Optimal Control of Switched Autonomous Systems with a Prespecified Sequence of Active Subsystems*

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Abstract

In this paper, optimal control problems for switched autonomous systems are studied. In particular, we focus on problems in which a prespecified sequence of active subsystems is given and propose an approach to finding the optimal switching instants. The approach derives the derivatives of the cost with respect to the switching instants and uses nonlinear optimization techniques to locate the optimal switching instants. The approach is then applied to general quadratic problems for switched linear autonomous systems and to reachability problems. Examples illustrate the results.

1 Introduction

A switched system is a particular kind of hybrid system that consists of several subsystems and a switching law specifying the active subsystem at each time instant. Examples of switched systems can be found in chemical processes, automotive systems, and electrical circuit systems, etc.

Recently, many results for optimal control of switched systems have appeared in the literature (e.g., [3, 7, 8, 9, 10]). Most of them consider problems which seek for the solution of both the optimal continuous input and the optimal switching sequence. Approaches to such problems include ones based on discretization of the time and state space [7, 8] and ones that are not based discretizations [9, 10]. Many of these approaches find approximations to local optimal solutions.

In this paper, we focus on optimal control problems for a class of switched systems, namely, switched autonomous systems where each subsystem is autonomous (i.e., with no continuous input). For such problems, we develop an effective approach for finding accurate numerical values of local optimal solutions instead of approximations. In particular, we focus on problems in which a prespecified sequence of active subsystems is given. Such problems arise naturally in multimodal control and in logic-based control systems whose controllers are switched between several given controllers. In this paper, we consider general autonomous subsystems and general performance costs. We note that the cost is actually a function of the switching instants for such problems and propose to use constrained nonlinear optimization techniques to locate the optimal switching instants. To apply nonlinear optimization techniques, we need to first determine the values of the derivatives of the cost with respect to the switching instants. An approach similar to that in [10] is proposed in this paper for their derivations and is presented in detail. One of the main results of the paper is Theorem 3.1 which gives us the expressions of the derivatives. Note here the approach provides us with accurate values of the derivatives (while in [10], only approximate values to the

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derivatives of the optimal cost with respect to the switching instants are obtained). The approach is then applied to general quadratic problems for switched linear autonomous systems. The computation of the derivatives can be further simplified by utilizing the special structure of such problems. Finally, we apply the optimal control approach to reachability problems. Using the approach, the reachability switching instants can be determined if a final state is reachable from an initial state.

Similar problems have also been looked into by other researchers. The closest problem formulation and results to ours, as far as we know, are reported by Giua *et al* in [5, 6] which study switched linear autonomous systems. However, there are some differences between our results and those in [5, 6]. First, the results in [5, 6] are for linear subsystems and quadratic costs, while our results apply to general subsystems and costs. Second, [5, 6] study infinite horizon problems, while this paper studies finite horizon problems. Third, the approach in this paper can be applied to reachability problems, while the approach in [5, 6] fits better for stability problems. Besides the above differences, we should also point out that Giua *et al* proposes closed-loop global optimal solutions (due to the special problem structure) in [5, 6] and our approach in this paper obtains open-loop local optimal solutions (due to the general problem formulation). This is not surprising because of the differences in problem formulation and time horizon. For the general problem formulation in this paper, the cost function is generally nonconvex, therefore we expect the solutions found by our approach to be local optimal solutions in general. Overall, we believe our results are new and contribute to the understanding and the solution of optimal control problems of switched systems.

The structure of the paper is as follows. In Section 2, we formulate the optimal control problems and propose an algorithm for solving them. In Section 3, detailed derivations are presented to show how to obtain the derivatives of the cost with respect to the switching instants. In Section 4, the result is applied to general quadratic problems for switched linear autonomous systems. In Section 5, we show how to apply the optimal control result to reachability problems. Examples are given in Section 6. Section 7 concludes the paper.

2 Problem Formulation

In this paper, we consider the following *switched autonomous systems*, i.e., switched systems which consist of autonomous subsystems (i.e., without continuous input)

$$\dot{x} = f_i(x,t), \quad f_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad i \in I = \{1, 2, \cdots, M\}. \tag{2.1}$$

For a switched system, one can control its state trajectory evolution by choosing appropriate switching sequences. Here a *switching sequence* σ in $[t_0, t_f]$ is defined as

$$\sigma = ((t_0, i_0), (t_1, i_1), (t_2, i_2), \cdots, (t_K, i_K)), \tag{2.2}$$

with $0 \leq K < \infty$, $t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_K \leq t_f$, and $i_k \in I$, $k = 0, 1, 2, \cdots, K$. σ tells us that the switched system switches to subsystem i_k at time instant t_k. Note that one feature of a switched system is that its continuous state does not have discontinuities at the switching instants.

In the following, we assume that a prespecified sequence of active subsystems is given (i.e., the untimed sequence $(i_0, i_1, i_2, \cdots, i_K)$ is given). Furthermore, we assume without loss of generality that the untimed sequence is $(1, 2, \cdots, K, K+1)$, i.e., subsystem k is active in $[t_{k-1}, t_k)$. Note that we can always do this by relabeling the subsystem indices and even expanding the collection of subsystems (i.e., two subsystems may actually refer to the same actual subsystem). We consider the following optimal control problem.

Problem 2.1 (Optimal Control Problem) *Consider a switched autonomous system with subsystems* $f_i(x,t)$, $i \in I$. *Assume that a prespecified sequence of active subsystems* $(1, 2, \cdots, K, K+1)$ *is given. Find* *optimal switching instants $t_1, \cdots, t_K$ ($t_0 \leq t_1 \leq \cdots \leq t_K \leq t_f$) such that the corresponding continuous state trajectory x departs from a given initial state $x(t_0) = x_0$ and the cost*

$$J(t_1, \cdots, t_K) = \psi\big(x(t_f)\big) + \int_{t_0}^{t_f} L(x,t)\ dt \tag{2.3}$$

is minimized. Here t_0, t_f, x_0 are given. □

Problem 2.1 is an optimal control problem in Bolza form. As in the usual practice of formulating optimal control problems (see [1]), in the sequel, we assume that f_i's are continuously differentiable; L and ψ are assumed to have twice continuous derivatives.

Remark 2.1 Due to the smoothness assumptions for f_i's, L, and ψ, we can observe that a small disturbance of $t_1, \cdots, t_K$ will only cause a small disturbance of J value. Furthermore, it is not difficult to show that the cost J is a continuously differentiable function of $t_1, \cdots, t_K$. □

2.1 An Algorithm

Note that Problem 2.1 is actually a constrained multivariable optimization problem

$$\begin{array}{c} \min_{\hat{t}} J(\hat{t}) \\ \text{subject to } \hat{t} \in T \end{array} \tag{2.4}$$

where $T \triangleq \{\hat{t} = (t_1, t_2, \cdots, t_K)^T | t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_K \leq t_f\}$. The following algorithm can be adopted to solve such a nonlinear optimization problem.

Algorithm 2.1

(1). Set the iteration index $j = 0$. Choose an initial $\hat{t}^j$.

(2). Find $J(\hat{t}^j)$, $\frac{\partial J}{\partial \hat{t}}(\hat{t}^j)$ and $\frac{\partial^2 J}{\partial \hat{t}^2}(\hat{t}^j)$.

(3). Use some first-order or second-order feasible direction method (e.g., the gradient projection method or the constrained Newton's method) to update $\hat{t}^j$ to be $\hat{t}^{j+1} = \hat{t}^j + \alpha^j d\hat{t}^j$ (here the stepsize α^j is chosen using the Armijo's rule [2]). Set the iteration index $j = j + 1$.

(4). Repeat Steps (2), (3) and (4), until a prespecified termination condition is satisfied (e.g. $\|\frac{\partial J}{\partial \hat{t}}(\hat{t}^j)\|_2 < \epsilon$ where ϵ is a given small number). □

In order to apply the above algorithm, one needs to find the values of the derivatives $\frac{\partial J}{\partial \hat{t}}$ and $\frac{\partial^2 J}{\partial \hat{t}^2}$ (step (2)). Let us elaborate more on step (2) in the sequel.

3 Differentiations of the Cost Function

In this section, we propose an approach to Problem 2.1 which finds the values of the derivatives $\frac{\partial J}{\partial \hat{t}}$ and $\frac{\partial^2 J}{\partial \hat{t}^2}$ based on direct differentiations of the cost function. The idea of the method is similar to that in [10].

Assume that we have a nominal $\hat{t} = (t_1, \cdots, t_K)^T$ and the corresponding nominal state trajectory $x(t)$. For such nominal values, the cost is

$$J(t_1, \cdots, t_K) = \psi\big(x(t_f)\big) + \int_{t_0}^{t_1} L(x,t)\ dt + \cdots + \int_{t_K}^{t_f} L(x,t)\ dt \tag{3.1}$$

Since x_0 and t_0 are given in Problem 2.1, J will not be a function of them. Next we define the value function at the k-th switching instant as

$$J^k\big(x(t_k), t_k, t_{k+1}, \cdots, t_K\big) = \psi\big(x(t_f)\big) + \int_{t_k}^{t_{k+1}} L(x,t)\ dt + \cdots + \int_{t_K}^{t_f} L(x,t)\ dt. \tag{3.2}$$

Note that, unlike J, J^k for $k \geq 1$ will be a function of t_k and of the initial state $x(t_k)$ which depends on the trajectory before t_k. In order to make our presentation clear, in the sequel, we denote $\frac{\partial J^k}{\partial x}$ for a function J^k as a row vector J_x^k, $\frac{\partial^2 J^k}{\partial x^2}$ as an $n \times n$ matrix J_{xx}^k and so on.

3.1 Single Switching

Let us first consider the case of a single switching. Assume that we are given a nominal t_1 and a corresponding nominal trajectory $x(t)$, we denote by $\hat{x}(t)$ the state trajectory after a variation dt_1 has taken place. In the sequel, we adopt the following notational convention. We write f, f_x and f_t with a superscript $1-$ (resp. $1+$) whenever the corresponding active vector field at t_1- (resp. t_1+) is used for evaluation at $\big(x(t_1), t_1\big)$. Examples of this convention are $f^{1-} \triangleq f_1\big(x(t_1), t_1\big)$, $f^{1+} \triangleq f_2\big(x(t_1), t_1\big)$, $f_t^{1-} \triangleq \frac{\partial f_1}{\partial t}\big(x(t_1), t_1\big)$, and $f^{1+} \triangleq \frac{\partial f_2}{\partial t}\big(x(t_1), t_1\big)$, etc. Also, we simply write $J^1 \triangleq J^1\big(x(t_1), t_1\big)$, $L^1 \triangleq L\big(x(t_1), t_1\big)$, $J_x^1 \triangleq J_x^1\big(x(t_1), t_1\big)$, $L_x^1 \triangleq L_x\big(x(t_1), t_1\big), \cdots$ (be careful to distinguish the values J^1, L^1, J_x^1, and $L_x^1, \cdots$ from the functions $J^1\big(x(t_1), t_1\big)$, $L(x,t)$, $J_x^1\big(x(t_1), t_1\big)$, and $L_x(x,t), \cdots$).

It is not difficult to see that

$$J(t_1) = \int_{t_0}^{t_1} L(x,t)\ dt + J^1\big(x(t_1), t_1\big). \tag{3.3}$$

For a small variation dt_1 of t_1, we have

$$J(t_1 + dt_1) = \int_{t_0}^{t_1+dt_1} L(\hat{x}, t)\ dt + J^1\big(\hat{x}(t_1 + dt_1), t_1 + dt_1\big). \tag{3.4}$$

There are two terms in (3.4). Let us consider the second order Taylor expansion of each term. In the following derivations we denote

$$dx(t_1) \triangleq \hat{x}(t_1 + dt_1) - x(t_1) = f^{1-} dt_1 + \frac{1}{2}(f_t^{1-} + f_x^{1-} f^{1-}) dt_1^2 + o(dt_1^2). \tag{3.5}$$

Consider the first term in (3.4), if $dt_1 \geq 0$, we have

$$\begin{aligned} \int_{t_0}^{t_1+dt_1} L(\hat{x}, t)\ dt &= \int_{t_0}^{t_1} L(x,t)\ dt + \int_{t_1}^{t_1+dt_1} L(\hat{x}, t)\ dt \\ &= \int_{t_0}^{t_1} L(x,t)\ dt + L^1 dt_1 + \frac{1}{2} dt_1 L_x^1 dx(t_1) + \frac{1}{2} L_t^1 dt_1^2 + (\text{higher order terms}). \end{aligned} \tag{3.6}$$

Note that in deriving (3.6), we have used the relationship $\hat{x}(t_1) = x(t_1)$. If $dt_1 < 0$, we have

$$\begin{aligned} \int_{t_0}^{t_1+dt_1} L(\hat{x}, t)\ dt &= \int_{t_0}^{t_1} L(x,t)\ dt + \int_{t_1}^{t_1+dt_1} L(x,t)\ dt \\ &= \int_{t_0}^{t_1} L(x,t)\ dt + L^1 dt_1 + \frac{1}{2} dt_1 L_x^1 dx(t_1) + \frac{1}{2} L_t^1 dt_1^2 + (\text{higher order terms}). \end{aligned} \tag{3.7}$$

which has the same expression as (3.6) for $dt_1 \geq 0$ although the derivation is slightly different.

For the second term in (3.4), we have the second order expansion

$$\begin{aligned} J^1\big(\hat{x}(t_1+dt_1),t_1+dt_1\big) &= J^1 + J_x^1 dx(t_1) + J_{t_1}^1 dt_1 + \frac{1}{2}\big(dx(t_1)\big)^T J_{xx}^1 dx(t_1) + \frac{1}{2} J_{t_1t_1}^1 dt_1^2 \\ &\quad + dt_1 J_{t_1x}^1 dx(t_1) + \text{(higher order terms)}. \end{aligned} \tag{3.8}$$

In order to express (3.4) into second order expansion with respect to dt_1, we substitute (3.5) into (3.6) and (3.8) and sum them to obtain

$$\begin{aligned} J(t_1+dt_1) &= J(t_1) + L^1 dt_1 + \frac{1}{2} dt_1 L_x^1 dx(t_1) + \frac{1}{2} L_t^1 dt_1^2 + J_x^1 dx(t_1) + J_{t_1}^1 dt_1 \\ &\quad + \frac{1}{2}(dx(t_1))^T J_{xx}^1 dx(t_1) + \frac{1}{2} J_{t_1t_1}^1 dt_1^2 + dt_1 J_{t_1x}^1 dx(t_1) + \text{(higher order terms)} \\ &= J(t_1) + (L^1 + J_{t_1}^1 + J_x^1 f^{1-}) dt_1 + \frac{1}{2}\big(L_x^1 f^{1-} + L_t^1 + J_x^1(f_t^{1-} + f_x^{1-} f^{1-}) \\ &\quad + (f^{1-})^T J_{xx}^1 f^{1-} + J_{t_1t_1}^1 + 2J_{t_1x}^1 f^{1-}\big) dt_1^2 + o(dt_1^2) \\ &\triangleq J(t_1) + J_{t_1} dt_1 + \frac{1}{2} J_{t_1t_1} dt_1^2 + o(dt_1^2) \end{aligned} \tag{3.9}$$

for all dt_1 (no matter $dt_1 \geq 0$ or $dt_1 < 0$ we get the same expression).

Now let us consider $J^1\big(x(t_1),t_1\big)$ which is the value function for the given nominal t_1. The following dynamic programming equation holds for it

$$J_{t_1}^1 = -J_x^1 f^{1+} - L^1. \tag{3.10}$$

Note that (3.10) can be derived similarly to the HJB equation. However, the difference between it and the HJB equation is that (3.10) holds for any trajectory that is not necessarily optimal (for more details see [4]).

By differentiating (3.10), we obtain

$$J_{t_1x}^1 = -(f^{1+})^T J_{xx}^1 - J_x^1 f_x^{1+} - L_x^1 \tag{3.11}$$

$$\begin{aligned} J_{t_1t_1}^1 &= -J_{t_1x}^1 f^{1+} - J_x^1 f_t^{1+} - L_t^1 \\ &= (f^{1+})^T J_{xx}^1 f^{1+} + (J_x^1 f_x^{1+} + L_x^1) f^{1+} - J_x^1 f_t^{1+} - L_t^1. \end{aligned} \tag{3.12}$$

By substituting (3.10), (3.11) and (3.12) into (3.9), we can write J_{t_1} and $J_{t_1t_1}$ in the following form

$$J_{t_1} = J_x^1(f^{1-} - f^{1+}), \tag{3.13}$$

$$\begin{aligned} J_{t_1t_1} &= J_x^1(f_t^{1-} - f_t^{1+}) - (J_x^1 f_x^{1+} + L_x^1)(f^{1-} - f^{1+}) \\ &\quad + J_x^1(f_x^{1-} - f_x^{1+}) f^{1-} + (f^{1-} - f^{1+})^T J_{xx}^1 (f^{1-} - f^{1+}) \end{aligned} \tag{3.14}$$

3.2 Two or More Switchings

In order to construct a second-order optimization algorithm for switched systems with two or more switchings, we need more information to derive the derivatives of J with respect to the t_k's. Let us first consider the case of two switchings. Assume that a system switches from subsystem 1 to 2 at t_1 and from subsystem 2 to 3 at t_2 ($t_0 \leq t_1 \leq t_2 \leq t_f$). The cost then is

$$J(t_1,t_2) = \int_{t_0}^{t_1} L(x,t)\, dt + J^1\big(x(t_1),t_1,t_2\big) \tag{3.15}$$

$$= \int_{t_0}^{t_2} L(x,t)\, dt + J^2\big(x(t_2),t_2\big). \tag{3.16}$$

Using (3.15), by holding t_2 fixed, J_{t_1}, $J_{t_1t_1}$ can be derived similarly to that in subsection 3.1. In the same manner, J_{t_2}, $J_{t_2t_2}$ can be derived using (3.16). However, we need additional information to derive $J_{t_1t_2}$. Arguments from the calculus of variations are used in the followings to derive $J_{t_1t_2}$. Let us first define the important notion of incremental change which will be used in the sequel.

Definition 3.1 (Incremental Change) *Given any variations dt_1 and dt_2, we define $\delta x(t), \min\{t_1, t_1 + dt_1\} \le t \le \max\{t_2, t_2 + dt_2\}$ to be* the incremental change *of the state due to dt_1 and dt_2. In detail, it is defined as follows (see figure 1).*

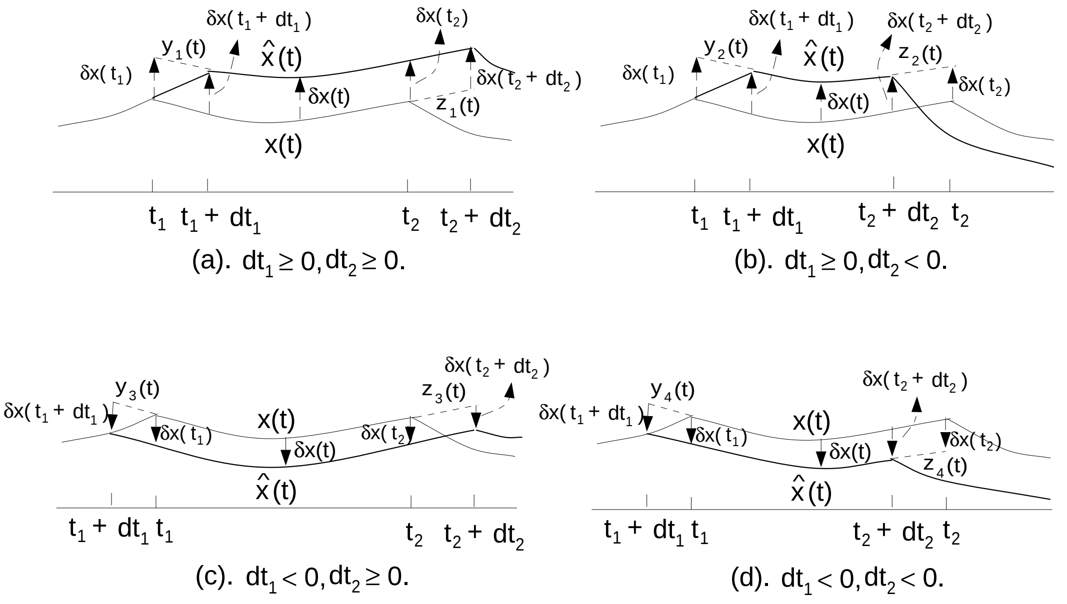

Figure 1: The incremental change $\delta x(t)$ for (a). $dt_1 \ge 0$, $dt_2 \ge 0$; (b). $dt_1 \ge 0$, $dt_2 < 0$; (c). $dt_1 < 0$, $dt_2 \ge 0$; (d). $dt_1 < 0$, $dt_2 < 0$.

Case 1: $dt_1 \ge 0, dt_2 \ge 0$ *(see figure 1(a))*

In this case, $\delta x(t)$ is defined to be

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), \ t \in [t_1 + dt_1, t_2] \\ y_1(t) - x(t), \ t \in [t_1, t_1 + dt_1] \\ \hat{x}(t) - z_1(t), \ t \in [t_2, t_2 + dt_2] \end{cases} \tag{3.17}$$

where $y_1(t)$ is the solution of

$$\begin{cases} \dot{y}_1(t) = f_2\big(y_1(t), t\big), \ t \in [t_1, t_1 + dt_1] \\ y_1(t_1 + dt_1) = \hat{x}(t_1 + dt_1) \end{cases} \tag{3.18}$$

and $z_1(t)$ is the solution of

$$\begin{cases} \dot{z}_1(t) = f_2\big(z_1(t), t\big), \ t \in [t_2, t_2 + dt_2] \\ z_1(t_2) = x(t_2). \end{cases} \tag{3.19}$$

Case 2: $dt_1 \ge 0, dt_2 < 0$ *(see figure 1(b))*

In this case, $\delta x(t)$ is defined to be

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), \ t \in [t_1 + dt_1, t_2 + dt_2] \\ y_2(t) - x(t), \ t \in [t_1, t_1 + dt_1] \\ z_2(t) - x(t), \ t \in [t_2 + dt_2, t_2] \end{cases} \tag{3.20}$$

where $y_2(t)$ is the solution of

$$\begin{cases} \dot{y}_2(t) = f_2\big(y_2(t), t\big), \ t \in [t_1, t_1 + dt_1] \\ y_2(t_1 + dt_1) = \hat{x}(t_1 + dt_1) \end{cases} \tag{3.21}$$

and $z_2(t)$ is the solution of

$$\begin{cases} \dot{z}_2(t) = f_2\big(z_2(t), t\big), \ t \in [t_2 + dt_2, t_2] \\ z_2(t_2 + dt_2) = \hat{x}(t_2 + dt_2). \end{cases} \tag{3.22}$$

Case 3: $dt_1 < 0, dt_2 \geq 0$ *(see figure 1(c))*

In this case, $\delta x(t)$ is defined to be

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), \ t \in [t_1, t_2] \\ \hat{x}(t) - y_3(t), \ t \in [t_1 + dt_1, t_1] \\ \hat{x}(t) - z_3(t), \ t \in [t_2, t_2 + dt_2] \end{cases} \tag{3.23}$$

where $y_3(t)$ is the solution of

$$\begin{cases} \dot{y}_3(t) = f_2\big(y_3(t), t\big), \ t \in [t_1 + dt_1, t_1] \\ y_3(t_1) = x(t_1) \end{cases} \tag{3.24}$$

and $z_3(t)$ is the solution of

$$\begin{cases} \dot{z}_3(t) = f_2\big(z_3(t), t\big), \ t \in [t_2, t_2 + dt_2] \\ z_3(t_2) = x(t_2). \end{cases} \tag{3.25}$$

Case 4: $dt_1 < 0, dt_2 < 0$ *(see figure 1(d))*

In this case, $\delta x(t)$ is defined to be

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), \ t \in [t_1, t_2 + dt_2] \\ \hat{x}(t) - y_4(t), \ t \in [t_1 + dt_1, t_1] \\ z_4(t) - x(t), \ t \in [t_2 + dt_2, t_2] \end{cases} \tag{3.26}$$

where $y_4(t)$ is the solution of

$$\begin{cases} \dot{y}_4(t) = f_2\big(y_4(t), t\big), \ t \in [t_1 + dt_1, t_1] \\ y_4(t_1) = x(t_1) \end{cases} \tag{3.27}$$

and $z_4(t)$ is the solution of

$$\begin{cases} \dot{z}_4(t) = f_2\big(z_4(t), t\big), \ t \in [t_2 + dt_2, t_2] \\ z_4(t_2 + dt_2) = \hat{x}(t_2 + dt_2). \end{cases} \tag{3.28}$$

□

Remark 3.1 Note that $\delta x(t)$ defines the difference between $\hat{x}(t)$ and $x(t)$ in the time interval where subsystem 2 is active. Moreover, by extending the trajectories $\hat{x}$ and x under the dynamics of subsystem 2 to the time interval $\min\{t_1, t_1 + dt_1\} \leq t \leq \max\{t_2, t_2 + dt_2\}$ where at least one of $\hat{x}(t)$ and $x(t)$ evolves along subsystem 2, $\delta x(t)$ even defines the difference for this time interval. □

In the followings, the expressions of $\delta x(t_2)$ and $dx(t_2)$ are derived.

Lemma 3.1 *The expressions of $\delta x(t_2)$ and $\delta x(t_2+dt_2)$ are*

$$
\begin{aligned}
\delta x(t_2) &= A(t_2,t_1)(f^{1-}-f^{1+})dt_1+o(dt_1), && (3.29)\\
\delta x(t_2+dt_2) &= A(t_2,t_1)(f^{1-}-f^{1+})dt_1+f_x^{2-}A(t_2,t_1)(f^{1-}-f^{1+})dt_1dt_2\\
&\quad +(\textit{other terms in } dt_1^2, dt_2^2 \textit{ and higher order terms}), && (3.30)
\end{aligned}
$$

where $A(t_2,t_1)$ is the state transition matrix for the variational time-varying equation

$$\dot{y}(t)=\frac{\partial f\big(x(t),t\big)}{\partial x}y(t) \tag{3.31}$$

for $y(t)$ from t_1 to t_2; in (3.31), f is the corresponding active subsystem vector field (here it is f_2) in $[t_1,t_2]$ and $x(t)$ is the current nominal state trajectory.

Proof: See Appendix A. □

In fact, from the proof of Lemma 3.1 (see Appendix A), we can observe that $\delta x(t)=A(t,t_1)\delta x(t_1)+o(dt_1)$ for any $t\in[\min\{t_1,t_1+dt_1\},\max\{t_2,t_2+dt_2\}]$. The following important principle can be obtained directly from this observation. We refer to it as *the forward decoupling principle.* It reveals some intrinsic relationship among different switching instants.

The Forward Decoupling Principle:

(a). The value of the incremental change $\delta x(t_1)$ at t_1 does not depend on dt_2.

(b). The value of the incremental change $\delta x(t_2)$ at t_2 does depend on dt_1. □

The forward decoupling principle tells us that a variation of an earlier switching instant will affect the value of the incremental change at a later switching instant, but not vice versa.

Lemma 3.2 *The expression of $dx(t_2)$ (i.e., $\hat{x}(t_2+dt_2)-x(t_2)$) is*

$$
\begin{aligned}
dx(t_2) &= A(t_2,t_1)(f^{1-}-f^{1+})dt_1+f_x^{2-}A(t_2,t_1)(f^{1-}-f^{1+})dt_1dt_2+f^{2-}dt_2\\
&\quad +(\textit{other terms in } dt_1^2, dt_2^2 \textit{ and higher order terms}). && (3.32)
\end{aligned}
$$

Proof: The proof follows directly from the fact that

$$dx(t_2)=\delta x(t_2+dt_2)+f_2\big(x(t_2),t_2\big)dt_2+o(dt_2) \tag{3.33}$$

for all four cases of the signs of dt_1, dt_2. □

Remark 3.2 It is very important to point out that in the expression of $dx(t_2)$, we deliberately express the term $f_x^{2-}A(t_2,t_1)(f^{1-}-f^{1+})dt_1dt_2$ explicitly because it will contribute to the coefficient of dt_1dt_2 as can be seen below. □

Now that we have the expressions for $\delta x(t_2)$, $\delta x(t_2+dt_2)$ and $dx(t_2)$, we are ready to derive the coefficient for dt_1dt_2 in the expansion of

$$J(t_1+dt_1,t_2+dt_2)=\int_{t_0}^{t_2+dt_2}L\big(\hat{x}(t),t\big)\,dt+J^2\big(\hat{x}(t_2+dt_2),t_2+dt_2\big). \tag{3.34}$$

There are two terms in (3.34). Let us look at their Taylor expansions one by one in order to find each term's contribution to the coefficient of dt_1dt_2.

For the first term in (3.34), we have the following Lemma.

Lemma 3.3 *The contribution of* $\int_{t_0}^{t_2+dt_2} L(\hat{x},t)\,dt$ *to the coefficient of* dt_1dt_2 *is*

$$L_x^2 A(t_2,t_1)(f^{1-}-f^{1+}). \tag{3.35}$$

Proof: See Appendix A. □

For the second term in (3.34), similar to the single switching case, we can obtain its Taylor expansion as

$$\begin{aligned} J^2\big(\hat{x}(t_2+dt_2),t_2+dt_2\big) &= J^2 + J_x^2 dx(t_2) + J_{t_2}^2 dt_2 + \frac{1}{2}\big(dx(t_2)\big)^T J_{xx}^2 dx(t_2) + \frac{1}{2}J_{t_2t_2}^2 dt_2^2 \\ &\quad + dt_2 J_{t_2x}^2 dx(t_2) + (\text{higher order terms}). \end{aligned} \tag{3.36}$$

In (3.36), the terms that will possibly contribute to the coefficient of dt_1dt_2 are those containing $dx(t_2)$. They are

$$J_x^2 dx(t_2),\ \frac{1}{2}\big(dx(t_2)\big)^T J_{xx}^2 dx(t_2),\ dt_2 J_{t_2x}^2 dx(t_2). \tag{3.37}$$

Substituting the expression of $dx(t_2)$ into (3.37) and summing them, we obtain the contribution of the second term to the coefficient of dt_1dt_2 as

$$\big(J_x^2 f_x^{2-} + (f^{2-})^T J_{xx}^2 + J_{t_2x}^2\big)A(t_2,t_1)(f^{1-}-f^{1+}). \tag{3.38}$$

Summing (3.35) and (3.38) and also substituting into the sum the expression of $J_{t_2x}^2$ which can be obtained similarly to the expression of $J_{t_1x}^1$ in (3.11), we conclude that the coefficient of dt_1dt_2 (i.e., $J_{t_1t_2}$ in the expansion of $J(t_1+dt_1,t_2+dt_2)$) is

$$\begin{aligned} J_{t_1t_2} &= \big(L_x^2 + J_x^2 f_x^{2-} + (f^{2-})^T J_{xx}^2 + J_{t_2x}^2\big)A(t_2,t_1)(f^{1-}-f^{1+}) \\ &= \big(J_x^2(f_x^{2-}-f_x^{2+}) + (f^{2-}-f^{2+})^T J_{xx}^2\big)A(t_2,t_1)(f^{1-}-f^{1+}). \end{aligned} \tag{3.39}$$

Remark 3.3 The above result still holds even when $t_1 = t_2$ (we can consider $t_2 > t_1$ first and then let $t_2 \to t_1$ to prove this). □

The above derivations can similarly be extended to the case of K switchings to relate $\delta x(t_l)$ and dt_k ($k < l$). The expression for $J_{t_kt_l}$ can be obtained in the similar manner. We summarize and extend the result obtained in this section into the following theorem.

Theorem 3.1 *For a switched system with* K *switchings,*

$$\begin{aligned} & J(t_1+dt_1,t_2+dt_2,\cdots,t_K+dt_K) \\ = \ & J(t_1,t_2,\cdots,t_K) + \sum_{k=1}^{K} J_{t_k}dt_k + \frac{1}{2}\sum_{k=1}^{K} J_{t_kt_k}dt_k^2 + \sum_{1\le k<l\le K} J_{t_kt_l}dt_kdt_l \\ & +(\textit{higher order terms}) \end{aligned} \tag{3.40}$$

where

$$J_{t_k} = J_x^k(f^{k-}-f^{k+}), \tag{3.41}$$

$$\begin{aligned} J_{t_kt_k} &= J_x^k(f_t^{k-}-f_t^{k+}) - (J_x^k f_x^{k+} + L_x^k)(f^{k-}-f^{k+}) \\ &\quad + J_x^k(f_x^{k-}-f_x^{k+})f^{k-} + (f^{k-}-f^{k+})^T J_{xx}^k (f^{k-}-f^{k+}) \end{aligned} \tag{3.42}$$

$$J_{t_kt_l} = \big(J_x^l(f_x^{l-}-f_x^{l+}) + (f^{l-}-f^{l+})^T J_{xx}^l\big)A(t_l,t_k)(f^{k-}-f^{k+}). \tag{3.43}$$

□

3.3 Computation of $A(t_l, t_k)$, J_x^k, and J_{xx}^k

In order to use Theorem 3.1 to compute the values of J_{t_k}, $J_{t_k t_k}$ and $J_{t_k t_l}$, the values of $A(t_l, t_k)$, J_x^k and J_{xx}^k need to be known. However, given nominal $\hat{t}$ and x, these values are not readily available. In general, numerical methods need to be used to compute their values. An efficient numerical method based on solving additional initial value ordinary differential equations (ODEs) is developed in this subsection.

First note that $A(t_l, t_k)$ is the state transition matrix for

$$\dot{y}(t) = \frac{\partial f(x,t)}{\partial x} y(t), \tag{3.44}$$

where f is the vector field of the corresponding active subsystem at each time instant (i.e., $f = f_j$ for $t \in [t_{j-1}, t_j)$, $j = k+1, \cdots, l$). To find its value, we can first find the solution $y^{(1)}(t), \cdots, y^{(n)}(t)$ corresponding to initial conditions

$$y^{(1)}(t_k) = e_1, \ \cdots, \ y^{(n)}(t_k) = e_n \tag{3.45}$$

respectively, where e_j is the unit column vector with all 0's except that the j-th element being 1, $j = 1, 2, \cdots, n$. From linear systems theory, $A(t_l, t_k)$ is equal to the square matrix whose j-th column is $y^{(j)}(t_l)$, i.e.

$$A(t_l, t_k) = [y^{(1)}(t_l), \cdots, y^{(n)}(t_l)]. \tag{3.46}$$

To obtain the value of J_x^k, note that

$$J^k\big(x(t_k), t_k\big) = \psi\big(x(t_f)\big) + \int_{t_k}^{t_f} L\big(x(t), t\big)\, dt. \tag{3.47}$$

If $x(t_k)$ has a variation $\delta x(t_k)$, then

$$\begin{aligned}
J^k\big(x(t_k) + \delta x(t_k), t_k\big) &= \psi\Big(x(t_f) + A(t_f, t_k)\delta x(t_k) + \big(\text{higher order terms in } \delta x(t_k)\big)\Big) \\
&\quad + \int_{t_k}^{t_f} L\Big(x(t) + A(t, t_k)\delta x(t_k) + \big(\text{higher order terms in } \delta x(t_k)\big), t\Big)\, dt \\
&= \psi\big(x(t_f)\big) + \int_{t_k}^{t_f} L(x,t)\, dt + \psi_x\big(x(t_f)\big)A(t_f, t_k)\delta x(t_k) \\
&\quad + \int_{t_k}^{t_f} \big(L_x(x,t)A(t,t_k)\delta x(t_k)\big)\, dt + \big(\text{higher order terms in } \delta x(t_k)\big) \\
&= J^k\big(x(t_k), t_k\big) + \Big(\psi_x\big(x(t_f)\big)A(t_f, t_k) + \int_{t_k}^{t_f} L_x(x,t)A(t,t_k)\, dt\Big)\delta x(t_k) \\
&\quad + \big(\text{higher order terms in } \delta x(t_k)\big).
\end{aligned} \tag{3.48}$$

Hence

$$J_x^k = \psi_x\big(x(t_f)\big)A(t_f, t_k) + \int_{t_k}^{t_f} L_x(x,t)A(t,t_k)\, dt. \tag{3.49}$$

Now if we apply the similar procedure by varying $x(t_k)$ as in (3.49) to $J_x^k\big(x(t_k), t_k\big)$, we can obtain

$$J_{xx}^k = A^T(t_f, t_k)\psi_{xx}\big(x(t_f)\big)A(t_f, t_k) + \int_{t_k}^{t_f} A^T(t, t_k)L_{xx}(x,t)A(t,t_k)\, dt. \tag{3.50}$$

From the above discussions, we find that $A(t_l, t_k)$ can be obtained by solving ODEs (3.44) along with initial conditions (3.45). $A(t_f, t_k)$ can be obtained in the same fashion. J_x^k and J_{xx}^k are in integral forms

(3.49) and (3.50) which can easily be rewritten as ODEs. Hence, we can find the values of $A(t_f, t_k)$, J_x^k and J_{xx}^k at once by solving the following initial value ODEs from t_k to t_f (along with the system differential equation $\dot{x} = f(x,t)$)

$$\begin{aligned}
\dot{A}(t,t_k) &= \frac{\partial f(x,t)}{\partial t} A(t,t_k), \ A(t_k,t_k) = [e_1, e_2, \cdots, e_n], && (3.51)\\
\dot{\eta}_1 &= L_x(x,t)A(t,t_k), \ \eta_1(t_k) = 0_{1\times n}, && (3.52)\\
\dot{\eta}_2 &= A^T(t,t_k)L_{xx}(x,t)A(t,t_k), \ \eta_2(t_k) = 0_{n\times n}, && (3.53)
\end{aligned}$$

and taking into consideration that

$$\begin{aligned}
J_x^k &= \psi_x\big(x(t_f)\big)A(t_f,t_k) + \eta_1(t_f), && (3.54)\\
J_{xx}^k &= A^T(t_f,t_k)\psi_{xx}\big(x(t_f)\big)A(t_f,t_k) + \eta_2(t_f). && (3.55)
\end{aligned}$$

Remark 3.4 (Computational Cost) All other terms in (3.41)-(3.43) except for $A(t_l, t_k)$, J_x^k and J_{xx}^k are readily available once the nominal trajectory $x(t)$ is known. Therefore the main computational cost for J_{t_k}, $J_{t_k t_k}$, J_{t_k,t_l} occurs in the computation of $A(t_l,t_k)$, J_x^k and J_{xx}^k. The above method we propose reduces the computation of $A(t_l,t_k)$ to solving initial value ODEs (3.44) for any $k < l$ and the computation of J_x^k and J_{xx}^k to solving initial value ODEs (3.51)-(3.53) for $k = 1, 2, \cdots, K$. Hence we altogether need to solve $\frac{(K-1)K}{2} + K = \frac{K(K+1)}{2}$ sets of initial value ODEs. With today's powerful ODE solvers (e.g., `ode45` function in MATLAB), these equations can be solved efficiently and accurately. For our purpose of efficient optimization of open-loop solutions of optimal switching instants, such computation suffices. Moreover, for general quadratic problems for switched autonomous linear systems which we will elaborate on in the next section, the computational costs of these values can be reduced greatly. □

4 General Quadratic Problems for Switched Linear Autonomous Systems

In this section, we apply the approach developed in Section 3 to a special class of problems, namely, general quadratic problems for switched autonomous linear systems. In particular, we show that due to the special structure of the problem, the computation of $A(t_l,t_k)$, J_x^k and J_{xx}^k can be further simplified.

Problem 4.1 *Consider a switched system with linear autonomous subsystems $\dot{x} = A_i x, i \in I$. Assume a prespecified sequence of active subsystems $(1, 2, \cdots, K, K+1)$ is given. Find optimal switching instants $t_1, \cdots, t_K$ $(t_0 \le t_1 \le \cdots \le t_K \le t_f)$ such that the cost in general quadratic form*

$$J = \frac{1}{2}x(t_f)^T Q_f x(t_f) + M_f x(t_f) + W_f + \int_{t_0}^{t_f} \Big(\frac{1}{2}\big(x(t)\big)^T Q x(t) + M x(t) + W\Big)\, dt \tag{4.1}$$

is minimized. Here t_0, t_f and $x(t_0) = x_0$ are given; Q_f, M_f, W_f, Q, M, W are matrices of appropriate dimensions with $Q_f \ge 0$, $Q \ge 0$. □

In comparison to the cost functionals of standard LQR control problems, here the cost J has the terms $M_f x(t_f)$, W_f, Mx, and W. In this way, we can address more general problems such as tracking of a constant trajectory, controlling the final state to a target, etc.

In view of the special structure of Problem 4.1, we can readily observe that

$$A(t_l,t_k) = e^{A_l(t_l - t_{l-1})} \cdots e^{A_{k+1}(t_{k+1}-t_k)}. \tag{4.2}$$

for any $k < l$.

The computation of J_x^k and J_{xx}^k is discussed next. Assume a nominal $\hat{t}$ is given. If for any $x \in \mathbb{R}^n$ and any $t \in [t_0, t_f]$ we denote by $\tilde{J}(x,t)$ the cost incurred if the system starts from the state x at time instant t and evolves according to the portion of the switching sequence generated by $\hat{t}$ in $[t, t_f]$. In other words,

$$\tilde{J}(x,t) = \frac{1}{2}\big(x(t_f)\big)^T Q_f x(t_f) + M_f x(t_f) + W_f + \int_t^{t_f} \Big(\frac{1}{2}\big(x(t)\big)^T Q x(t) + M x(t) + W\Big)\, dt \tag{4.3}$$

where $x(t) = x$. Dynamic programming approach similar to (3.10) can be applied to $\tilde{J}(x,t)$ to obtain

$$\tilde{J}(x,t) = \frac{1}{2} x^T P(t) x + S(t) x + T(t) \tag{4.4}$$

where $P(t) = P^T(t)$ and

$$\begin{aligned} -\dot{P} &= PA + A^T P + Q, \quad P(t_f) = Q_f, && (4.5)\\ -\dot{S} &= SA + M, \quad S(t_f) = M_f, && (4.6)\\ -\dot{T} &= W, \quad T(t_f) = W_f, && (4.7) \end{aligned}$$

where $A = A(t)$ equals the A_i of the corresponding active subsystem at each time instant t.

From the definitions of the functions $\tilde{J}$ and J^k, if $\hat{t}$ is fixed, we have

$$\begin{aligned} J^k\big(x(t_k), t_k, \cdots, t_K\big) &= \tilde{J}\big(x(t_k), t_k\big), && (4.8)\\ J_x^k\big(x(t_k), t_k, \cdots, t_K\big) &= \tilde{J}_x\big(x(t_k), t_k\big), && (4.9)\\ J_{xx}^k\big(x(t_k), t_k, \cdots, t_K\big) &= \tilde{J}_{xx}\big(x(t_k), t_k\big). && (4.10) \end{aligned}$$

Therefore the values of J_x^k and J_{xx}^k can be obtained as

$$\begin{aligned} J_x^k &= \tilde{J}_x\big(x(t_k), t_k\big) = \big(x(t_k)\big)^T P(t_k) + S(t_k), && (4.11)\\ J_{xx}^k &= \tilde{J}_{xx}\big(x(t_k), t_k\big) = P(t_k). && (4.12) \end{aligned}$$

Remark 4.1 (Computational Cost) The computation of $A(t_l, t_k)$'s using (4.2) is straightforward and do not resort to ODE solver. The computation of J_x^k and J_{xx}^k using (4.11) and (4.12) relies on the values of $P(t_k)$'s and $S(t_k)$'s which are easy to obtain by solving the initial value ODEs (4.5)-(4.7) backward in time only once. Therefore, the computational cost for Problem 4.1 is greatly reduced as opposed to the general case in subsection 3.3. □

5 Reachability Problems

The optimal control approach discussed above can also be applied to the following class of reachability problems.

Problem 5.1 (Reachability Problem) *Given a switched autonomous system, does there exist a switching sequence such that the state trajectory x departs from $x(t_0) = x_0$ and meets x_f at some t_f? Here t_0, x_0, x_f are given; t_f is not given.* □

Note that x_f is reachable from x_0, if and only if the following optimal control problem achieves its minimum at $J = 0$. The problem is having a free final time t_f and seeks to minimize the cost

$$J = \frac{1}{2}\|x(t_f) - x_f\|_2^2, \tag{5.1}$$

here t_0, x_0, x_f are given. In general, the optimal control problem is difficult to solve due to the large number of possible patterns of switching sequences. But if we assume that a prespecified sequence of active subsystems is given, the problem can be handled by using optimal control methodologies. For example, we can assume subsystem k being active in $[t_{k-1}, t_k)$ (subsystem $K+1$ in $[t_k, t_{K+1}]$ with $t_{K+1} = t_f$). In this case, we can minimize J with respect to the switching instants and the final time t_f. In other words, the reachability problem can be formulated as an optimal control problem which seeks for optimal values of $t_1, \cdots, t_K, t_f$ such that

$$J(t_1, \cdots, t_K, t_f) = \frac{1}{2}\|x(t_f) - x_f\|_2^2 \tag{5.2}$$

is minimized. In this case, ideally the minimum cost should be 0 if x_f is reachable from x_0 by the given order of active subsystems. In practice,if the optimal value of J is found to be smaller than a predefined small tolerance $\epsilon > 0$, then we regard x_f as reachable from x_0 and regard the corresponding optimal $t_1, \cdots, t_K, t_f$ as the reachability switching instants.

Remark 5.1 Note that since our approach for optimal control finds local optimal solutions, an optimal value of J greater than ϵ does not necessarily imply that x_f is not reachable from x_0. In the case that x_f is reachable from x_0, another trial of initial guess of switching instants may lead to the global optimal solution with $J < \epsilon$. Therefore the optimal control approach can only be used as a sufficient condition for judging reachability. However, whenever x_f is determined to be reachable from x_0, our approach also provides the explicit sequence $(t_1, \cdots, t_K, t_f)$ for achieving it. This is the strength of the approach. □

To minimize $J(t_1, \cdots, t_K, t_f)$ with respect to $(t_1, \cdots, t_K, t_f)$, we can use Algorithm 2.1. To apply the algorithm, the derivatives of J first need to be computed. The derivative values J_{t_k}, $J_{t_k t_k}$ and $J_{t_k t_l}$ can be obtained using the expressions stated in Theorem 3.1. However, we note here since $t_{K+1} = t_f$ is free, we also need to derive J_{t_f}, $J_{t_f t_f}$ and $J_{t_k t_f}$. These values can be obtained following the idea of the derivation in Section 3. It is not difficult to show that

$$\begin{aligned}
J_{t_f} &= L^{K+1} + J^{K+1}_{t_{K+1}} + J^{K+1}_x f^{(K+1)-}, && (5.3)\\
J_{t_f t_f} &= L^{K+1}_x f^{(K+1)-} + L^{K+1}_t + J^{K+1}_x (f^{(K+1)-}_t + f^{(K+1)-}_x f^{(K+1)-}) \\
&\quad + (f^{(K+1)-})^T J^{K+1}_{xx} f^{(K+1)-} + J^{K+1}_{t_{K+1} t_{K+1}} + 2J^{K+1}_{t_{K+1} x} f^{(K+1)-}, && (5.4)\\
J_{t_k t_f} &= \big(L^{K+1}_x + J^{K+1}_x f^{(K+1)-}_x + (f^{(K+1)-})^T J^{K+1}_{xx} + J^{K+1}_{t_{K+1} x}\big) A(t_{K+1}, t_k)(f^{k-} - f^{k+}). && (5.5)
\end{aligned}$$

Note that for this optimal control problem, we have

$$\begin{aligned}
J^{K+1} &= \frac{1}{2}\|x(t_f) - x_f\|_2^2, && (5.6)\\
J^{K+1}_x &= \big(x(t_f) - x_f\big)^T, && (5.7)\\
J^{K+1}_{t_{K+1}} &= 0, && (5.8)\\
J^{K+1}_{xx} &= I_{n\times n}, && (5.9)\\
J^{K+1}_{t_{K+1} t_{K+1}} &= 0, && (5.10)\\
J^{K+1}_{t_{K+1} x} &= 0_{1\times n}, && (5.11)\\
L^{(K+1)} &= 0, && (5.12)\\
L^{(K+1)}_x &= 0_{1\times n}, && (5.13)\\
L^{(K+1)}_t &= 0. && (5.14)
\end{aligned}$$

Using (5.6)-(5.14), we can simplify (5.3)-(5.5) as

$$J_{t_f} = \left(x(t_f) - x_f\right)^T f^{(K+1)-}, \tag{5.15}$$

$$\begin{aligned} J_{t_f t_f} &= \left(x(t_f) - x_f\right)^T (f_t^{(K+1)-} + f_x^{(K+1)-} f^{(K+1)-}) \\ &\quad + (f^{(K+1)-})^T f^{(K+1)-}, \end{aligned} \tag{5.16}$$

$$J_{t_k t_f} = \left(\left(x(t_f) - x_f\right)^T f_x^{(K+1)-} + (f^{(K+1)-})^T\right) A(t_f, t_k)(f^{k-} - f^{k+}). \tag{5.17}$$

6 Examples

In this section, we present two examples to illustrate the effectiveness of the approach developed in this paper.

Example 6.1 Consider a switched autonomous system consisting of

$$\text{subsystem 1:} \begin{cases} \dot{x}_1 = x_1 + 0.5 \sin x_2 \\ \dot{x}_2 = -0.5 \cos x_1 - x_2 \end{cases} \tag{6.1}$$

$$\text{subsystem 2:} \begin{cases} \dot{x}_1 = 0.3 \sin x_1 + 0.5 x_2 \\ \dot{x}_2 = -0.5 x_1 + 0.3 \cos x_2 \end{cases} \tag{6.2}$$

$$\text{subsystem 3:} \begin{cases} \dot{x}_1 = -x_1 - 0.5 \cos x_2 \\ \dot{x}_2 = 0.5 \sin x_1 + x_2 \end{cases} \tag{6.3}$$

Assume that $t_0 = 0$, $t_f = 3$ and the system switches at $t = t_1$ from subsystem 1 to 2 and at $t = t_2$ from subsystem 2 to 3 ($0 \le t_1 \le t_2 \le 3$). We want to find optimal switching instants t_1, t_2 such that the cost

$$J = \frac{1}{2}x_1^2(3) + \frac{1}{2}x_2^2(3) + \frac{1}{2}\int_0^3 x_1^2(t) + x_2^2(t)\ dt$$

is minimized. Here $x_1(0) = 1$ and $x_2(0) = 3$.

For this problem, we choose initial nominal $t_1 = 1$, $t_2 = 1.5$. We derive the derivatives of J using the result in Theorem 3.1. By using the Algorithm 2.1 with the constrained Newton's method, after 9 iterations we find that the optimal switching instants are $t_1 = 0.5466$, $t_2 = 2.0337$ and the corresponding optimal cost is 9.9933. The corresponding state trajectory is shown in figure 2. Figure 3 shows the plot of the cost function for different $0 \le t_1 \le t_2 \le 3$. By comparing the J value for different t_1 and t_2, we verify that the solution we obtain is the global optimal (although it is difficult to tell from the cost surface, our computation shows us so). $\square$

Example 6.2 (A Reachability Problem) Consider a switched system consisting of

$$\text{subsystem 1: } \dot{x} = A_1 x = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x, \tag{6.4}$$

$$\text{subsystem 2: } \dot{x} = A_2 x = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} x. \tag{6.5}$$

Assume that at $t_0 = 0$, the system state departs from the initial condition $x_1(0) = 1$ and $x_2(0) = 1$ and evolves following the dynamics of subsystem 1. Also assume that the system switches once at t_1 from subsystem 1 to 2. We want to find a t_1 and a t_f ($0 \le t_1 \le t_f$) such that the system state arrives at $[e^3,\ e^3]^T$ at t_f.

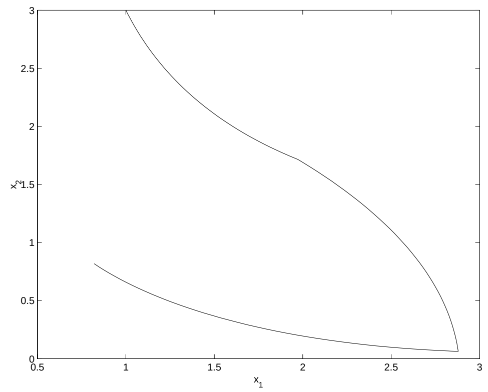

Figure 2: The state trajectory for Example 6.1.

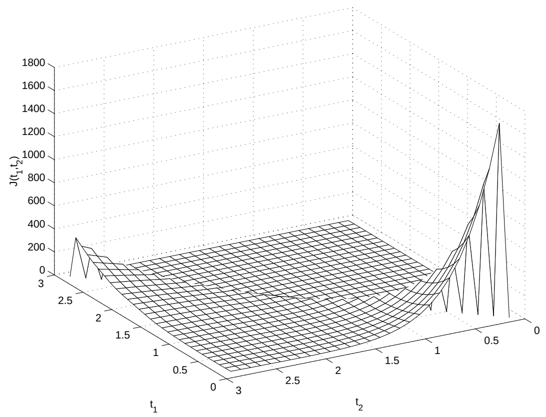

Figure 3: The cost for Example 6.1 for different (t_1, t_2)'s $(0 \le t_1 \le t_2 \le 3)$.

This reachability problem can be posed as an optimal control problem with unknown t_f and cost $J = \frac{1}{2}\big((x_1(t_f) - e^3)^2 + (x_2(t_f) - e^3)^2\big)$. We choose initial nominal $t_1 = 0.7$, $t_f = 1.7$. The values of J_{t_1}, J_{t_f}, $J_{t_1 t_1}$, $J_{t_f t_f}$ and $J_{t_1 t_f}$ can be derived using the formulae (3.41)-(3.42) and (5.15)-(5.17). We use Algorithm 2.1 with the constrained Newton's method to search for an optimal solution. After 8 iterations we find that the optimal switching instants are $t_1 = 1.0000$, $t_2 = 2.0000$ and the corresponding optimal cost is 6.3109×10^{-29}. The corresponding state trajectory is shown in figure 4. Figure 5 shows the plot of the cost function for different $0 \le t_1 \le t_f$. By comparing the J value for different t_1 and t_f, we verify that the solution we obtain is the global optimal (although it is difficult to tell from the cost surface, our computation shows us so).

It is worth noting that for this example we can verify the correctness of (5.15)-(5.17). For example, the expression of $J_{t_1 t_f}$ can be derived from (5.17) as (here $K = 1$)

$$\begin{aligned} J_{t_1 t_f} &= \Big(\big(x(t_f) - x_f\big)^T f_x^{2-} + (f^{2-})^T\Big) A(t_f, t_1)(f^{1-} - f^{1+}) \\ &= \Big(\big(x(t_f) - x_f\big)^T A_2 + \big(A_2 x(t_f)\big)^T\Big) A(t_f, t_1)(A_1 - A_2)x(t_1). \end{aligned} \tag{6.6}$$

We can substitute $x(t_1) = [e^{t_1},\ e^{2t_1}]^T$, $x(t_f) = [e^{2t_f - t_1},\ e^{t_f + t_1}]^T$, $x_f^T = [e^3,\ e^3]$, $A(t_f, t_1) = e^{A_2(t_f - t_1)}$, and A_1, A_2 into (6.6) and obtain

$$J_{t_1 t_f} = -4e^{4t_f - 2t_1} + 2e^{2t_f - t_1 + 3} + 2e^{2t_f + 2t_1} - e^{t_f + t_1 + 3}. \tag{6.7}$$

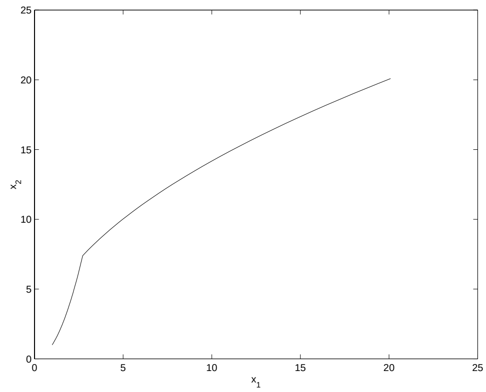

Figure 4: The state trajectory for Example 6.2.

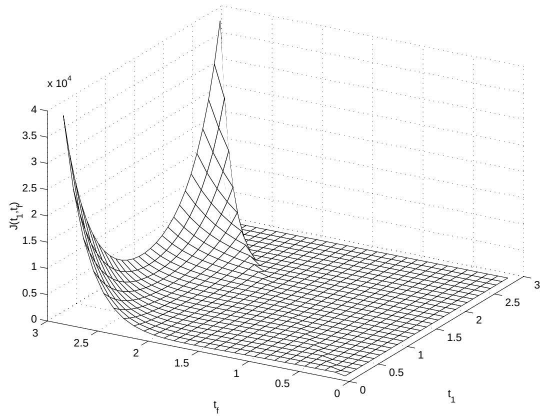

Figure 5: The cost for Example 6.2 for different (t_1, t_f)'s $(0 \le t_1 \le t_f)$.

The correctness of (6.7) can be verified by directly differentiating the expression of J

$$
\begin{aligned}
J &= \frac{1}{2}\big((e^{2t_f - t_1} - e^3)^2 + (e^{t_f + t_1} - e^3)^2\big) \\
\frac{\partial J}{\partial t_1} &= -e^{4t_f - 2t_1} + e^{2t_f - t_1 + 3} + e^{2t_f + 2t_1} - e^{t_f + t_1 + 3} \\
\frac{\partial^2 J}{\partial t_1 \partial t_f} &= -4e^{4t_f - 2t_1} + 2e^{2t_f - t_1 + 3} + 2e^{2t_f + 2t_1} - e^{t_f + t_1 + 3}.
\end{aligned} \tag{6.8}
$$

Similarly, we can also verify the correctness of the expressions of J_{t_1}, J_{t_f}, $J_{t_1 t_1}$, $J_{t_f t_f}$ by direct differentiations of J. □

7 Conclusion

In this paper, we proposed an approach for solving optimal control problems for switched autonomous systems with prespecified sequences of active subsystems. In particular, we derive the derivatives of the cost with respect to the switching instants and use nonlinear optimization techniques to locate the optimal switching instants. It is also shown in the paper that the computational duty can be eased for general quadratic problems for switched linear autonomous systems. Finally reachability problems can also be studied using the optimal control techniques. If a system is reachable given a prespecified sequence of active subsystems, optimal control methods can be used to locate the corresponding switching instants. Further research topics include the search for optimal switching sequences when the active subsystems are not prespecified, and the application of the approach to hybrid systems with state discontinuities at the switching instants.

Appendix A: Some Proofs for Section 3.2

Proof of Lemma 3.1: Although the results in the Lemma hold for all cases in the definition of $\delta x(t)$, we need to discuss each case in order to show the validity of them.

Case 1: $dt_1 \geq 0$, $dt_2 \geq 0$ (see figure 1(a))

$$\begin{aligned}
\delta x(t_1+dt_1) &= \hat{x}(t_1+dt_1) - x(t_1+dt_1) \\
&= \int_{t_1}^{t_1+dt_1} f_1\big(\hat{x}(t),t\big)\ dt - \int_{t_1}^{t_1+dt_1} f_2\big(x(t),t\big)\ dt \\
&= f_1\big(\hat{x}(t_1),t_1\big)dt_1 - f_2\big(x(t_1),t_1\big)dt_1 + o(dt_1) \\
&= (f^{1-} - f^{1+})dt_1 + o(dt_1). \qquad \text{(A.1)}
\end{aligned}$$

We then conclude from the property of the variational equation that

$$\begin{aligned}
\delta x(t_2) &= A(t_2,t_1+dt_1)\delta x(t_1+dt_1) + o\big(\|\delta x(t_1+dt_1)\|_2\big) \\
&= \big(A(t_2,t_1) + A_{t_1}dt_1 + o(dt_1)\big)\big((f^{1-} - f^{1+})dt_1 + o(dt_1)\big) + o(dt_1) \\
&= A(t_2,t_1)(f^{1-} - f^{1+})dt_1 + o(dt_1), \qquad \text{(A.2)} \\
\delta x(t_2+dt_2) &= \hat{x}(t_2+dt_2) - z_1(t_2+dt_2) \\
&= \Big(\hat{x}(t_2) + \int_{t_2}^{t_2+dt_2} f_2\big(\hat{x}(t),t\big)\ dt\Big) - \Big(z_1(t_2) + \int_{t_2}^{t_2+dt_2} f_2\big(z_1(t),t\big)\ dt\Big) \\
&= \delta x(t_2) + \int_{t_2}^{t_2+dt_2} \Big(f_2\big(\hat{x}(t),t\big) - f_2\big(z_1(t),t\big)\Big)\ dt \\
&= \delta x(t_2) + \Big(f_2\big(\hat{x}(t_2),t_2\big) - f_2\big(x(t_2),t_2\big)\Big)dt_2 + o(dt_2) \\
&= \delta x(t_2) + f_x^{2-}\delta x(t_2)dt_2 + o(dt_2) \\
&= A(t_2,t_1)(f^{1-} - f^{1+})dt_1 + f_x^{2-}A(t_2,t_1)(f^{1-} - f^{1+})dt_1dt_2 \\
&\quad +(\text{other terms in } dt_1^2, dt_2^2 \text{ and higher order terms}). \qquad \text{(A.3)}
\end{aligned}$$

Case 2: $dt_1 \geq 0$, $dt_2 < 0$ (see figure 1(b))

The arguments for proving (A.1) in Case 1 can be applied in this case to show its validity. In this case,

$$\begin{aligned}
\delta x(t_2+dt_2) &= z_2(t_2+dt_2) - x(t_2+dt_2) \\
&= \Big(z_2(t_2) + \int_{t_2}^{t_2+dt_2} f_2\big(z_2(t),t\big)\ dt\Big) - \Big(x(t_2) + \int_{t_2}^{t_2+dt_2} f_2\big(x(t),t\big)\ dt\Big) \\
&= \delta x(t_2) + \int_{t_2}^{t_2+dt_2} \Big(f_2\big(z_2(t),t\big) - f_2\big(x(t),t\big)\Big)\ dt \\
&= \delta x(t_2) + \Big(f_2\big(z_2(t_2),t_2\big) - f_2\big(x(t_2),t_2\big)\Big)dt_2 + o(dt_2) \\
&= \delta x(t_2) + f_x^{2-}\delta x(t_2)dt_2 + o(dt_2) \\
&= A(t_2,t_1)(f^{1-} - f^{1+})dt_1 + f_x^{2-}A(t_2,t_1)(f^{1-} - f^{1+})dt_1dt_2 \\
&\quad +(\text{other terms in } dt_1^2, dt_2^2 \text{ and higher order terms}). \qquad \text{(A.4)}
\end{aligned}$$

Case 3: $dt_1 < 0$, $dt_2 \geq 0$ (see figure 1(c))

In this case, we have

$$\begin{aligned}
\delta x(t_1) &= \hat{x}(t) - x(t) \\
&= \int_{t_1+dt_1}^{t_1} f_2\big(\hat{x}(t),t\big)\ dt - \int_{t_1+dt_1}^{t_1} f_1\big(x(t),t\big)\ dt \\
&= f_2\big(x(t_1+dt_1),t_1+dt_1\big)(-dt_1) - f_1\big(x(t_1+dt_1),t_1+dt_1\big)(-dt_1) + o(dt_1) \\
&= f_1\big(x(t_1),t_1\big)dt_1 - f_2\big(x(t_1),t_1\big)dt_1 + o(dt_1) \\
&= (f^{1-} - f^{1+})dt_1 + o(dt_1). \qquad \text{(A.5)}
\end{aligned}$$

In the derivations of the third to the fourth equations in (A.5), we use the relationship

$$x(t_1+dt_1) = x(t_1) + f^{1-}dt_1 + o\big(\|\delta x(t_1)\|_2\big), \tag{A.6}$$

and the Taylor expressions of f_2 and f_1. Therefore, we have

$$\begin{aligned}
\delta x(t_2) &= A(t_2,t_1)\delta x(t_1) + o(dt_1)\\
&= A(t_2,t_1)(f^{1-}-f^{1+})dt_1 + o(dt_1) && \text{(A.7)}\\
\delta x(t_2+dt_2) &= \hat{x}(t_2+dt_2) - z_3(t_2+dt_2)\\
&= \Big(\hat{x}(t_2) + \int_{t_2}^{t_2+dt_2} f_2\big(\hat{x}(t),t\big)\ dt\Big) - \Big(z_3(t_2) + \int_{t_2}^{t_2+dt_2} f_2\big(z_3(t),t\big)\ dt\Big)\\
&= \delta x(t_2) + \int_{t_2}^{t_2+dt_2}\Big(f_2\big(\hat{x}(t),t\big) - f_2\big(z_3(t),t\big)\Big)\ dt\\
&= \delta x(t_2) + \Big(f_2\big(\hat{x}(t_2),t_2\big) - f_2\big(x(t_2),t_2\big)\Big)dt_2 + o(dt_2)\\
&= \delta x(t_2) + f_x^{2-}\delta x(t_2)dt_2 + o(dt_2)\\
&= A(t_2,t_1)(f^{1-}-f^{1+})dt_1 + f_x^{2-}A(t_2,t_1)(f^{1-}-f^{1+})dt_1dt_2\\
&\quad +(\text{other terms in } dt_1^2, dt_2^2 \text{ and higher order terms}). && \text{(A.8)}
\end{aligned}$$

Case 4: $dt_1 < 0$, $dt_2 < 0$ (see figure 1(a))

The arguments for proving (A.7) in Case 3 can be applied in this case to show its validity. In this case, we have

$$\begin{aligned}
\delta x(t_2+dt_2) &= z_4(t_2+dt_2) - x(t_2+dt_2)\\
&= \Big(z_4(t_2) + \int_{t_2}^{t_2+dt_2} f_2\big(z_4(t),t\big)\ dt\Big) - \Big(x(t_2) + \int_{t_2}^{t_2+dt_2} f_2\big(x(t),t\big)\ dt\Big)\\
&= \delta x(t_2) + \int_{t_2}^{t_2+dt_2}\Big(f_2\big(z_4(t),t\big) - f_2\big(x(t),t\big)\Big)\ dt\\
&= \delta x(t_2) + \Big(f_2\big(z_4(t_2),t_2\big) - f_2\big(x(t_2),t_2\big)\Big)dt_2 + o(dt_2)\\
&= \delta x(t_2) + f_x^{2-}\delta x(t_2)dt_2 + o(dt_2)\\
&= A(t_2,t_1)(f^{1-}-f^{1+})dt_1 + f_x^{2-}A(t_2,t_1)(f^{1-}-f^{1+})dt_1dt_2\\
&\quad +(\text{other terms in } dt_1^2, dt_2^2 \text{ and higher order terms}). && \text{(A.9)}
\end{aligned}$$

□

Proof of Lemma 3.3: We first note that

$$\int_{t_0}^{t_2+dt_2} L(\hat{x},t)dt = \int_{t_0}^{\max\{t_1,t_1+dt_1\}} L(\hat{x},t)dt + \int_{\max\{t_1,t_1+dt_1\}}^{t_2+dt_2} L(x+\delta x,t)dt. \tag{A.10}$$

In the light of the forward decoupling principle, the first term in (A.10) will not depend on dt_2; therefore, it will not contribute to the coefficient of dt_1dt_2.

For the second term, we discuss as follows.

Case 1: $dt_2 \geq 0$ (see figure 1(a) and (c))

In this case, we have

$$\int_{\max\{t_1,t_1+dt_1\}}^{t_2+dt_2} L(\hat{x},t)\ dt = \int_{\max\{t_1,t_1+dt_1\}}^{t_2} L(x+\delta x,t)\ dt + \int_{t_2}^{t_2+dt_2} L(\hat{x},t)\ dt. \tag{A.11}$$

The first term in (A.11) will not be contributing due to the reason that

$$\delta x(t) = A(t,t_1)(f^{1-}-f^{1+})dt_1 + o(dt_1), \tag{A.12}$$

for $t \in [\max\{t_1,t_1+dt_1\},t_2]$ and therefore it does not depend on dt_2.

The second term is shown to be

$$\begin{aligned}
\int_{t_2}^{t_2+dt_2} L\big(\hat{x}(t),t\big)\ dt &= L\big(\hat{x}(t_2),t_2\big)dt_2 + o(dt_2)\\
&= L\big(x(t_2),t_2\big)dt_2 + L_x^2\delta x(t_2)dt_2\\
&\quad +(\text{other terms in } dt_2^2 \text{ and terms higher than order 2}). && \text{(A.13)}
\end{aligned}$$

By substituting the expression of $\delta x(t_2)$ into (A.13), we obtain the coefficient of $dt_1 dt_2$ contributed by this term as

$$L_x^2 A(t_2, t_1)(f^{1-} - f^{1+}). \tag{A.14}$$

Case 2: $dt_2 < 0$ (see figure 1(b) and (d))

In this case, since $x(t) + \delta x(t) = \hat{x}(t)$ for $t \in [\max\{t_1, t_1 + dt_1\}, t_2 + dt_2]$, we have

$$\int_{\max\{t_1, t_1+dt_1\}}^{t_2+dt_2} L(\hat{x}, t)\, dt = \int_{\max\{t_1, t_1+dt_1\}}^{t_2+dt_2} L(x + \delta x, t)\, dt = \int_{\max\{t_1, t_1+dt_1\}}^{t_2} L(x + \delta x, t)\, dt + \int_{t_2}^{t_2+dt_2} L(x + \delta x, t)\, dt. \tag{A.15}$$

Similar to Case 1, the first term in (A.15) will not be contributing. The second term is shown to be

$$\begin{aligned} \int_{t_2}^{t_2+dt_2} L(x + \delta x, t)\, dt &= L\big(x(t_2) + \delta x(t_2), t_2\big) dt_2 + o(dt_2) \\ &= L\big(x(t_2), t_2\big) dt_2 + L_x^2 \delta x(t_2) dt_2 \\ &\quad + (\text{other terms in } dt_2^2 \text{ and terms higher than order 2}). \end{aligned} \tag{A.16}$$

Therefore, by substituting the expression of $\delta x(t_2)$ into (A.16), we obtain the same coefficient (A.14). □

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