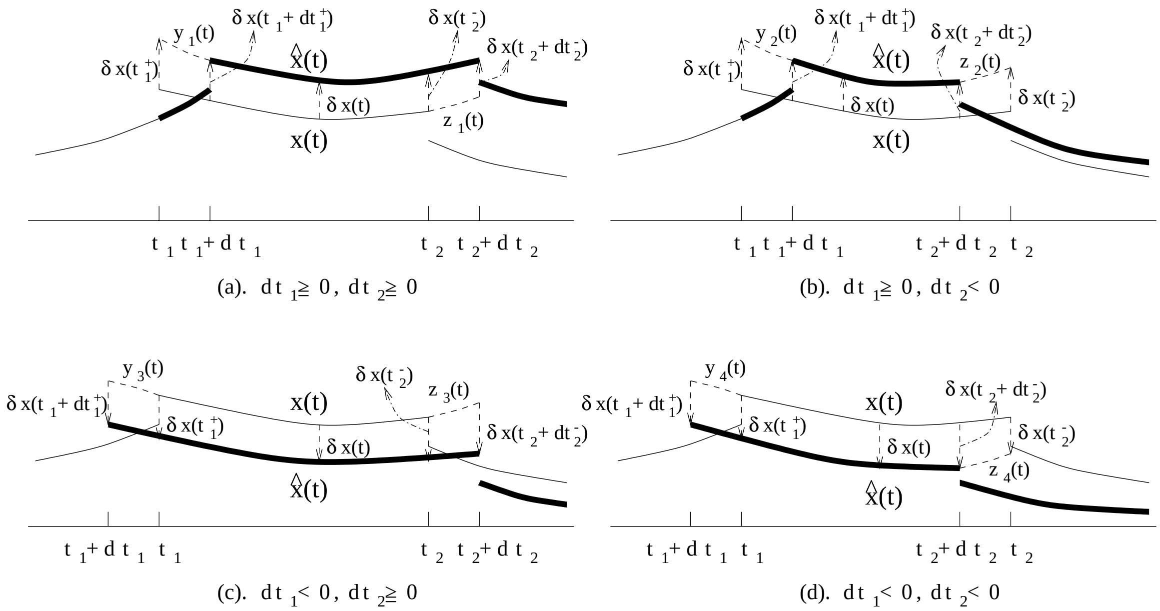

Figure 1: The incremental change $\delta x(t)$ for (a). $dt_1 \geq 0$, $dt_2 \geq 0$; (b). $dt_1 \geq 0$, $dt_2 < 0$; (c). $dt_1 < 0$, $dt_2 \geq 0$; (d). $dt_1 < 0$, $dt_2 < 0$.

and $z_1(t)$ *is the solution of*

$$\begin{cases} \dot{z}_1(t) = f_2\big(z_1(t)\big), \ t \in [t_2^-, t_2 + dt_2^-] \\ z_1(t_2^-) = x(t_2^-). \end{cases} \tag{3.26}$$

Case 2: $dt_1 \geq 0, dt_2 < 0$ *(see figure 1(b))*

In this case, $\delta x(t)$ *is defined to be*

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), \ t \in [t_1 + dt_1^+, t_2 + dt_2^-] \\ y_2(t) - x(t), \ t \in [t_1^+, t_1 + dt_1^+] \\ z_2(t) - x(t), \ t \in [t_2 + dt_2^-, t_2^-] \end{cases} \tag{3.27}$$

where $y_2(t)$ *is the solution of*

$$\begin{cases} \dot{y}_2(t) = f_2\big(y_2(t)\big), \ t \in [t_1^+, t_1 + dt_1^+] \\ y_2(t_1 + dt_1^+) = \hat{x}(t_1 + dt_1^+) \end{cases} \tag{3.28}$$

and $z_2(t)$ *is the solution of*

$$\begin{cases} \dot{z}_2(t) = f_2\big(z_2(t)\big), \ t \in [t_2 + dt_2^-, t_2^-] \\ z_2(t_2 + dt_2^-) = \hat{x}(t_2 + dt_2^-). \end{cases} \tag{3.29}$$

Case 3: $dt_1 < 0, dt_2 \geq 0$ *(see figure 1(c))*

In this case, $\delta x(t)$ *is defined to be*

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), \ t \in [t_1^+, t_2^-] \\ \hat{x}(t) - y_3(t), \ t \in [t_1 + dt_1^+, t_1^+] \\ \hat{x}(t) - z_3(t), \ t \in [t_2^-, t_2 + dt_2^-] \end{cases} \tag{3.30}$$

where $y_3(t)$ *is the solution of*

$$\begin{cases} \dot{y}_3(t) = f_2\big(y_3(t)\big), \ t \in [t_1 + dt_1^+, t_1^+] \\ y_3(t_1^+) = x(t_1^+) \end{cases} \tag{3.31}$$

and $z_3(t)$ *is the solution of*

$$\begin{cases} \dot{z}_3(t) = f_2\big(z_3(t)\big), \ t \in [t_2^-, t_2 + dt_2^-] \\ z_3(t_2^-) = x(t_2^-). \end{cases} \tag{3.32}$$

Case 4: $dt_1 < 0, dt_2 < 0$ *(see figure 1(d))*

In this case, $\delta x(t)$ *is defined to be*

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), \ t \in [t_1^+, t_2 + dt_2^-] \\ \hat{x}(t) - y_4(t), \ t \in [t_1 + dt_1^+, t_1^+] \\ z_4(t) - x(t), \ t \in [t_2 + dt_2^-, t_2^-] \end{cases} \tag{3.33}$$

where $y_4(t)$ *is the solution of*

$$\begin{cases} \dot{y}_4(t) = f_2\big(y_4(t)\big), \ t \in [t_1 + dt_1^+, t_1^+] \\ y_4(t_1^+) = x(t_1^+) \end{cases} \tag{3.34}$$

and $z_4(t)$ *is the solution of*

$$\begin{cases} \dot{z}_4(t) = f_2\big(z_4(t)\big), \ t \in [t_2 + dt_2^-, t_2^-] \\ z_4(t_2 + dt_2^-) = \hat{x}(t_2 + dt_2^-). \end{cases} \tag{3.35}$$

□

Remark 3.1 Note that $\delta x(t)$ defines the difference between $\hat{x}(t)$ and $x(t)$ in the time interval where subsystem 2 is active. Moreover, by extending the trajectories $\hat{x}$ and x under the dynamics of subsystem 2 to the time interval $[\min\{t_1^+, t_1 + dt_1^+\}, \max\{t_2^-, t_2 + dt_2^-\}]$ in which at least one of $\hat{x}(t)$ and $x(t)$ evolves along subsystem 2, $\delta x(t)$ even defines the difference for this interval. □

In the followings, the expressions for $\delta x(t_2^-)$, $dx(t_2^-)$, and $dx(t_2^+)$ are derived.

Lemma 3.1 *The expressions of* $\delta x(t_2^-)$ *and* $\delta x(t_2 + dt_2^-)$ *are as follows*

$$\delta x(t_2^-) = A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+})dt_1 + o(dt_1), \tag{3.36}$$

$$\begin{aligned} \delta x(t_2 + dt_2^-) = {} & A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+})dt_1 + f_x^{2-} A(t_2, t_1)(\gamma_x^{1-} f^{1-} - f^{1+})dt_1 dt_2 \\ & +(\textit{terms in } dt_1^2, dt_2^2 \textit{ and H.O.T.}), \end{aligned} \tag{3.37}$$

where $A(t_2^-, t_1^+)$ *is the state transition matrix for the variational equation*

$$\dot{y}(t) = \frac{\partial f_2\big(x(t)\big)}{\partial x} y(t) \tag{3.38}$$

for $y(t), t \in [t_1^+, t_2^-]$*; in (3.38),* x *is the current nominal state.*

Proof: See Appendix A. □

In fact, from the proof of Lemma 3.1 (see Appendix A), we can observe that $\delta x(t) = A(t, t_1^+)\delta x(t_1^+) + \big(\text{H.O.T. in } \delta x(t_1^+)\big) = A(t, t_1^+)\delta x(t_1^+) + o(dt_1)$ for any $t \in [\min\{t_1^+, t_1 + dt_1^+\}, \max\{t_2^-, t_2 + dt_2^-\}]$. The following important principle can be obtained directly from this observation. We refer to it as *the forward decoupling principle.* It reveals some intrinsic relationship among different switching instants.

The Forward Decoupling Principle:

(a). The value of the incremental change $\delta x(t_1^+)$ at t_1^+ does not depend on dt_2.

(b). The value of the incremental change $\delta x(t_2^-)$ at t_2^- does depend on dt_1. □

The forward decoupling principle tells us that a variation of an earlier switching instant will affect the value of the incremental change at a later switching instant, but not vice versa.

Lemma 3.2 *The expressions of $dx(t_2^-)$ (i.e., $\hat{x}(t_2+dt_2^-)-x(t_2^-)$) and $dx(t_2^+)$ (i.e., $\hat{x}(t_2+dt_2^+)-x(t_2^+)$) are*

$$\begin{aligned} dx(t_2^-) &= A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-}-f^{1+})dt_1 + f_x^{2-}A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-}-f^{1+})dt_1dt_2 + f^{2-}dt_2 \\ &\quad +(\textit{terms in } dt_1^2, dt_2^2 \textit{ and H.O.T.}), \qquad (3.39) \\ dx(t_2^+) &= \gamma_x^{2-}A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-}-f^{1+})dt_1 + (\gamma_x^{2-}f_x^{2-}+\xi^{2-})A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-}-f^{1+})dt_1dt_2 \\ &\quad +\gamma_x^{2-}f^{2-}dt_2 + (\textit{terms in } dt_1^2, dt_2^2 \textit{ and H.O.T.}) \qquad (3.40) \end{aligned}$$

where ξ^{2-} is defined similarly to ξ^{1-} in (3.14) as

$$\xi^{2-} \triangleq \begin{bmatrix} (f^{2-})^T \frac{\partial^2 \gamma^2_{(1)}\left(x(t_2^-)\right)}{\partial x^2} \\ \vdots \\ (f^{2-})^T \frac{\partial^2 \gamma^2_{(n)}\left(x(t_2^-)\right)}{\partial x^2} \end{bmatrix} \qquad (3.41)$$

with $\gamma^2_{(j)}$ referring to the j-th element of the vector-valued function γ^2.

Proof: See Appendix A. □

Remark 3.2 It is very important to point out that in the expressions of $dx(t_2^-)$ and $dx(t_2^+)$, we deliberately express the terms $f_x^{2-}A(t_2^-,t_1^-)(\gamma_x^{1-}f^{1-}-f^{1+})dt_1dt_2$ and $(\gamma_x^{2-}f_x^{2-}+\xi^{2-})A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-}-f^{1+})dt_1dt_2$ explicitly because they will contribute to the coefficient of dt_1dt_2. □

Now that we have the expressions for $\delta x(t_2^-)$, $\delta x(t_2+dt_2^-)$, $dx(t_2^-)$, and $dx(t_2^+)$, we are ready to derive the coefficient for dt_1dt_2 in the expansion of

$$\begin{aligned} J(t_1+dt_1,t_2+dt_2) &= \int_{t_0}^{t_1+dt_1^-} L\big(\hat{x}(t)\big)\ dt + \psi^1\big(\hat{x}(t_1+dt_1^-)\big) \\ &\quad + \int_{t_1+dt_1^+}^{t_2+dt_2^-} L\big(\hat{x}(t)\big)\ dt + \psi^2\big(\hat{x}(t_2+dt_2^-)\big) + J^2\big(\hat{x}(t_2+dt_2^+), t_2+dt_2\big). \qquad (3.42) \end{aligned}$$

There are five terms in (3.42). Let us look at each term's Taylor expansion in order to find its contribution to the coefficient of dt_1dt_2.

By using the forward decoupling principle, we can conclude that none of $\delta x(t_1^-)$, $\delta x(t_1^+)$, $dx(t_1^-)$, and $dx(t_1^+)$ will depend on dt_2. Consequently the Taylor expansion of the first two terms will not have terms in dt_2, dt_2^2 and dt_1dt_2. Therefore the first two terms will not contribute to the coefficient of dt_1dt_2.

For the third term in (3.42), we have the following Lemma.

Lemma 3.3 *The contribution of $\int_{t_0}^{t_2+dt_2^-} L(\hat{x})\ dt$ to the coefficient of dt_1dt_2 is*

$$L_x^{2-}A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-}-f^{1+}). \qquad (3.43)$$

Proof: See Appendix A. □

The fourth term in (3.42) can be expanded as

$$\begin{aligned}\psi^2\big(\hat{x}(t_2+dt_2^-)\big) &= \psi^2\big(x(t_2^-)+dx(t_2^-)\big)\\ &= \psi^{2-}+\psi_x^{2-}dx(t_2^-)+\frac{1}{2}\big(dx(t_2^-)\big)^T\psi_{xx}^{2-}dx(t_2^-)+\text{H.O.T.}\end{aligned}\tag{3.44}$$

Therefore the contribution to the coefficient of dt_1dt_2 by the fourth term is

$$\big(\psi_x^{2-}f_x^{2-}+(f^{2-})^T\psi_{xx}^{2-}\big)A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-}-f^{1+}).\tag{3.45}$$

For the fifth term in (3.42), similar to the single switching case, we can obtain its Taylor expansion as

$$\begin{aligned}J^2(\hat{x}(t_2+dt_2^+),t_2+dt_2) &= J^{2+}+J_x^{2+}dx(t_2^+)+J_{t_2}^{2+}dt_2+\frac{1}{2}\big(dx(t_2^+)\big)^TJ_{xx}^{2+}dx(t_2^+)+\frac{1}{2}J_{t_2t_2}^{2+}dt_2^2\\ &\quad +dt_2J_{t_2x}^{2+}dx(t_2^+)+\text{H.O.T.}\end{aligned}\tag{3.46}$$

In (3.46), the terms that will possibly contribute to the coefficient of dt_1dt_2 are those containing $dx(t_2^+)$. They are

$$J_x^{2+}dx(t_2^+),\ \frac{1}{2}\big(dx(t_2^+)\big)^TJ_{xx}^{2+}dx(t_2^+),\ dt_2J_{t_2x}^{2+}dx(t_2^+).\tag{3.47}$$

Substituting the expression of $dx(t_2^+)$ into (3.47) and summing them, we obtain the contribution of the fifth term to the coefficient of dt_1dt_2 as

$$\big(J_x^{2+}(\gamma_x^{2-}f_x^{2-}+\xi^{2-})+(f^{2-})^T(\gamma_x^{2-})^TJ_{xx}^{2+}\gamma_x^{2-}+J_{t_2x}^{2+}\gamma_x^{2-}\big)A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-}-f^{1+}).\tag{3.48}$$

Summing (3.43), (3.45), and (3.48) and also substituting into the sum the expression of $J_{t_2x}^{2+}$ which can be obtained similarly to the expression of $J_{t_1x}^{1+}$ in (3.18), we conclude that the coefficient of dt_1dt_2 (i.e., J_{t1t2} in the expansion of $J(t_1+dt_1,t_2+dt_2)$ is

$$\begin{aligned}J_{t_1t_2} &= \big(L_x^{2-}+\psi_x^{2-}f_x^{2-}+(f^{2-})^T\psi_{xx}^{2-}+J_x^{2+}(\gamma_x^{2-}f_x^{2-}+\xi^{2-})+(f^{2-})^T(\gamma_x^{2-})^TJ_{xx}^{2+}\gamma_x^{2-}\\ &\quad +J_{t_2x}^{2+}\gamma_x^{2-}\big)A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-}-f^{1+})\\ &= \big(L_x^{2-}-L_x^{2+}\gamma_x^{2-}+\psi_x^{2-}f_x^{2-}+(f^{2-})^T\psi_{xx}^{2-}+J_x^{2+}(\gamma_x^{2-}f_x^{2-}+\xi^{2-}-f_x^{2+}\gamma_x^{2-})\\ &\quad +(\gamma_x^{2-}f^{2-}-f^{2+})^TJ_{xx}^{2+}\gamma_x^{2-}\big)A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-}-f^{1+}).\end{aligned}\tag{3.49}$$

Remark 3.3 The above results still holds even when $t_1=t_2$ (we can consider $t_2>t_1$ first and then let $t_2\to t_1$ to prove this). □

The above result can also be similarly extended to the case of K switchings to relate $dx(t_l^-)$, $dx(t_l^+)$ to dt_l and dt_k $(k<l)$. The expression for $J_{t_kt_l}$ can similarly be obtained. We summarize and generalize the results obtained in this section into the following theorem.

Theorem 3.1 *The cost J in Problem 2.1 satisfies*

$$\begin{aligned}&J(t_1+dt_1,t_2+dt_2,\cdots,t_K+dt_K)\\ =\ &J(t_1,t_2,\cdots,t_K)+\sum_{k=1}^K J_{t_k}dt_k+\frac{1}{2}\sum_{k=1}^K J_{t_kt_k}dt_k^2+\sum_{1\le k<l\le K}J_{t_kt_l}dt_kdt_l\\ &+(\textit{higher order terms})\end{aligned}\tag{3.50}$$

where

$$\begin{aligned} J_{t_k} &= L^{k-} - L^{k+} + \psi_x^{k-} f^{k-} + J_x^{k+}(\gamma_x^{k-} f^{k-} - f^{k+}), & (3.51)\\ J_{t_k t_k} &= (L_x^{k-} - L_x^{k+}\gamma_x^{k-}) f^{k-} + \psi_x^{k-} f_x^{k-} f^{k-} + (f^{k-})^T \psi_{xx}^{k-} f^{k-} \\ &\quad + J_x^{k+}(\gamma_x^{k-} f_x^{k-} + \xi^{k-} - f_x^{k+}\gamma_x^{k-}) f^{k-} - (J_x^{k+} f_x^{k+} + L_x^{k+})(\gamma_x^{k-} f^{k-} - f^{k+}) \\ &\quad + (\gamma_x^{k-} f^{k-} - f^{k+})^T J_{xx}^{k+} (\gamma_x^{k-} f^{k-} - f^{k+}), & (3.52) \end{aligned}$$

for any $k = 1, \cdots, K$, *and*

$$\begin{aligned} J_{t_k t_l} &= \big(L_x^{l-} - L_x^{l+}\gamma_x^{l-} + \psi_x^{l-} f_x^{l-} + (f^{l-})^T \psi_{xx}^{l-} + J_x^{l+}(\gamma_x^{l-} f_x^{l-} + \xi^{l-} - f_x^{l+}\gamma_x^{l-}) \\ &\quad + (\gamma_x^{l-} f^{l-} - f^{l+})^T J_{xx}^{l+} \gamma_x^{l-}\big) H(t_l^-, t_k^+)(\gamma_x^{k-} f^{k-} - f^{k+}), & (3.53) \end{aligned}$$

for any $1 \le k < l \le K$. *Here* $H(t_l^-, t_k^+)$ *is the state transition matrix under state jumps*

$$H(t_l^-, t_k^+) = A(t_l^-, t_{l-1}^+)\gamma_x^{(l-1)-} A(t_{l-1}^-, t_{l-2}^+) \bullet \cdots \bullet \gamma_x^{(k+1)-} A(t_{k+1}^-, t_k^+) \quad (3.54)$$

where $A(t_{j+1}^-, t_j^+)$, $k \le j \le l-1$ *is the state transition matrix for the time interval* $[t_j^+, t_{j+1}^-]$ *for the variational equation*

$$\dot{y}(t) = \frac{\partial f_{j+1}\big(x(t)\big)}{\partial x} y(t). \quad (3.55)$$

Also here

$$\xi^{k-} \triangleq \begin{bmatrix} (f^{k-})^T \frac{\partial \gamma_{(1)}^k\big(x(t_k^-)\big)}{\partial x} \\ \vdots \\ (f^{k-})^T \frac{\partial \gamma_{(n)}^k\big(x(t_k^-)\big)}{\partial x} \end{bmatrix}, \ k = 1, \cdots, K, \quad (3.56)$$

with $\gamma_{(j)}^k$ *referring to the* j*-th element of the vector-valued function* γ^k. □

Remark 3.4 In general in the interval $[t_k^+, t_l^-]$, there will be discontinuous jumps and they must be taken into consideration when we consider the incremental change $\delta(x)$ in this interval, hence $H(t_l^-, t_k^+)$ appears in (3.53) (instead of $A(t_l^-, t_k^+)$) if we follow the similar derivations as in the two switchings case. In the special case when $l = k+1$, $H(t_l^-, t_k^+)$ is reduced to be $A(t_{k+1}^-, t_k^+)$. □

3.3 Computation of $H(t_l^-, t_k^+)$, J_x^{k+}, and J_{xx}^{k+}

In order to compute J_{t_k}, $J_{t_k t_k}$ and $J_{t_k t_l}$ using Theorem 3.1, we need to know the values of $H(t_l^-, t_k^+)$, J_x^{k+} and J_{xx}^{k+}. However, given nominal $\hat{t}$ and x, these values are not readily available. In general, numerical methods need to be used to compute their values. An efficient numerical method based on solving additional initial value ordinary differential equations (ODEs) with jumps is developed in this subsection.

First note that if $l = k+1$ then $H(t_l^-, t_k^+)$ is equal to $A(t_{k+1}^-, t_k^+)$, which is the state transition matrix for

$$\dot{y}(t) = \frac{\partial f_{k+1}\big(x(t)\big)}{\partial x} y(t). \quad (3.57)$$

To find its value, we can first find the solution $y^{(1)}(t), \cdots, y^{(n)}(t)$ corresponding to initial conditions

$$y^{(1)}(t_k^+) = e_1, \ \cdots, \ y^{(n)}(t_k^+) = e_n \quad (3.58)$$

respectively, where e_j is the unit column vector with all 0's except that the j-th element being 1, $j = 1, 2, \cdots, n$. From linear systems theory, $A(t_{k+1}^-, t_k^+)$, is equal to the square matrix whose j-th column is $y^{(j)}(t_{k+1}^-)$, i.e., in this case

$$H(t_l^-, t_k^+) = A(t_{k+1}^-, t_k^-) = [y^{(1)}(t_{k+1}^-), \cdots, y^{(n)}(t_{k+1}^-)]. \quad (3.59)$$

If $l > k$, the similar method can be adopted to compute $H(t_l^-, t_k^+)$. Instead of solving initial value ODEs for $y^{(j)}$'s, $y^{(j)}(t)$'s are now obtained by solve the following ODEs with jumps with initial conditions (3.58).

$$\begin{cases} \dot{y}(t) = \frac{\partial f_{j+1}\big(x(t)\big)}{\partial x} y(t), \text{ for } t_j^+ \le t \le t_{j+1}^-, \\ y(t_j^+) = \gamma_x^{j-} y(t_j^-), k < j < l. \end{cases} \tag{3.60}$$

We then have

$$H(t_l^-, t_k^+) = [y^{(1)}(t_l^-), \cdots, y^{(n)}(t_l^-)]. \tag{3.61}$$

To obtain the value of J_x^k, note that

$$J^k\big(x(t_k^+), t_k, \cdots, t_K\big) = \psi\big(x(t_f)\big) + \sum_{j=k}^{K} \int_{t_j^+}^{t_{j+1}^-} L\big(x(t)\big)\, dt + \sum_{j=k+1}^{K} \psi^j\big(x(t_j^-)\big). \tag{3.62}$$

Note that, for simplicity of notation, we regard t_f as t_{K+1}^- in (3.62).

If $x(t_k^+)$ has a variation $\delta x(t_k^+)$, then

$$\begin{aligned} & J^k\big(x(t_k^+) + \delta x(t_k^+), t_k, \cdots, t_K\big) \\ = \;& \psi\big(x(t_f) + H(t_f, t_k^+)\delta x(t_k^+) + \text{H.O.T.}\big) + \sum_{j=k}^{K} \int_{t_j^+}^{t_{j+1}^-} L\big(x(t) + H(t, t_k^+)\delta x(t_k^+) + \text{H.O.T}\big)\, dt \\ & + \sum_{j=k+1}^{K} \psi^j\big(x(t_j^-) + H(t_j^-, t_k^+)\delta x(t_k^+) + \text{H.O.T.}\big) \\ = \;& J^k\big(x(t_k^+), t_k, \cdots, t_K\big) + \Big(\psi_x\big(x(t_f)\big)H(t_f, t_k^+) + \sum_{j=k}^{K} \int_{t_j^+}^{t_{j+1}^-} L_x\big(x(t)\big)H(t, t_k^+)\, dt \\ & + \sum_{j=k+1}^{K} \psi_x^j\big(x(t_j^-)\big)H(t_j^-, t_k^+)\Big)\delta x(t_k^+) + \text{H.O.T.} \end{aligned} \tag{3.63}$$

Hence

$$J_x^{k+} = \psi_x\big(x(t_f)\big)H(t_f, t_k^+) + \sum_{j=k}^{K} \int_{t_j^+}^{t_{j+1}^-} L_x\big(x(t)\big)H(t, t_k^+)\, dt + \sum_{j=k+1}^{K} \psi_x^j\big(x(t_j^-)\big)H(t_j^-, t_k^+). \tag{3.64}$$

Now if we apply the similar procedure by varying $x(t_k^+)$ as in (3.63) to $J_x^k\big(x(t_k^+), t_k, \cdots, t_K\big)$, we can obtain

$$\begin{aligned} J_{xx}^{k+} = \;& H^T(t_f, t_k^+)\psi_{xx}\big(x(t_f)\big)H(t_f, t_k^+) + \sum_{j=k}^{K} \int_{t_j^+}^{t_{j+1}^-} H^T(t, t_k^+)L_{xx}\big(x(t)\big)H(t, t_k^+)\, dt \\ & + \sum_{j=k+1}^{K} H^T(t_j^-, t_k^+)\psi_{xx}^j\big(x(t_j^-)\big)H(t_j^-, t_k^+). \end{aligned} \tag{3.65}$$

From the above discussions, we find that $H(t_l^-, t_k^+)$ can be obtained by solving ODEs with jumps (3.60) along with initial conditions (3.58). $H(t_f, t_k^+)$ can be obtained in the same fashion. J_x^{k+} and J_{xx}^{k+} are in the forms (3.64) and (3.65) which can be easily rewritten as ODEs with jumps. By solving the following initial value ODEs with jumps from t_k^+ to t_f (along with the hybrid system ODEs with jumps which provides us with the state trajectory)

$$\begin{cases} \dot{H}(t,t_k^+) = \frac{\partial f_{j+1}\big(x(t)\big)}{\partial x} H(t,t_k^+), \ t_j^+ \le t \le t_{j+1}^-, \\ H(t_j^+,t_k^+) = \gamma_x^{j-} H(t_j^-,t_k^+), \end{cases} \tag{3.66}$$

$$\begin{cases} \dot{\eta}_1 = L_x\big(x(t)\big)H(t,t_k^+), \ t_j^+ \le t \le t_{j+1}^-, \\ \eta_1(t_j^+) = \eta_1(t_j^-) + \psi_x^j\big(x(t_j^-)\big)H(t_j^-,t_k^+), \end{cases} \tag{3.67}$$

$$\begin{cases} \dot{\eta}_2 = H^T(t,t_k^+)L_{xx}\big(x(t)\big)H(t,t_k^+), \ t_j^+ \le t \le t_{j+1}^-, \\ \eta_2(t_j^+) = \eta_2(t_j^-) + H^T(t_j^-,t_k^+)\psi_{xx}^j\big(x(t_j^-)\big)H(t_j^-,t_k^+), \end{cases} \tag{3.68}$$

along with initial conditions (3.58) and

$$\eta_1(t_k) = 0_{1\times n}, \tag{3.69}$$

$$\eta_2(t_k) = 0_{n\times n}, \tag{3.70}$$

we can find the values of $H(t_f,t_k^+)$, J_x^{k+} and J_{xx}^{k+} from

$$J_x^{k+} = \psi_x\big(x(t_f)\big)H(t_f,t_k^+) + \eta_1(t_f), \tag{3.71}$$

$$J_{xx}^{k+} = H^T(t_f,t_k^+)\psi_{xx}\big(x(t_f)\big)H(t_f,t_k^+) + \eta_2(t_f). \tag{3.72}$$

Remark 3.5 (Computational Cost) All other terms in (3.51)-(3.53) except for $H(t_l^-,t_k^+)$, J_x^{k+} and J_{xx}^{k+} are readily available once the nominal trajectory $x(t)$ is known. Therefore the main computational cost for J_{t_k}, $J_{t_k t_k}$, J_{t_k,t_l} occurs in the computation of $H(t_l^-,t_k^+)$, J_x^{k+} and J_{xx}^{k+}. The above method we propose reduces the computation of $H(t_l^-,t_k^+)$ to solving initial value ODEs with jumps (3.60) for any $k < l$ and the computation of J_x^{k+} and J_{xx}^{k+} to solving initial value ODEs with jumps (3.66)-(3.68) for $k = 1, 2, \cdots, K$. Hence we altogether need to solve $\frac{(K-1)K}{2} + K = \frac{K(K+1)}{2}$ sets of initial value ODEs with jumps. With today's powerful ODE solvers (e.g., `ode45` function in MATLAB), these equations can be solved efficiently and accurately. For our purpose of efficient optimization of open-loop solutions of optimal switching instants, such computation suffices. Moreover, for general quadratic problems for switched autonomous linear systems which we will elaborate on in the next section, the computational costs of these values can be reduced greatly. □

4 General Quadratic Problems for Hybrid Autonomous Systems with Linear Subsystems and Linear State Jumps

In this section, we apply the approach developed in Section 3 to a special class of problems, namely, general quadratic problems for hybrid autonomous systems with linear subsystems and linear state jumps. In particular, we show that due to the special structure of the problem, the computation of $H(t_l^-,t_k^+)$, J_x^{k+} and J_{xx}^{k+} can further be simplified.

Problem 4.1 *Consider a hybrid autonomous system with linear subsystems $\dot{x} = A_i x, i \in I$. Assume a prespecified sequence of active subsystems $(1, 2, \cdots, K, K+1)$ is given. Also assume that when the system switches from subsystem k to $k+1$ ($k = 1, \cdots, K$), there is a discontinuous jump of the continuous state which has the linear relationship*

$$x(t_k^+) = \gamma^k\big(x(t_k^-)\big) = \Theta_k x(t_k^-) + \Gamma_k \tag{4.1}$$

where Θ_k, Γ_k are matrices of appropriate dimensions. Find optimal switching instants $t_1, \cdots, t_K$ ($t_0 \leq t_1 \leq \cdots \leq t_K \leq t_f$) such that the cost in general quadratic form

$$J(t_1, \cdots, t_K) = \psi\big(x(t_f)\big) + \int_{t_0}^{t_f} L(x)\ dt + \sum_{k=1}^{K} \psi^k\big(x(t_k^-)\big) \tag{4.2}$$

where

$$\begin{aligned} \psi\big(x(t_f)\big) &= \frac{1}{2}\big(x(t_f)\big)^T Q_f x(t_f) + M_f x(t_f) + W_f, & (4.3)\\ L(x) &= \frac{1}{2}\big(x(t)\big)^T Q x(t) + M x(t) + W, & (4.4)\\ \psi^k\big(x(t_k^-)\big) &= \frac{1}{2}\big(x(t_k^-)\big)^T Q_k x(t_k^-) + M_k x(t_k^-) + W_k, & (4.5) \end{aligned}$$

is minimized. Here t_0, t_f and $x(t_0) = x_0$ are given; Q_f, M_f, W_f, Q, M, W are matrices of appropriate dimensions with $Q_f \geq 0$, $Q \geq 0$. Q_k, M_k, W_k, ($k = 1, \cdots, K$), are matrices of appropriate dimensions which form the quadratic terms for the cost of discontinuous jumps from subsystem k to $k+1$ and $Q_k \geq 0$. $\square$

In view of the special structure of Problem 4.1, we can readily observe that

$$A(t_{k+1}^-, t_k^+) = e^{A_{k+1}(t_{k+1}-t_k)} \tag{4.6}$$

for any $k = 1, \cdots, K$. Moreover,

$$\begin{aligned} H(t_l^-, t_k^+) &= e^{A_l(t_l-t_{l-1})}\gamma_x^{(l-1)-} e^{A_{l-1}(t_{l-1}-t_{l-2})} \bullet \cdots \bullet \gamma_x^{(k+1)-} e^{A_{k+1}(t_{k+1}-t_k)} \\ &= e^{A_l(t_l-t_{l-1})}\Theta_{l-1} e^{A_{l-1}(t_{l-1}-t_{l-2})} \bullet \cdots \bullet \Theta_{k+1} e^{A_{k+1}(t_{k+1}-t_k)}. \end{aligned} \tag{4.7}$$

The computation of J_x^{k+} and J_{xx}^{k+} is discussed next. Assume a nominal $\hat{t}$ is given. If for any $x \in \mathbb{R}^n$ and any $t \in [t_0, t_f]$ we denote by $\tilde{J}(x,t)$ the cost incurred if the system starts from the state x at time instant t and evolves according to the portion of the switching sequence generated by $\hat{t}$ in $[t, t_f]$. In other words,

$$\tilde{J}(x,t) = \psi\big(x(t_f)\big) + \int_t^{t_f} L\big(x(\tau)\big)\ d\tau + \sum_{k \text{ with } t_k \in [t, t_f]} \psi^k\big(x(t_k^-)\big) \tag{4.8}$$

where $x(t) = x$. Dynamic programming approach similar to (3.10) can be applied to $\tilde{J}(x,t)$ to obtain

$$\tilde{J}(x,t) = \frac{1}{2}x^T P(t) x + S(t) x + T(t) \tag{4.9}$$

where $P(t) = P^T(t)$ and $P(t)$, $S(t)$, $T(t)$ obey the following differential equations with jumps

$$\begin{cases} -\dot{P} = P A_{j+1} + A_{j+1}^T P + Q, \ t_j^+ \leq t \leq t_{j+1}^-, \\ P(t_j^-) = \Theta_j^T P(t_j^+)\Theta_j + Q_j, \end{cases} \tag{4.10}$$

$$\begin{cases} -\dot{S} = S A_{j+1} + M, \ t_j^+ \leq t \leq t_{j+1}^-, \\ S(t_j^-) = \Gamma_j^T P(t_j^+)\Theta_j + S(t_j^+)\Theta_j + M_j, \end{cases} \tag{4.11}$$

$$\begin{cases} -\dot{T} = W, \ t_j^+ \leq t \leq t_{j+1}^-, \\ T(t_j^-) = \frac{1}{2}\Gamma_j^T P(t_j^+)\Gamma_j + S(t_j^+)\Gamma_j + T(t_j^+) + W_j, \end{cases} \tag{4.12}$$

along with initial conditions

$$
\begin{aligned}
P(t_f) &= Q_f, &(4.13)\\
S(t_f) &= M_f, &(4.14)\\
T(t_f) &= W_f. &(4.15)
\end{aligned}
$$

From the definitions of the functions $\tilde{J}$ and J^k, if $\hat{t}$ is fixed, we have

$$
\begin{aligned}
J^k\big(x(t_k^+), t_k, \cdots, t_K\big) &= \tilde{J}\big(x(t_k^+), t_k^+\big), &(4.16)\\
J_x^k\big(x(t_k^+), t_k, \cdots, t_K\big) &= \tilde{J}_x\big(x(t_k^+), t_k^+\big), &(4.17)\\
J_{xx}^k\big(x(t_k^+), t_k, \cdots, t_K\big) &= \tilde{J}_{xx}\big(x(t_k^+), t_k^+\big). &(4.18)
\end{aligned}
$$

Therefore the values of J_x^{k+} and J_{xx}^{k+} can be obtained as

$$
\begin{aligned}
J_x^{k+} &= \tilde{J}_x\big(x(t_k^+), t_k^+\big) = \big(x(t_k^+)\big)^T P(t_k^+) + S(t_k^+), &(4.19)\\
J_{xx}^{k+} &= \tilde{J}_{xx}\big(x(t_k^+), t_k^+\big) = P(t_k^+). &(4.20)
\end{aligned}
$$

Remark 4.1 (Computational Cost) The computation of $H(t_l^-, t_k^+)$'s using (4.7) is straightforward and do not resort to an ODE solver. The computation of J_x^{k+} and J_{xx}^{k+} using (4.19) and (4.20) relies on the values of $P(t_k^+)$'s and $S(t_k^+)$'s which are easy to obtain by solving the initial value ODEs with jumps (4.10)-(4.15) backward in time only once. Therefore, the computational cost for Problem 4.1 is greatly reduced as opposed to the general case in subsection 3.3. □

5 Reachability Problems

The optimal control approach discussed above can also be applied to the following important class of reachability problems.

Problem 5.1 (Reachability Problem) *Given a hybrid autonomous system with state jumps, does there exist a switching sequence such that the state trajectory x departs from $x(t_0) = x_0$ and meets x_f at some t_f? Here t_0, x_0, x_f are given; t_f is not given.* □

Note that x_f is reachable from x_0, if and only if the following optimal control problem achieves its minimum at $J = 0$. The problem is having a free final time t_f and seeks to minimize the cost

$$J = \frac{1}{2}\|x(t_f) - x_f\|_2^2, \tag{5.1}$$

here t_0, x_0, x_f are given. In general, the optimal control problem is difficult to solve due to the large number of possible patterns of switching sequences. But if we assume that a prespecified sequence of active subsystems is given, the problem can be handled by using optimal control methodologies. For example, we can assume subsystem k being active in $[t_{k-1}, t_k)$ (subsystem $K+1$ in $[t_k, t_f]$. In this case, we can minimize J with respect to the switching instants and the final time t_f. In other words, the reachability problem can be formulated as an optimal control problem which seeks for optimal values of $t_1, \cdots, t_K, t_f$ such that

$$J(t_1, \cdots, t_K, t_f) = \frac{1}{2}\|x(t_f) - x_f\|_2^2 \tag{5.2}$$

is minimized. In this case, ideally the minimum cost should be 0 if x_f is reachable from x_0 by the given order of active subsystems. In practice, if the optimal value of J is found to be smaller than a predefined small tolerance $\epsilon > 0$, then we regard x_f as reachable from x_0 and regard the corresponding optimal $t_1, \cdots, t_K, t_f$ as the reachability switching instants.

Remark 5.1 Note that since our approach for optimal control finds local optimal solutions, an optimal value of J greater than ϵ does not necessarily imply that x_f is not reachable from x_0. In the case that x_f is reachable from x_0, another trial of initial guess of switching instants may lead to the global optimal solution with $J < \epsilon$. Therefore the optimal control approach can only be used as a sufficient condition for determining reachability. However, whenever x_f is determined to be reachable from x_0, our approach also provides the explicit sequence $(t_1, \cdots, t_K, t_f)$ that achieves it. This is the strength of the approach. □

To minimize $J(t_1, \cdots, t_K, t_f)$ with respect to $(t_1, \cdots, t_K, t_f)$, we can use Algorithm 2.1. To apply the algorithm, the derivatives of J first need to be computed. The derivative values J_{t_k}, $J_{t_k t_k}$ and $J_{t_k t_l}$ can be obtained using the expressions stated in Theorem 3.1. However, we note here since t_f is free, we also need to derive J_{t_f}, $J_{t_f t_f}$ and $J_{t_k t_f}$. These values can be obtained following the idea of the derivation in Section 3. We define

$$J^f\big(x(t_f)\big) \triangleq \frac{1}{2}\|x(t_f) - x_f\|_2^2, \tag{5.3}$$

and consider the Taylor expansion of

$$J(t_1, \cdots, t_K, t_f) = J^f\big(x(t_f)\big). \tag{5.4}$$

By fixing $t_1, \cdots, t_K$, we can expand J into second order expansions with respect to t_f and obtain

$$J_{t_f} = J_x^f f^f, \tag{5.5}$$

$$J_{t_f t_f} = J_x^f f_x^f f^f + (f^f)^T J_{xx}^f f^f, \tag{5.6}$$

Similarly to the derivations in Section 3.2, we can derive

$$J_{t_k t_f} = \big(J_x^f f_x^f + (f^f)^T J_{xx}^f\big) H(t_f, t_k^+)(\gamma_x^{k-} f^{k-} - f^{k+}). \tag{5.7}$$

In the above expression, we have

$$J^f = \frac{1}{2}\|x(t_f) - x_f\|_2^2, \tag{5.8}$$

$$J_x^f = \big(x(t_f) - x_f\big)^T, \tag{5.9}$$

$$J_{xx}^f = I_{n\times n}, \tag{5.10}$$

$$f^f = f_{K+1}\big(x(t_f)\big), \tag{5.11}$$

$$f_x^f = \frac{\partial f_{K+1}\big(x(t_f)\big)}{\partial x}, \tag{5.12}$$

Using (5.8)-(5.12), we can simplify (5.5)-(5.7) as

$$J_{t_f} = \big(x(t_f) - x_f\big)^T f_{K+1}\big(x(t_f)\big), \tag{5.13}$$

$$J_{t_f t_f} = \big(x(t_f) - x_f\big)^T \frac{\partial f_{K+1}\big(x(t_f)\big)}{\partial x} f_{K+1}\big(x(t_f)\big) + \Big(f_{K+1}\big(x(t_f)\big)\Big)^T f_{K+1}\big(x(t_f)\big), \tag{5.14}$$

$$J_{t_k t_f} = \left(\big(x(t_f) - x_f\big)^T \frac{\partial f_{K+1}\big(x(t_f)\big)}{\partial x} + \Big(f_{K+1}\big(x(t_f)\big)\Big)^T\right) H(t_f, t_k^+)(\gamma_x^{k-} f^{k-} - f^{k+}). \tag{5.15}$$

6 Examples

In this section, we present two examples to illustrate the effectiveness of the approach developed in this paper. The examples are computed using MATLAB implementation of our approach. Our approach and implementation can solve these examples very efficiently.

Example 6.1 Consider a hybrid autonomous system consisting of

$$\text{subsystem 1:} \begin{cases} \dot{x}_1 = x_1 + 0.5 \sin x_2 \\ \dot{x}_2 = -0.5 \cos x_1 - x_2 \end{cases} \tag{6.1}$$

$$\text{subsystem 2:} \begin{cases} \dot{x}_1 = 0.3 \sin x_1 + 0.5 x_2 \\ \dot{x}_2 = -0.5 x_1 + 0.3 \cos x_2 \end{cases} \tag{6.2}$$

$$\text{subsystem 3:} \begin{cases} \dot{x}_1 = -x_1 - 0.5 \cos x_2 \\ \dot{x}_2 = 0.5 \sin x_1 + x_2 \end{cases} \tag{6.3}$$

Assume that $t_0 = 0$, $t_f = 3$ and the system switches at $t = t_1$ from subsystem 1 to 2 and at $t = t_2$ from subsystem 2 to 3 ($0 \le t_1 \le t_2 \le 3$). Also assume that the system has the state jump

$$\begin{cases} x_1(t_1^+) = x_1(t_1^-) + 0.2 \\ x_2(t_1^+) = x_2(t_1^-) + 0.2 \end{cases} \tag{6.4}$$

when switching from subsystem 1 to 2 and

$$\begin{cases} x_1(t_2^+) = x_1(t_2^-) + 0.2 \\ x_2(t_2^+) = x_2(t_2^-) - 0.2 \end{cases} \tag{6.5}$$

when switching from subsystem 2 to 3. We want to find optimal switching instants t_1, t_2 such that the cost

$$J = \frac{1}{2} x_1^2(3) + \frac{1}{2} x_2^2(3) + \frac{1}{2} \int_0^3 \left(x_1^2(t) + x_2^2(t) \right) dt + \sum_{k=1}^{2} \left(\frac{1}{2} x_1^2(t_k^-) + \frac{1}{2} x_2^2(t_k^-) \right) \tag{6.6}$$

is minimized. Here $x_1(0) = 1$ and $x_2(0) = 3$.

For this problem, we choose initial nominal $t_1 = 1$, $t_2 = 1.5$. We derive the derivatives of J using the result in Theorem 3.1. The computation of $H(t_2^-, t_1^+)$, J_x^{1+}, J_{xx}^{1+}, J_x^{2+}, and J_{xx}^{2+} is based on results in Section 3.3. By using the Algorithm 2.1 with the constrained Newton's method, after 8 iterations we find that the optimal switching instants are $t_1 = 0.4847$, $t_2 = 1.9273$ and the corresponding optimal cost is 18.8310. It takes about 4 seconds for MATLAB to compute the solution. The corresponding state trajectory is shown in figure 2. Figure 3 shows the plot of the cost function for different $0 \le t_1 \le t_2 \le 3$. By comparing the J value for different t_1 and t_2, we verify that the solution we obtain is the global optimal (although it is difficult to tell from the cost surface, our computation shows us so). □

Example 6.2 (A Reachability Problem) Consider a hybrid autonomous system consisting of

$$\text{subsystem 1: } \dot{x} = A_1 x = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x, \tag{6.7}$$

$$\text{subsystem 2: } \dot{x} = A_2 x = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} x. \tag{6.8}$$

Assume that at $t_0 = 0$, the system state departs from the initial condition $x_1(0) = 1$ and $x_2(0) = 1$ and evolves following the dynamics of subsystem 1. Assume that the system switches once at t_1 from subsystem 1 to 2. Also assume that the system has the state jump

$$x(t_1^+) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} x(t_1^-) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{6.9}$$

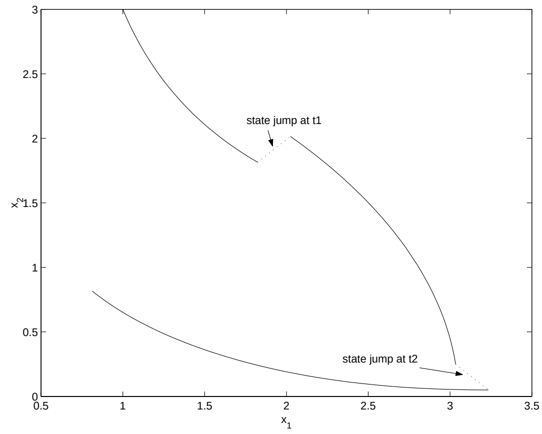

Figure 2: The state trajectory for Example 6.1.

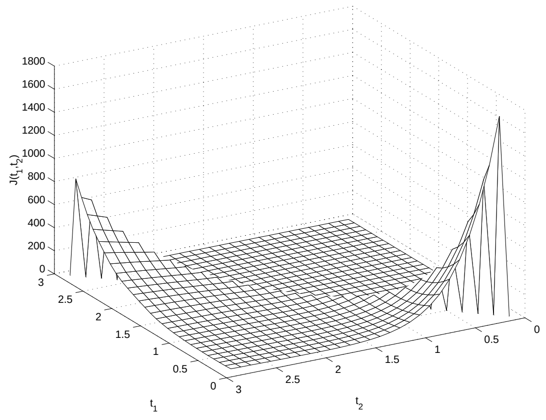

Figure 3: The cost for Example 6.1 for different (t_1, t_2)'s $(0 \leq t_1 \leq t_2 \leq 3)$.

when switching from subsystem 1 to 2. We want to find a t_1 and a t_f $(0 \leq t_1 \leq t_f)$ such that the system state arrives at $[2e^3 + e^2,\ e^3 + e]^T$ at t_f.

This reachability problem can be posed as an optimal control problem with unknown t_f and cost $J = \frac{1}{2}\big((x_1(t_f) - (2e^3 + e^2))^2 + (x_2(t_f) - (e^3 + e))^2\big)$. We choose initial nominal $t_1 = 0.8$, $t_f = 1.8$. The values of J_{t_1}, J_{t_f}, $J_{t_1 t_1}$, $J_{t_f t_f}$ and $J_{t_1 t_f}$ can be derived using the formulae (3.51)-(3.53) and (5.13)-(5.15). The computation of $H(t_f, t_1^+)$, J_x^{1+}, and J_{xx}^{1+} can be done efficiently based on results in Section 4 because this problem is a quadratic optimal control problem with linear subsystems and linear state jumps. We use Algorithm 2.1 with the constrained Newton's method to search for an optimal solution. After 8 iterations we find that the optimal switching instants are $t_1 = 1.0000$, $t_2 = 2.0000$ and the corresponding optimal cost is 2.7603×10^{-10}. It takes about 3.5 seconds for MATLAB to compute the solution. The corresponding state trajectory is shown in figure 4. Figure 5 shows the plot of the cost function for different $0 \leq t_1 \leq t_f$. By comparing the J value for different t_1 and t_f, we verify that the solution we obtain is the global optimal (although it is difficult to tell from the cost surface, our computation shows us so).

It is worth noting that for this example we can verify the correctness of (5.13)-(5.15). For example, the expression of $J_{t_1 t_f}$ can be derived from (5.15) as (here $K = 1$)

$$\begin{aligned} J_{t_1 t_f} &= \left(\big(x(t_f) - x_f\big)^T \frac{\partial f_{K+1}\big(x(t_f)\big)}{\partial x} + \Big(f_{K+1}\big(x(t_f)\big)\Big)^T \right) H(t_f, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+}) \\ &= \Big(\big(x(t_f) - x_f\big)^T A_2 + \big(A_2 x(t_f)\big)^T \Big) H(t_f, t_1^+)\big(\gamma_x^{1-} A_1 x(t_1^-) - A_2 x(t_1^+)\big). \end{aligned} \tag{6.10}$$

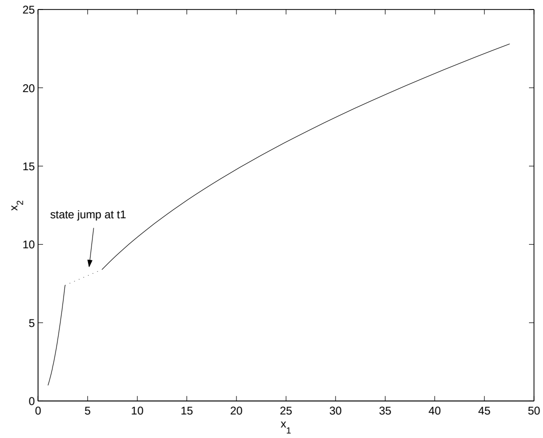

Figure 4: The state trajectory for Example 6.2.

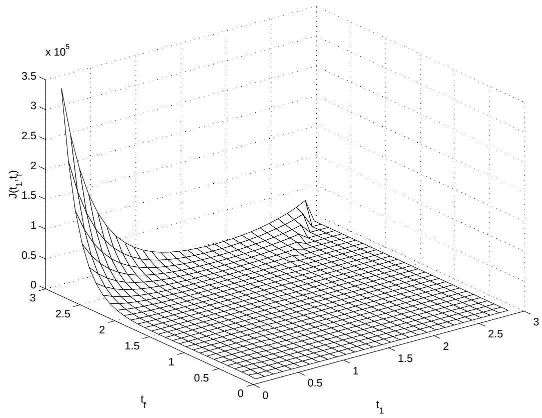

Figure 5: The cost for Example 6.2 for different (t_1, t_f)'s $(0 \leq t_1 \leq t_f)$.

We can substitute $x(t_1^-) = [e^{t_1},\ e^{2t_1}]^T$, $x(t_1^+) = [2e^{t_1}+1,\ e^{2t_1}+1]^T$, $x(t_f) = [2e^{2t_f-t_1} + e^{2t_f-2t_1},\ e^{t_f+t_1} + e^{t_f-t_1}]^T$, $x_f^T = [2e^3 + e^2,\ e^3 + e]$, $H(t_f, t_1^+) = e^{A_2(t_f-t_1)}$, $\gamma_x^{1-} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, A_1, and A_2 into (6.10) and obtain

$$\begin{aligned} J_{t_1 t_f} &= (8e^{2t_f-t_1} + 4e^{2t_f-2t_1} - 4e^3 - 2e^2)(-2e^{2t_f-t_1} - 2e^{2t_f-2t_1}) \\ &\quad + (2e^{t_f+t_1} + 2e^{t_f-t_1} - e^3 - e)(e^{t_f+t_1} - e^{t_f-t_1}). \end{aligned} \tag{6.11}$$

The correctness of (6.11) can be verified by directly differentiating the expression of J

$$J = \frac{1}{2}\big((2e^{2t_f-t_1} + e^{2t_f-2t_1} - 2e^3 - e^2)^2 + (e^{t_f+t_1} + e^{t_f-t_1} - e^3 - e)^2\big), \tag{6.12}$$

$$\begin{aligned} \frac{\partial J}{\partial t_1} &= (2e^{2t_f-t_1} + e^{2t_f-2t_1} - 2e^3 - e^2)(-2e^{2t_f-t_1} - 2e^{2t_f-2t_1}) \\ &\quad + (e^{t_f+t_1} + e^{t_f-t_1} - e^3 - e)(e^{t_f+t_1} - e^{t_f-t_1}), \end{aligned} \tag{6.13}$$

$$\begin{aligned} \frac{\partial^2 J}{\partial t_1 \partial t_f} &= (8e^{2t_f-t_1} + 4e^{2t_f-2t_1} - 4e^3 - 2e^2)(-2e^{2t_f-t_1} - 2e^{2t_f-2t_1}) \\ &\quad + (2e^{t_f+t_1} + 2e^{t_f-t_1} - e^3 - e)(e^{t_f+t_1} - e^{t_f-t_1}). \end{aligned} \tag{6.14}$$

Similarly, we can also verify the correctness of the expressions of J_{t_1}, J_{t_f}, $J_{t_1 t_1}$, $J_{t_f t_f}$ by direct differentiations of J. □

7 Conclusion

In this paper, we propose an approach for solving optimal control problems for hybrid autonomous systems with state jumps given prespecified sequences of active subsystems. In particular, we derive the derivatives of the cost with respect to the switching instants and use nonlinear optimization techniques to locate the optimal switching instants. It is also shown in the paper that the computational burden can be eased in the case of general quadratic problems for hybrid autonomous systems with linear subsystems and linear state jumps. Finally it is shown that reachability problems can also be studied using the optimal control techniques. If a system is reachable given a prespecified sequence of active subsystems, optimal control methods can be used to locate the corresponding switching instants. The approach developed in the paper has been implemented using MATLAB and can be requested from the authors. The software we developed can solve the optimal control problems studied in this paper very efficiently. A future research topic is the development of methods for searching for optimal switching sequences when the sequence of active subsystems are not prespecified.

Appendix A: Some Proofs for Section 3.2

PROOF OF LEMMA 3.1: Although the results in the Lemma hold for all cases in the definition of $\delta x(t)$, we need to discuss each case in order to show the validity of them.
Case 1: $dt_1 \geq 0$, $dt_2 \geq 0$

$$\begin{aligned}
\delta x(t_1 + dt_1^+) &= \hat{x}(t_1 + dt_1^+) - x(t_1 + dt_1^+) \\
&= \gamma^1\Big(x(t_1^-) + \int_{t_1^-}^{t_1+dt_1^-} f_1\big(\hat{x}(t)\big)\, dt\Big) - \Big(\gamma^1\big(x(t_1^-)\big) + \int_{t_1^+}^{t_1+dt_1^+} f_2\big(x(t)\big)\, dt\Big) \\
&= \gamma^1\Big(x(t_1^-) + f_1\big(x(t_1^-)\big)dt_1 + o(dt_1)\Big) - \Big(\gamma^1\big(x(t_1^-)\big) + f_2\big(x(t_1^+)\big)dt_1 + o(dt_1)\Big) \\
&= (\gamma_x^{1-} f^{1-} - f^{1+})dt_1 + o(dt_1).
\end{aligned} \tag{A.1}$$

We then conclude from the property of the variational equation that

$$\begin{aligned}
\delta x(t_2^-) &= A(t_2^-, t_1 + dt_1^+)\delta x(t_1 + dt_1^+) + \big(\text{H.O.T. in } \delta x(t_1 + dt_1^+)\big) \\
&= \big(A(t_2^-, t_1^+) + A_{t_1}(t_2^-, t_1^+)dt_1 + o(dt_1)\big)\big((\gamma_x^{1-} f^{1-} - f^{1+})dt_1 + o(dt_1)\big) + o(dt_1) \\
&= A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+})dt_1 + o(dt_1),
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\delta x(t_2 + dt_2^-) &= \hat{x}(t_2 + dt_2^-) - z_1(t_2 + dt_2^-) \\
&= \Big(\hat{x}(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} f_2\big(\hat{x}(t)\big)\, dt\Big) - \Big(z_1(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} f_2\big(z_1(t)\big)\, dt\Big) \\
&= \delta x(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} \Big(f_2\big(\hat{x}(t)\big) - f_2\big(z_1(t)\big)\Big)\, dt \\
&= \delta x(t_2^-) + \Big(f_2\big(\hat{x}(t_2^-)\big) - f_2\big(z_1(t_2^-)\big)\Big)dt_2 + o(dt_2) \\
&= \delta x(t_2^-) + f_x^{2-}\delta x(t_2^-)dt_2 + o(dt_2) \\
&= A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+})dt_1 + f_x^{2-}A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+})dt_1 dt_2 \\
&\quad +(\text{terms in } dt_1^2, dt_2^2 \text{ and H.O.T.}).
\end{aligned} \tag{A.3}$$

Case 2: $dt_1 \geq 0$, $dt_2 < 0$

The arguments for proving (A.1) in Case 1 can be applied in this case to show its validity. In this case,

$$\begin{aligned}
\delta x(t_2+dt_2^-) &= z_2(t_2+dt_2^-) - x(t_2+dt_2^-) \\
&= \Big(z_2(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} f_2\big(z_2(t)\big)\,dt\Big) - \Big(x(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} f_2\big(x(t)\big)\,dt\Big) \\
&= \delta x(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} \Big(f_2\big(z_2(t)\big) - f_2\big(x(t)\big)\Big)\,dt \\
&= \delta x(t_2^-) + \Big(f_2\big(z_2(t_2^-)\big) - f_2\big(x(t_2^-)\big)\Big)dt_2 + o(dt_2) \\
&= \delta x(t_2^-) + f_x^{2-}\delta x(t_2^-)dt_2 + o(dt_2) \\
&= A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1 + f_x^{2-}A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1dt_2 \\
&\quad +(\text{terms in } dt_1^2, dt_2^2 \text{ and H.O.T.}). \qquad\qquad (A.4)
\end{aligned}$$

Case 3: $dt_1 < 0$, $dt_2 \geq 0$

In this case, we have

$$\begin{aligned}
\delta x(t_1^+) &= \hat{x}(t_1^+) - x(t_1^+) \\
&= \Big(\gamma^1\big(x(t_1+dt_1^-)\big) + \int_{t_1+dt_1^+}^{t_1^+} f_2\big(\hat{x}(t)\big)\,dt\Big) - \gamma^1\Big(x(t_1+dt_1^-) + \int_{t_1+dt_1^-}^{t_1^-} f_1\big(x(t)\big)\,dt\Big) \\
&= \Big(\gamma^1\big(x(t_1+dt_1^-)\big) + f_2\big(\hat{x}(t_1+dt_1^+)\big)(-dt_1) + o(dt_1)\Big) - \gamma^1\Big(x(t_1+dt_1^-) + f_1\big(x(t_1^-)\big)(-dt_1) + o(dt_1)\Big) \\
&= -f_2\Big(\gamma^1\big(x(t_1+dt_1^-)\big)\Big)dt_1 + \gamma_x^1\big(x(t_1+dt_1^-)\big)f^{1-}dt_1 + o(dt_1) \\
&= -f_2\Big(\gamma^1\big(x(t_1^-)\big) + O(dt_1)\Big)dt_1 + \gamma_x^1\big(x(t_1^-) + O(dt_1)\big)f^{1-}dt_1 + o(dt_1) \\
&= \Big(\gamma_x^1\big(x(t_1^-)\big)f^{1-} - f_2\Big(\gamma_1\big(x(t_1^-)\big)\Big)\Big)dt_1 + o(dt_1) \\
&= (\gamma_x^{1-}f^{1-} - f^{1+})dt_1 + o(dt_1). \qquad\qquad (A.5)
\end{aligned}$$

In the derivations of the third to the last equations in (A.5), we use the relationship

$$x(t_1+dt_1^-) = x(t_1^-) + f^{1-}dt_1 + o(dt_1) = x(t_1^-) + O(dt_1), \qquad (A.6)$$

and the Taylor expression of f_2. Therefore, we have

$$\begin{aligned}
\delta x(t_2^-) &= A(t_2^-,t_1^+)\delta x(t_1^+) + \big(\text{H.O.T. in } \delta x(t_1^+)\big) \\
&= A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1 + o(dt_1) \qquad\qquad (A.7) \\
\delta x(t_2+dt_2^-) &= \hat{x}(t_2+dt_2^-) - z_3(t_2+dt_2^-) \\
&= \Big(\hat{x}(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} f_2\big(\hat{x}(t)\big)\,dt\Big) - \Big(z_3(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} f_2\big(z_3(t)\big)\,dt\Big) \\
&= \delta x(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} \Big(f_2\big(\hat{x}(t)\big) - f_2\big(x_3(t)\big)\Big)\,dt \\
&= \delta x(t_2^-) + \Big(f_2\big(\hat{x}(t_2^-)\big) - f_2\big(z_3(t_2^-)\big)\Big)dt_2 + o(dt_2) \\
&= \delta x(t_2^-) + f_x^{2-}\delta x(t_2^-)dt_2 + o(dt_2) \\
&= A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1 + f_x^{2-}A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1dt_2 \\
&\quad +(\text{terms in } dt_1^2, dt_2^2 \text{ and H.O.T.}). \qquad\qquad (A.8)
\end{aligned}$$

Case 4: $dt_1 < 0$, $dt_2 < 0$

The arguments for proving (A.7) in Case 3 can be applied in this case to show its validity. In this case, we have

$$\begin{aligned}
\delta x(t_2+dt_2^-) &= z_4(t_2+dt_2^-) - x(t_2+dt_2^-) \\
&= \Big(z_4(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} f_2\big(z_4(t)\big)\,dt\Big) - \Big(x(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} f_2\big(x(t)\big)\,dt\Big) \\
&= \delta x(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} \Big(f_2\big(z_4(t)\big) - f_2\big(x(t)\big)\Big)\,dt \\
&= \delta x(t_2^-) + \Big(f_2\big(z_4(t_2^-)\big) - f_2\big(x(t_2^-)\big)\Big)dt_2 + o(dt_2) \\
&= \delta x(t_2^-) + f_x^{2-}\delta x(t_2^-)dt_2 + o(dt_2) \\
&= A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1 + f_x^{2-}A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1dt_2 \\
&\quad +(\text{terms in } dt_1^2, dt_2^2 \text{ and H.O.T.}). \qquad (A.9)
\end{aligned}$$

□

PROOF OF LEMMA 3.2: (3.39) follows directly from the fact that

$$dx(t_2^-) = \delta x(t_2+dt_2^-) + f_2\big(x(t_2^-)\big)dt_2 + o(dt_2). \qquad (A.10)$$

To prove (3.40), we note that

$$\begin{aligned}
dx(t_2^+) &= \gamma^2\big(\hat{x}(t_2+dt_2^-)\big) - \gamma^2\big(x(t_2^-)\big) \\
&= \gamma^2\big(x(t_2^-) + dx(t_2^-)\big) - \gamma^2\big(x(t_2^-)\big) \\
&= \gamma_x^{2-}dx(t_2^-) + \frac{1}{2}\begin{bmatrix} \big(dx(t_2^-)\big)^T \frac{\partial^2\gamma_{(1)}^2\left(x(t_2^-)\right)}{\partial x^2} dx(t_2^-) \\ \vdots \\ \big(dx(t_2^-)\big)^T \frac{\partial^2\gamma_{(n)}^2\left(x(t_2^-)\right)}{\partial x^2} dx(t_2^-) \end{bmatrix} + \big(\text{H.O.T. in } dx(t_2^-)\big). \qquad (A.11)
\end{aligned}$$

Now since

$$\begin{aligned}
&\frac{1}{2}\begin{bmatrix} \big(dx(t_2^-)\big)^T \frac{\partial^2\gamma_{(1)}^2\left(x(t_2^-)\right)}{\partial x^2} dx(t_2^-) \\ \vdots \\ \big(dx(t_2^-)\big)^T \frac{\partial^2\gamma_{(n)}^2\left(x(t_2^-)\right)}{\partial x^2} dx(t_2^-) \end{bmatrix} \\
= &\begin{bmatrix} (f^{2-})^T \frac{\partial^2\gamma_{(1)}^2\left(x(t_2^-)\right)}{\partial x^2} \\ \vdots \\ (f^{2-})^T \frac{\partial^2\gamma_{(n)}^2\left(x(t_2^-)\right)}{\partial x^2} \end{bmatrix} A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1dt_2 + (\text{terms in } dt_1^2, dt_2^2 \text{ and H.O.T.}) \\
= &\ \xi^{2-}A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1dt_2 + (\text{terms in } dt_1^2, dt_2^2 \text{ and H.O.T.}), \qquad (A.12)
\end{aligned}$$

we can substitute (A.12) into (A.11) to obtain (3.40). □

PROOF OF LEMMA 3.3: We first note that

$$\int_{t_1+dt_1^+}^{t_2+dt_2^-} L(\hat{x})\,dt = \begin{cases} \int_{\max\{t_1^+,t_1+dt_1^+\}}^{t_2+dt_2^-} L(\hat{x})\,dt, & \text{if } dt_1 \geq 0, \\ \int_{t_1+dt_1^+}^{\max\{t_1^+,t_1+dt_1^+\}} L(\hat{x})\,dt + \int_{\max\{t_1^+,t_1+dt_1^+\}}^{t_2+dt_2^-} L(\hat{x})\,dt, & \text{if } dt_1 < 0. \end{cases} \qquad (A.13)$$

In the light of the forward decoupling principle, the term $\int_{t_1+dt_1^+}^{\max\{t_1^+,t_1+dt_1^+\}} L(\hat{x})\,dt$ in the case of $dt_1 < 0$ will not depend on dt_2; therefore, it will not contribute to the coefficient of dt_1dt_2. So we conclude that no matter $dt_1 \geq 0$ or $dt_1 < 0$, we only need to consider the term $\int_{\max\{t_1^+,t_1+dt_1^+\}}^{t_2+dt_2^-} L(\hat{x})\,dt$. For this term, we discuss as follows.

Case 1: $dt_2 \geq 0$

In this case, we have

$$\int_{\max\{t_1^+,t_1+dt_1^+\}}^{t_2+dt_2^-} L(\hat{x})\,dt = \int_{\max\{t_1^+,t_1+dt_1^+\}}^{t_2^-} L(x+\delta x)\,dt + \int_{t_2^-}^{t_2+dt_2^-} L(\hat{x})\,dt. \qquad (A.14)$$

The first term in (A.14) will not be contributing due to the reason that

$$\delta x(t) = A(t, t_1^-)(\gamma_x^{1-} f^{1-} - f^{1+})dt_1 + o(dt_1), \tag{A.15}$$

for $t \in [\max\{t_1^+, t_1 + dt_1^+\}, t_2^-]$ and therefore they do not depend on dt_2.

The second term is shown to be

$$\begin{aligned}\int_{t_2^-}^{t_2+dt_2^-} L(\hat{x})\ dt &= L\big(\hat{x}(t_2^-)\big)dt_2 + o(dt_2)\\ &= L^{2-}dt_2 + L_x^{2-}\delta x(t_2^-)dt_2 + \Big(\text{terms in } \big(\delta x(t_2^-)\big)^2 dt_2, dt_2^2 \text{ and H.O.T.}\Big).\end{aligned} \tag{A.16}$$

By substituting the expression of $\delta x(t_2^-)$ into (A.16), we obtain the coefficient of $dt_1 dt_2$ contributed by this term as

$$L_x^{2-} A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+}). \tag{A.17}$$

Case 2: $dt_2 < 0$

In this case, since $x(t) + \delta x(t) = \hat{x}(t)$ for $t \in [\max\{t_1^+, t_1 + dt_1^+\}, t_2 + dt_2^-]$, we have

$$\begin{aligned}\int_{\max\{t_1^+, t_1+dt_1^+\}}^{t_2+dt_2^-} L(\hat{x})\ dt &= \int_{\max\{t_1^-, t_1+dt_1^-\}}^{t_2+dt_2^-} L(x+\delta x)\ dt\\ &= \int_{\max\{t_1^+, t_1+dt_1^+\}}^{t_2^-} L(x+\delta x)\ dt + \int_{t_2^-}^{t_2+dt_2^-} L(x+\delta x)\ dt.\end{aligned} \tag{A.18}$$

Similar to Case 1, the first term in (A.18) will not be contributing. The second term is shown to be

$$\begin{aligned}\int_{t_2^-}^{t_2+dt_2^-} L(x+\delta x)\ dt &= L(x(t_2^-) + \delta x(t_2^-))dt_2 + o(dt_2)\\ &= L^{2-}dt_2 + L_x^{2-}\delta x(t_2^-)dt_2 + \Big(\text{terms in } \big(\delta x(t_2^-)\big)^2 dt_2, dt_2^2 \text{ and H.O.T.}\Big).\end{aligned} \tag{A.19}$$

Therefore, by substituting the expression of $\delta x(t_2^-)$ into (A.19), we obtain the same coefficient (A.17). □

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