

Solutions to the Exercises of Chapter 9

9A. Vectors and Forces

- $F_1 = 5 \sin 15^\circ \approx 1.29$ and $F_2 = 5 \cos 15^\circ \approx 4.83$.
- By the Pythagorean theorem, the length of the vector from 0 to $(2, 1)$ is $\sqrt{2^2 + 1^2} = \sqrt{5}$. So the magnitude of the force that it represents is $\sqrt{5}$. In the same way, the magnitude of the force represented by the vector from 0 to $(-1, -3)$ is $\sqrt{1^2 + 3^2} = \sqrt{10}$. The horizontal components of these two vectors are, respectively, the vectors from 0 to 2 on the x -axis and from 0 to -1 on the x -axis. So the resultant of the two horizontal components is the vector from 0 to 1 on the x -axis. The vertical components of the two vectors are the vectors from 0 to 1 on the y -axis and from 0 to -3 on the y -axis. So the resultant of the two vertical components is the vector from 0 to -2 on the y -axis. The resultant is obtained by combining the horizontal and vertical resultants. Doing so yields the vector from 0 to $(1, -2)$.
 - The magnitude of the vector from 0 to $(-2, 3)$ remains to be determined. This is $\sqrt{2^2 + 3^2} = \sqrt{13}$ by the Pythagorean theorem. The horizontal component is the vector from 0 to -2 on the x -axis and the vertical component is the vector from 0 to 3 on the y -axis. It follows from the discussion in (a) that the resultant of the three horizontal components is the vector from 0 to -1 on the x -axis, and that the resultant of the three vertical components is the vector, from 0 to 1 on the y -axis. It follows that the resultant of the three vectors is the vector from 0 to $(-1, 1)$.
- Because the system is in equilibrium, $F_1 \cos \theta_1 = F_2 \cos \theta_2$. So $10 \cos 30^\circ = F_2 \cos 60^\circ$. Therefore,

$$F_2 = \frac{10 \cos 30^\circ}{\cos 60^\circ} = \frac{10 \frac{\sqrt{3}}{2}}{\frac{1}{2}} = 10\sqrt{3} \approx 17.32 \text{ pounds.}$$

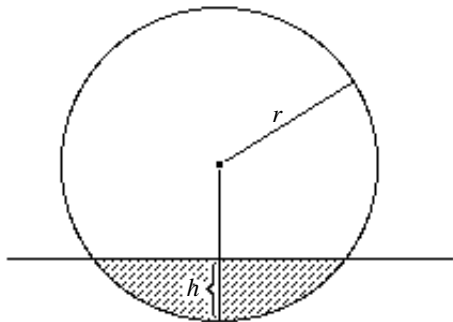
The sum of the magnitudes of the vertical components of F_1 and F_2 must be equal to F . So

$$F = F_1 \sin \theta_1 + F_2 \sin \theta_2 = 10 \sin 30^\circ + 10\sqrt{3} \sin 60^\circ = 10 \cdot \frac{1}{2} + 10\sqrt{3} \cdot \frac{\sqrt{3}}{2} = 5 + 15 = 20 \text{ pounds.}$$

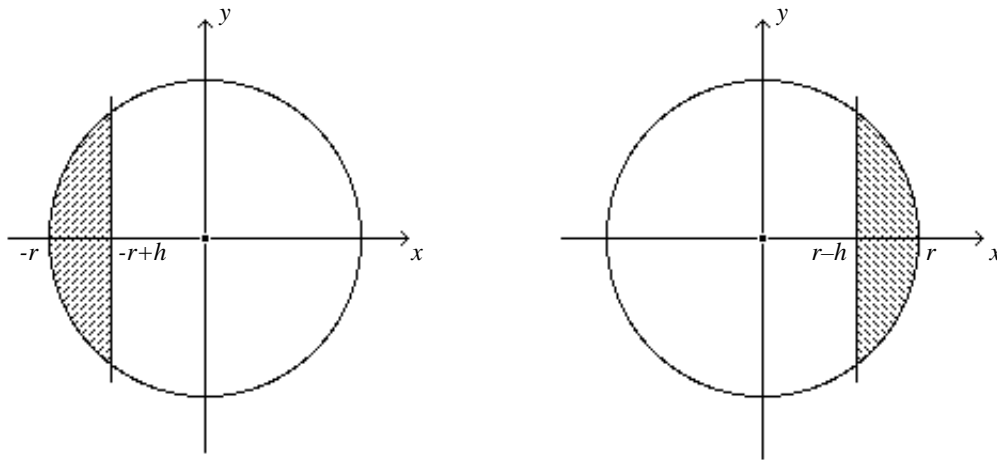
- By Archimedes's Law of Hydrostatics, the maximal buoyant force is the weight of the water that the entire hull displaces. This is $(200)(62.5) = 12,500$ pounds. So this is the largest weight that the boat plus its cargo can have. The maximum weight for the cargo is $12,500 - 8,000 = 4,500$ lbs.
- If the block weighs 5 pounds, it displaces $\frac{5}{62.5} = 0.08$ cubic feet of water. So if h is the amount in feet of the block that is submerged, then $(0.6)(0.6)h = 0.08$. So $h \approx 0.22$ feet. If the block weighs 15 pounds, then in the same way, $(0.6)(0.6)h = \frac{15}{62.5}$. So $h \approx 0.67$ feet. Because this is greater than the height of the block, it means that the block is completely submerged. It follows that the weight of the block is greater than the weight of the water that it displaces. So the block will sink.

Correction: In the introduction to Exercises 6 and 7 replace "contain" by "ask". In addition, line 3 of Exercise 7 should read "Exercises 8K" not "Exercises 8G".

6. As was the case with the floating block of wood of Section 9.1B, the weight of the basketball is equal to the weight of the water that it displaces. To compute the weight of the water that



is displaced, we need to compute the volume of the water that is displaced. Notice that this volume is obtained by rotating one revolution about the x axis the part of the circle of radius r centered at the origin that lies over the interval $[-r, -r + h]$ or, equivalently, over the interval



$[r - h, r]$. See the two figures above. By Chapter 5.6B this is equal to

$$\int_{-r}^{-r+h} \pi f(x)^2 dx \quad \text{and} \quad \int_r^{r-h} \pi f(x)^2 dx$$

respectively, with $f(x) = \sqrt{r^2 - x^2}$, the function whose graph is the top half of the circle. We will evaluate the first of these integrals and invite the student to work with the second. Observe that

$$\int_{-r}^{-r+h} \pi(r^2 - x^2) dx = \int_{-r}^{-r+h} (\pi r^2 - \pi x^2) dx = \pi r^2 x - \frac{\pi x^3}{3} \Big|_{-r}^{-r+h}$$

$$\begin{aligned}
&= \pi r^2(-r+h) - \frac{\pi}{3}(-r+h)^3 - (\pi r^2(-r) - \frac{\pi}{3}(-r)^3) \\
&= -\pi r^3 + \pi r^2 h - \frac{\pi}{3}(-r^3 + 3r^2 h - 3rh^2 + h^3) + \pi r^3 - \frac{\pi}{3}r^3 \\
&= \pi r^2 h - \pi r^2 h + \pi r h^2 - \frac{\pi}{3}h^3 = \pi r h^2 - \frac{\pi}{3}h^3.
\end{aligned}$$

It follows that the weight of the volume of water that the ball displaces is

$$(\pi r h^2 - \frac{\pi}{3}h^3) \frac{62.5 \text{ pounds}}{1 \text{ foot}^3} = 62.5(\pi r h^2 - \frac{\pi}{3}h^3) \text{ pounds.}$$

Since this is the weight of the basketball, we get that $62.5(\pi r h^2 - \frac{\pi}{3}h^3) = 1.3$ or, equivalently, that $\pi r h^2 - \frac{\pi}{3}h^3 = 0.0208$. Inserting $r = 0.39$ and rounding off to three decimal accuracy gives us

$$1.047h^3 - 1.225h^2 + 0.021 = 0.$$

7. To apply Newton's Method, we need the derivative $f'(x) = 3.141x^2 - 2.450x$ of $f(x)$. With $c_1 = 0.39$, we get

$$c_2 = 0.39 - \frac{f(0.39)}{f'(0.39)} = 0.39 - \frac{0.103216}{0.477754} = 0.173956.$$

Next,

$$\begin{aligned}
c_3 &= 0.173956 - \frac{f(0.173956)}{f'(0.173956)} = 0.173956 - \frac{0.010558}{0.331143} = 0.142072, \text{ and} \\
c_4 &= 0.142072 - \frac{f(0.142072)}{f'(0.142072)} = 0.142072 - \frac{0.000724}{0.284678} = 0.139530.
\end{aligned}$$

Plugging $x = 0.139530$ into $f(x)$ gives us $f(0.139530) = 0.000005 \approx 0$. Rounding off to the two significant figures that parallels the data gives $h = 0.14$.

8. With $CB = c = 10$, and $CF = a = 8$, we get

$$EC = x = \frac{a}{4c}(a + \sqrt{a^2 + 8c^2}) = \frac{8}{40}(8 + \sqrt{64 + 800}) \approx 7.479.$$

Notice that $\cos \theta_2 \approx \frac{7.479}{8} \approx 0.935$. By the inverse cosine button on your calculator (the theory of inverse functions will be taken up in Chapter 10.5), we get $\theta_2 = 20.80^\circ$. Observe next that $EF^2 + EC^2 = CF^2$. So

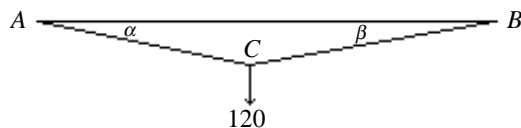
$$F \approx \sqrt{8^2 - 7.479^2} \approx \sqrt{8.064} \approx 2.840.$$

Because $\tan \theta_1 \approx \frac{2.840}{10 - 7.479} \approx 1.127$, we get by the inverse tan button of a calculator, that $\theta_1 \approx 48.41^\circ$. Finally,

$$BF^2 = BE^2 + EF^2 \approx (10 - x)^2 + (2.840)^2 \approx (2.521)^2 + (2.840)^2 \approx 14.42.$$

So $BF \approx 3.798$. The data supplied suggests that all these answers should be rounded off to two significant figures.

9. From a review of Section 9.1B, $T_1 = W = 100$ pounds, and $T_1 \cos \theta_2 = T_2 \cos \theta_1$. After inserting the results of Exercise 8, we get $T_2 \approx \frac{100 \cos 48.41}{\cos 20.80} \approx 71.29$ pounds.
10. By the symmetry of the situation, the tensions in the wires AC and BC are equal. If T is this tension, then its vertical component is $T \sin \alpha$. Hence $2T \sin \alpha = 200$. So $T = \frac{100}{\sin 5^\circ} = 1147$ pounds.
11. Let T_{AC} and T_{BC} be the tensions in the wires AC and BC respectively. From the figure,



$(\sin \alpha)(T_{AC}) + (\sin \beta)(T_{BC}) = 120$ and $T_{AC} \cos \alpha = T_{BC} \cos \beta$. So $T_{BC} = T_{AC} \frac{\cos \alpha}{\cos \beta}$. A look at the graph of the cosine over $[0, \frac{\pi}{2}]$ (refer to Figure 4.25) tells us that if $\alpha > \beta$, then $\cos \alpha < \cos \beta$, so that $T_{AC} > T_{BC}$. Because $T_{AC} \sin \alpha + T_{AC} \frac{\cos \alpha}{\cos \beta} \cdot \sin \beta = 120$, we get, $T_{AC}(\sin \alpha + \cos \alpha \tan \beta) = 120$, and hence $T_{AC} = \frac{120}{\sin \alpha + \cos \alpha \tan \beta}$. In the same way, $T_{BC} = \frac{120}{\sin \beta + \cos \beta \tan \alpha}$.

12. Let T be the tension in the string. Because the angle that the string makes with the horizontal is $\frac{\pi}{2} - \alpha$, the horizontal and vertical components of T are $T \cos(\frac{\pi}{2} - \alpha)$ and $T \sin(\frac{\pi}{2} - \alpha)$ respectively.
- i. The vertical component $T \sin(\frac{\pi}{2} - \alpha)$ has to counterbalance the weight of the sphere. So $T \sin(\frac{\pi}{2} - \alpha) = 50$ and hence

$$T = \frac{50}{\sin(\frac{\pi}{2} - \alpha)} = \frac{50}{\cos \alpha}.$$

- ii. This force F is the horizontal component $T \cos(\frac{\pi}{2} - \alpha)$ of the tension T . It is equal to

$$F = \frac{50 \cos(\frac{\pi}{2} - \alpha)}{\sin(\frac{\pi}{2} - \alpha)} = \frac{50}{\tan(\frac{\pi}{2} - \alpha)} = 50 \tan \alpha.$$

- iii. $T = \frac{50}{\cos 15^\circ} \approx 52$ pounds, and $F = 50 \tan 15^\circ \approx 13$ pounds.

The fact that the diameter of the sphere is 8 inches does not play a role in this problem.

9B. Archimedes and the Crown

13. Let's first solve the problem by assuming that $w_1 = \frac{1}{3}w$. If w pounds of gold loses f_1 pounds, then $\frac{1}{3}w$ pounds of gold will lose $\frac{1}{3}f_1$. Because $w_2 = \frac{2}{3}w$, we see similarly that the $w_2 = \frac{2}{3}w$ pounds of silver will lose $\frac{2}{3}f_2$ pounds. It follows that $f = \frac{1}{3}f_1 + \frac{2}{3}f_2$. We can now check whether the formula is correct. On the one hand, $\frac{w_1}{w_2} = \frac{\frac{1}{3}w}{\frac{2}{3}w} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$, and on the other,

$$\frac{f_2 - f}{f - f_1} = \frac{f_2 - \frac{1}{3}f_1 - \frac{2}{3}f_2}{\frac{1}{3}f_1 + \frac{2}{3}f_2 - f_1} = \frac{\frac{1}{3}(f_2 - f_1)}{\frac{2}{3}(f_2 - f_1)} = \frac{1}{2}.$$

To solve the problem in general, start with $w_1 = cw$. Because $w = w_1 + w_2 = cw + w_2$, we get that $w_2 = (1 - c)w$. Now simply repeat the argument above with c in place of $\frac{1}{3}$ and $1 - c$ in place of $\frac{2}{3}$.

14. By Archimedes's law of hydrostatics $f = 62.5v$, $f_1 = 62.5v_1$, and $f_2 = 62.5v_2$. Plug these values into the formula of Exercise 13 and cancel 62.5.

9C. Suspension Bridges

15. Notice that $w = \frac{9,200+1,500}{4} = \frac{10,700}{4} = 2675$, $d = \frac{1595.5}{2} = 797.75$, and $s = 128$. So $T_0 = \frac{1}{2} \frac{wd^2}{s} \approx 6.65 \times 10^6$ pounds, and $T_d = wd\sqrt{1 + \left(\frac{d}{2s}\right)^2} \approx 6.98 \times 10^6$ pounds. The angle α satisfies $\tan \alpha = \frac{2s}{d} = 0.32$. Using the inverse tan button of a calculator, $\alpha \approx 17.8^\circ$. The horizontal component of the tension must be $T_d \cos \alpha \approx 6.65 \times 10^6$ pounds.

16. Notice that $w = \frac{19,210+7,160}{4} = \frac{26,370}{4} = 6592.5$, $d = \frac{1600}{2} = 800$, and $s = 177$. So $T_d = wd\sqrt{1 + \left(\frac{d}{2s}\right)^2} \approx 1.30 \times 10^7$ pounds. The angle α satisfies $\tan \alpha = \frac{2s}{d} \approx 0.44$. Using the inverse tan button of a calculator, we get $\alpha \approx 23.9^\circ$. Let T_S be the tension in a cable over the side span at the tower. Since $T_d \cos 23.9^\circ = T_S \cos 22.7^\circ$, we find that $T_S \approx 1.29 \times 10^7$ pounds. It follows that the total compression on one tower is approximately

$$4(1.30 \times 10^7) \sin 23.9^\circ + 4(1.29 \times 10^7) \sin 22.7^\circ \approx 2.11 \times 10^7 + 1.99 \times 10^7 \approx 4.10 \times 10^7 \text{ pounds.}$$

17. Observe that $w = \frac{24,000+11,000}{4} = 8750$, $d = 735$, and $s = 160$. So $T_d = wd\sqrt{1 + \left(\frac{d}{2s}\right)^2} \approx 1.61 \times 10^7$ pounds. Since each wire has cross-sectional area $\pi\left(\frac{0.195}{2}\right)^2 \approx 0.0299$ square inches, each wire has an ultimate strength of $220,000 \frac{\text{pounds}}{\text{inch}^2} \times 0.0299 \approx 6570$ pounds. So the cable has an ultimate strength of $(37)(256)(6570) \approx 6.22 \times 10^7$ pounds. Therefore the safety factor is $\frac{6.22}{1.61} \approx 4$.

18. Notice that $w = \frac{21,300+4,000}{2} = \frac{25,300}{2} = 12,650$, $d = 2,100$, and $s = 470$. So $T_0 = \frac{1}{2} \frac{wd^2}{s} \approx 5.93 \times 10^7$ pounds, and $T_d = wd\sqrt{1 + \left(\frac{d}{2s}\right)^2} \approx 6.50 \times 10^7$ pounds. As in Exercise 17, the ultimate strength of each wire is $(220,000)\left(\pi\left(\frac{0.196}{2}\right)^2\right)$ pounds and that of the cable is $(61)(452)\left((220,000)\pi(0.098)^2\right) \approx 1.83 \times 10^8$ pounds. The safety factor is $\frac{18.3}{6.5} \approx 2.8$.

19. Notice that $w = \frac{8,680+4,000}{2} = 6340$, $d = 1400$, and $s = 280$. So $T_d = wd\sqrt{1 + \left(\frac{d}{2s}\right)^2} \approx 2.39 \times 10^7$ pounds. The angle α satisfies $\tan \alpha = \frac{2s}{d} = 0.40$. Using the inverse tan button of a calculator, we get $\alpha \approx 21.80^\circ$. Each cable over the center span generates a compression of $T_d \sin \alpha \approx 8.88 \times 10^6$ pounds.

20. Notice that $w = \frac{37,000+4,800}{4} = 10,450$, $d = 2130$, and $s = 385$. Thus, $T_d = wd\sqrt{1 + \left(\frac{d}{2s}\right)^2} \approx 6.55 \times 10^7$ pounds. The angle α satisfies $\tan \alpha = \frac{2s}{d} = 0.36$, so that $\alpha \approx 19.88^\circ$. The compression produced by all four cables is $4T_d \sin \alpha \approx 8.91 \times 10^7$ pounds.

21. Begin with a review of the units involved by consulting Exercises 7E. In this problem, $d = 995$

meters and $s = 201$ meters. As in the previous exercises, the important formula is

$$T_d = wd\sqrt{\left(\frac{d}{2s}\right)^2 + 1}.$$

The compression that one cable produces in the tower is $T_d \sin \alpha$, so it is $2T_d \sin \alpha$ for both. As an estimate, take the compression generated by the two cables over the side span to be $2T_d \sin \alpha$ as well. Therefore, $4T_d \sin \alpha \approx 980 \times 10^6$ newtons. Now recall that $\tan \alpha = \frac{2s}{d}$. So $\tan \alpha = \frac{402}{995} = 0.404$. Using the inverse tan button, we get $\alpha \approx 22.0^\circ$. It follows that

$$T_d \approx 654 \times 10^6 \text{ newtons.}$$

Because $w = \frac{1}{2}(\text{dead} + \text{live load})$, it remains to estimate w . From $wd\sqrt{\left(\frac{d}{2s}\right)^2 + 1} \approx 654 \times 10^6$, we get $2656w \approx 654 \times 10^6$. So $2w \approx 4.92 \times 10^5$ newtons/meter. Make use of Exercise 7E of Chapter 7 and check that this corresponds to approximately 33,700 pounds per foot.

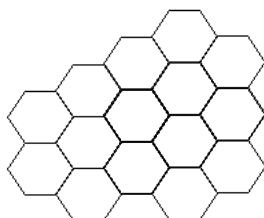
9D. About the Cables

Correction: Delete "The strands are arranged in one of ... have discussed" at the end of the statement of Exercise 22. This point was made earlier in the Exercise.

- 22.** Following the pattern indicated in Figure 9.53, add to the horizontal diagonal of 9 circles: two rows of 8 circles each, one above and one below the diagonal; two rows of 7 circles each above and below those; two rows of 6 circles each above and below; and, finally, two rows of 5 each above and below those. The resulting array of circles is a hexagonal array with a total of

$$9 + 2 \cdot 8 + 2 \cdot 7 + 2 \cdot 6 + 2 \cdot 5 = 61 \text{ circles.}$$

- 23.** Consider the differences between consecutive hexagonal numbers. Notice that $7 - 1 = 6$; $19 - 7 = 12$; $37 - 19 = 18$; $61 - 37 = 24$; and that the numbers 6, 12, 18, and 24 are consecutive multiples of 6. The next multiple of 6 is 30, so $61 + 30 = 91$ is the next hexagonal number. The hexagonal number $91 + 36 = 127$ is next, followed by $127 + 42 = 169$. Since each circle in the hexagonal configuration of Figure 9.53 represents a strand, observe that there are gaps between the strands in a cable constructed as described in Exercise 22. The advantage is that the hexagonal strands of the cable of the Akashi Straits Bridge can be (and



are) packed without gaps as shown. Incidentally, the pattern

$$9 + 2 \cdot 8 + 2 \cdot 7 + 2 \cdot 6 + 2 \cdot 5$$

that provided the 5th hexagonal number 61 (with diagonal 9) suggests a way to develop a formula for the n -th hexagonal number. Give it a try! Start with any odd number $2n - 1$ and use Gauss's addition process of Note 4 of Chapter 4 to get that the n -th hexagonal number is equal to $3n^2 - 3n + 1$.

Correction: Add ", where c is a constant," after $\sqrt{1 + cx^2}$ in the statement of Exercise 24.

24. The comparison as well as the approximation of $\sqrt{1 + x}$ was already discussed in Chapter 6.3. To get the approximation of $\sqrt{1 + cx^2}$ simply replace x by cx^2 in the approximation for $\sqrt{1 + x}$.

25. Refer to Figures 9.15 and 9.16 and recall that the curve representing the cable is the graph of $f(x) = \frac{s}{d^2}x^2$. Note that $f(d) = s$. By Chapter 5.6C, the length of the cable from its lowest point at $(0, 0)$ to the point (d, s) where it meets the tower is given by $\int_0^d \sqrt{1 + f'(x)^2} dx$. Because $f'(x) = \frac{2s}{d^2}x$, this equals

$$\int_0^d \sqrt{1 + \left(\frac{2s}{d^2}\right)^2 x^2} dx = \int_0^d \sqrt{1 + \frac{(2s)^2}{d^4} x^2} dx .$$

Correction: In the statement of Exercise 26: Insert "in Exercises 15-21" after "considered".

26. For the George Washington Bridge, $s = 327$ feet and $d = 1750$ feet. So

$$\left(\frac{2s}{d}\right)^2 \left(\frac{x}{d}\right)^2 \leq \left(\frac{2s}{d}\right)^2 \leq \left(\frac{654}{1750}\right)^2 < 0.14.$$

For the Brooklyn Bridge (see Exercise 15), $s = 128$ feet and $d = 798$ feet. So

$$\left(\frac{2s}{d}\right)^2 \left(\frac{x}{d}\right)^2 \leq \left(\frac{2s}{d}\right)^2 \leq \left(\frac{256}{798}\right)^2 < 0.11.$$

Combining Exercise 24 (with $c = \left(\frac{2s}{d^2}\right)^2$) and Exercise 25 we get

$$\sqrt{1 + \left(\frac{2s}{d^2}\right)^2 x^2} \approx 1 + \frac{1}{2} \left(\frac{2s}{d^2}\right)^2 x^2 - \frac{1}{8} \left(\frac{2s}{d^2}\right)^4 x^4 + \frac{1}{16} \left(\frac{2s}{d^2}\right)^6 x^6 - \frac{5}{128} \left(\frac{2s}{d^2}\right)^8 x^8.$$

Because an antiderivative of this polynomial is

$$F(x) = x + \frac{1}{6} \left(\frac{2s}{d^2}\right)^2 x^3 - \frac{1}{40} \left(\frac{2s}{d^2}\right)^4 x^5 + \frac{1}{112} \left(\frac{2s}{d^2}\right)^6 x^7 - \frac{5}{1152} \left(\frac{2s}{d^2}\right)^8 x^9,$$

we get

$$\begin{aligned} \int_0^d \sqrt{1 + \left(\frac{2s}{d^2}\right)^2} x^2 dx &\approx F(d) - F(0) = F(d) \\ &= d + \frac{1}{6} \left(\frac{2s}{d^2}\right)^2 d^3 - \frac{1}{40} \left(\frac{2s}{d^2}\right)^4 d^5 + \frac{1}{112} \left(\frac{2s}{d^2}\right)^6 d^7 - \frac{5}{1152} \left(\frac{2s}{d^2}\right)^8 d^9 \\ &= d + \frac{2s^2}{3d} - \frac{2s^4}{5d^3} + \frac{4s^6}{7d^5} - \frac{10s^8}{9d^7}. \end{aligned}$$

For the George Washington, $d = 1750$ and $s = 327$. So

$$\frac{2s^2}{3d} \approx 40.73 \quad \text{and} \quad \frac{2s^4}{5d^3} \approx 0.85.$$

The remaining terms of the approximation are too small to be relevant. So the length of the cable from the lowest point to the tower is approximately $1750 + 41 - 1 = 1790$ feet.

For the Brooklyn Bridge, $d = 798$ and $s = 128$. In this case,

$$\begin{aligned} \int_0^d \sqrt{1 + \left(\frac{2s}{d^2}\right)^2} x^2 dx &\approx d + \frac{2s^2}{3d} - \frac{2s^4}{5d^3} \\ &\approx 798 + 13.69 - 0.21 \approx 811 \text{ feet.} \end{aligned}$$

9E. Calculus of Rotation

27. When $t = 10$, the angle is $\theta(10) = \frac{1000}{125} + \frac{10}{5} = 8 + 2 = 10$. The average angular velocity between $t = 0$ and $t = 10$ is

$$\frac{\theta(10) - \theta(0)}{10 - 0} = \frac{10 - 0}{10 - 0} = 1 \text{ radians per second.}$$

The angular velocity at any time t is $w(t) = \theta'(t) = \frac{3}{125}t^2 + \frac{1}{5}$ radians per second. At the instant $t = 10$, the angular velocity is $w(10) = \frac{3}{125} \cdot 100 + \frac{1}{5} = 2.4 + 0.2 = 2.6$ radians per second. The average angular acceleration from $t = 0$ to $t = 10$ is

$$\frac{w(10) - w(0)}{10 - 0} = \frac{2.6 - 0.2}{10} = 0.24 \text{ radians/sec}^2.$$

The angular acceleration at any time t is $\alpha(t) = w'(t) = \frac{6}{125}t$ radians/sec². When $t = 10$, $\alpha = \frac{6}{125} \cdot 10 = 0.48$ radians per second².

Correction: In the statement of Exercise 28 replace "Pick a point on the outer diameter." by "Pick a point on the tire that comes into contact with the road."

28. Consider an instant at which the point on the surface of the tire touches the road and conclude that the point moves with a velocity of 100,000 meters per hour. Use of the formula $v(t) = rw(t)$ tells us that the angular velocity of the point is $w = \frac{1}{r}v = \frac{100,000}{0.325} \approx 307,692$ radians per

hour. Because 2π radians correspond to 1 revolution, the tire turns at a rate of $\frac{307,692}{2\pi} \approx 48971$ revolutions per hour, or approximately, $\frac{48971}{3600} \approx 13.60$ revolutions per second.

- 29.** Let s_A and s_B be the rotational speeds of pulleys A and B in revolutions per minute. Because 1 revolution corresponds to 2π radians, we see that $2\pi s_A = w_A$. Hence the angular velocity w_A of a point moving on the perimeter of pulley A is $w_A = 2\pi s_A = 2\pi(120) = 240\pi$. Similarly, $2\pi s_B = w_B$ where w_B is the rotational speed of pulley B in radians per minute. Consider the point on the fan belt during its motion around pulley A . Note that $v = r_A w_A$ where r_A is the radius of pulley A . Hence the velocity of a point moving on the fan belt is $v = r_A w_A = 4(240\pi) = 960\pi$ inches per minute. In the same way, $v = r_B w_B$ where r_B is the radius of pulley B . So $r_A w_A = r_B w_B$, and hence

$$\frac{r_A}{r_B} = \frac{w_B}{w_A} = \frac{2\pi s_B}{2\pi s_A} = \frac{s_B}{s_A}.$$

Because $r_A = 4$, $r_B = 6$, and $s_A = 120$, we get $s_B = \frac{4}{6} \cdot 120 = 80$ rpm.

- 30.** We will use the formula $\frac{s_B}{s_A} = \frac{r_A}{r_B}$ that was established in the solution of Exercise 29. Taking $s_A = 200$, $r_A = 4$, and $r_B = 10$, we get $s_B = \frac{r_A}{r_B} \cdot s_A = \frac{4}{10} \cdot 200 = 80$ rpm. It follows that the rotational speed s_C of pulley C is 80 rpm also. Because $\frac{s_C}{s_D} = \frac{r_D}{r_C}$, we get $s_D = \frac{r_C}{r_D} \cdot s_C = \frac{5}{7} \cdot 80 \approx 57.14$ rpm.

9F. The Rotational Effect of Forces

Exercise 31 and Exercises 34 to 36 all make use of Leibniz's strategy of summing up many very small quantities, in other words, of definite integrals. A review of Sections 5.3 to 5.5 might therefore be in order. Note also that the beams under discussion in Exercises 31 to 33 are assumed to be horizontal (as illustrated). See the Additional Exercises for Chapter 9 for what happens when this is not so.

- 31.** By Leibniz's strategy the sum of all the little torques $6.25x \, dx$ as x varies from 0 to 16 is given by the definite integral $\int_0^{16} 6.25x \, dx$. Because $6.25 \frac{x^2}{2}$ is an antiderivative of $6.25x$, we get

$$\int_0^{16} 6.25x \, dx = 6.25 \left. \frac{x^2}{2} \right|_0^{16} = 800$$

foot-pounds as required.

Note: Suppose more generally that a beam of length L weighs p pounds. It then weighs $\frac{p}{L}$ pounds per foot. It follows, as in the discussion above, that the torque it produces around one of its endpoints is

$$\int_0^L \frac{p}{L} x \, dx = \frac{p}{2L} x^2 \Big|_0^L = p \cdot \frac{L}{2}.$$

- 32.** By the last assertion of the solution of Exercise 31, the torque produced by the right side of the beam around A is $100 \cdot \frac{16}{2} = 800$ foot-pounds and that produced by the left side of the

beam is $50 \cdot \frac{8}{2} = 200$ foot-pounds. The torque on the left side produced by the weight of the person is $180x$ where x is the person's distance from A . It follows that the system will still be in balance when

$$200 + 180x = 800,$$

or when $x = \frac{600}{180} = 3\frac{1}{3}$ feet. This is the maximum distance that the person can walk beyond A without tilting the beam. As soon as $x > 3\frac{1}{3}$ feet the beam will begin to tip.

- 33.** Let W be the weight that has to be placed at A . The torque around B produced by the right side is maximum when the man stands at the right end of the beam. This is the torque that the left side must balance. Using the last assertion of the solution of Exercise 31, we get

$$16W + 320 \cdot \frac{16}{2} = (160)(20) + 400 \cdot \frac{20}{2}.$$

Therefore, $16W = 3200 + 4000 - 2560 = 4640$ foot-pounds. So $W = 290$ pounds. Any weight greater than 290 pounds will also do the job.

- 34.** The mass of the typical little piece is $\frac{m}{r}$ times its length dx , so it is $\frac{m}{r}dx$. Its distance from the axis of rotation is x . So it follows from the discussion of Section 9.3B that the moment of inertia of the typical piece is $(\frac{m}{r}dx)x^2$. By adding the moments of inertia of all the little pieces as x ranges from 0 to r , we get that the total moment of inertia is

$$\int_0^r \frac{m}{r}x^2 dx = \frac{m}{3r}x^3 \Big|_0^r = \frac{1}{3}mr^2.$$

The flaw is simply that the distance of the center of mass of an object from the axis of rotation has no direct bearing on the question of moment of inertia. This point is best illustrated by considering a rotating bicycle wheel. Its center of mass is on the axis of rotation. So its distance from the axis is zero. However, the wheel does resist rotation and hence has a non-zero moment of inertia.

- 35.** Because the circular piece has radius x , its length is $2\pi x$. Because it has thickness dx , its area and mass are approximately $2\pi x \cdot dx$ and $\frac{m}{\pi r^2}(2\pi x \cdot dx)$ as asserted. By the discussion in Section 9.3B about the moment of inertia of the "circle with mass", we get that the moment of inertia of our typical circular piece is $\text{mass} \times (\text{radius})^2 =$

$$\left(\frac{m}{\pi r^2} \cdot 2\pi x dx\right) \cdot x^2 = \frac{2m}{r^2}x^3 dx.$$

Adding up the moments of inertia of all the circular pieces as x varies from 0 to r , we get

$$\int_0^r \frac{2m}{r^2}x^3 dx = \frac{2m}{4r^2}x^4 \Big|_0^r = \frac{m}{2r^2} \cdot r^4 = \frac{1}{2}mr^2.$$

- 36.** Summing up the moments of inertia of all the discs as their x -coordinates range from $-r$ to r , we get that the total moment of inertia (i.e. that of the solid sphere) is

$$\int_{-r}^r \frac{3m}{8r^3}(r^2 - x^2)^2 dx.$$

This is the moment of inertia of the solid sphere of radius r and mass m . To evaluate this definite integral, we need an anti-derivative of

$$\frac{3m}{8r^3}(r^2 - x^2)^2 = \frac{3m}{8r^3}(r^4 - 2r^2x^2 + x^4).$$

Since this is $F(x) = \frac{3m}{8r^3} \left(r^4x - \frac{2}{3}r^2x^3 + \frac{x^5}{5} \right)$, we get

$$\begin{aligned} \int_{-r}^r \frac{3m}{8r^3}(r^2 - x^2)^2 dx &= F(r) - F(-r) \\ &= \frac{3m}{8r^3} \left(r^5 - \frac{2}{3}r^5 + \frac{r^5}{5} \right) - \frac{3m}{8r^3} \left(-r^5 + \frac{2}{3}r^5 - \frac{r^5}{5} \right) \\ &= \frac{3m}{8r^3} \left(2r^5 - \frac{4}{3}r^5 + \frac{2r^5}{5} \right) \\ &= \frac{3m}{8r^3} \left(\frac{30 - 20 + 6}{15} \right) r^5 = \frac{2}{5}mr^2. \end{aligned}$$

9G. Galileo's Experiment

37. The velocity of the ball at the bottom of the inclined plane is

$$v = \sqrt{\frac{10}{7}gh}$$

where h is the starting height. Because $v = r\omega$ where r is the radius and ω is the rotational velocity in radians per second, we get $\omega = \frac{1}{r}\sqrt{\frac{10}{7}gh}$. As 2π radians correspond to 1 revolution, the rotational velocity in revolutions per second is

$$\frac{1}{2\pi r} \sqrt{\frac{10}{7}gh}.$$

In M.K.S., $g = 9.81$, and $r = 0.01$, so

$$\frac{1}{2\pi r} \sqrt{\frac{10}{7}gh} = \frac{1}{0.02\pi} \sqrt{14.01} \sqrt{h} \approx 59.58 \sqrt{h}.$$

Taking $h = 0.282, 0.564, 0.752$, and 0.940 meters respectively, we get 31.64, 44.74, 51.67, and 57.76 revolutions per second.

38. We know that with a starting height of h meters, the ball is predicted to land at a distance of

$$R = 1.491\sqrt{h}$$

meters from the foot of the table. For $h = 1.10$, we get $R = (1.491)(1.049) = 1.564$ meters. Recall that in actuality the ball landed 0.018, 0.045, 0.04, and 0.036 meters short of the predicted distance. This is an average of about 0.035 meters. So the expectation is that the ball would have landed from about 1.52 meters to about 1.55 meters from the foot of the table.

Correction: It should be Stilman Drake not Stilmann Drake in Exerscise 39.

39. In terms of the height y_0 of the table, the ball is predicted to land at a distance of

$$\sqrt{\frac{20}{7}y_0h}$$

from the foot of the table given that it started its motion at a height of h above the table top. Taking $y_0 = 0.75$ meters and $h = 0.282, 0.564, 0.752,$ and 0.940 meters, respectively, we get the distances of $0.777, 1.099, 1.269,$ and 1.419 meters.

40. The expression $R = \sqrt{\frac{20}{7}y_0h}$ for the distance has no β in it. Therefore R does not depend on β . Neither does it depend on the gravitational acceleration g . This is why Galileo's experiment would have produced essentially the same distance measurements on the Moon, even though - see Exercises 7F - the gravitational acceleration is much less on the Moon than on Earth.

9H. Basic Optics

41. Refer to Figure 9.31. The medium containing A is air and that containing B is crown glass. The respective indices of refraction are $n_A = 1.00029$ and $n_B = 1.52$. So

$$\sin \beta = \frac{n_A}{n_B} \sin \alpha = \frac{1.00029}{1.52} \sin 30^\circ \approx (0.658)(0.5) \approx 0.329.$$

Using the inverse sine button of your calculator (in degree mode) we get $\beta \approx 19.2^\circ$.

42. Let α' be the angle of refraction of the incoming ray. So by Snell's law

$$\frac{\sin \alpha}{v_A} = \frac{\sin \alpha'}{v_B},$$

where v_A is the speed of light in air and v_B its speed in the plate glass. Notice that α' is also the angle of reflection at the silver coating. So α' is also the angle of incidence of the ray as it travels through the plate glass toward the interface between the plate glass and the air. Using Snell's law a second time, we get that

$$\frac{\sin \alpha'}{v_B} = \frac{\sin \beta}{v_A}.$$

Putting the two equations together, we see that $\sin \beta = \sin \alpha$. Because both α and β are between 0 and $\frac{\pi}{2}$, we get that $\beta = \alpha$ as required. (A look at the graph of the sine, see Figure 4.24, confirms this conclusion.)

Corrections: Exercises 43 and 44 require additional assumptions. To obtain the conclusions $\sin \alpha = n_2 \sin \beta_2$ and $\sin \alpha = n_k \sin \beta_k$, it is necessary to assume that $n_A = 1$, in other words that the light ray approaches the first slab through a vacuum. For the conclusions $\alpha_2 = \alpha$ and $\alpha_4 = \alpha$, it is required that the indices of refraction of the region above and below the array of plates are equal.

On the other hand - as the solutions of the exercises will show - it is not necessary to make any assumptions about the media between the plates. Finally, in this context, the word "translucent" might be better than "transparent."

- 43.** Suppose that the index of refraction of the medium above the two slabs is n_A and that of the medium below the slabs is n_B . Let n' be the index of refraction of the medium between the two slabs, and let α' be the angle that the path of the ray between the slabs makes with a perpendicular to the slabs. A look at Figure 9.62 and the repeated use of Snell's law shows that

$$n_A \sin \alpha = n_1 \sin \beta_1 = n' \sin \alpha' = n_2 \sin \beta_2 = n_B \sin \alpha_2 .$$

Assuming that the medium above the plates is a vacuum, so that $n_A = 1$, we get $\sin \alpha = n_2 \sin \beta_2$. Assuming that $n_A = n_B$, we get $\sin \alpha_2 = \sin \alpha$, and hence $\alpha_2 = \alpha$.

- 44.** Let n_A be the index of refraction of the medium above the slabs, and let n_B be that of the medium below them. Let n_1, n_2, n_3 , and n_4 be the indices of refraction of the first, second, third, and fourth slabs. Let $\alpha_1, \alpha_2, \alpha_3$, and α_4 be the respective angles of refraction of the ray in the first, second, third, and fourth slabs. Let n'_1, n'_2 , and n'_3 be the respective indices of refraction of the three media between the slabs, and let α'_1, α'_2 , and α'_3 , be the respective angles that the path of the ray between the slabs makes with a perpendicular to the slabs. By repeatedly applying Snell's law and a look at Figure 9.63, we get

$$\begin{aligned} n_A \sin \alpha &= n_1 \sin \beta_1 = n'_1 \sin \alpha_1 \\ &= n_2 \sin \beta_2 = n'_2 \sin \alpha_2 \\ &= n_3 \sin \beta_3 = n'_3 \sin \alpha_3 \\ &= n_4 \sin \beta_4 = n_B \sin \alpha_4 . \end{aligned}$$

Assuming that the medium above the slabs is a vacuum, so that $n_A = 1$, we get $\sin \alpha = n_4 \sin \beta_4$. Assuming that $n_A = n_B$, we get $\sin \alpha_4 = \sin \alpha$, and hence that $\alpha_4 = \alpha$.

- 45.** By the Lens Maker's equation,

$$\begin{aligned} \frac{1}{f} &= (n - 1) \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = (1.66 - 1) \left(\frac{1}{0.55} + \frac{1}{0.55} \right) \\ &\approx (0.66)(1.82 + 1.82) \approx 2.4 \text{ diopters} . \end{aligned}$$

So $f \approx 0.42$ meters.

- 46.** By the Lens Maker's equation,

$$\begin{aligned} \frac{1}{f} &= (n - 1) \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = (1.52 - 1) \left(\frac{1}{0.30} + \frac{1}{0.45} \right) \\ &\approx (0.52)(3.33 + 2.22) \approx (0.52)(5.56) \approx 2.9 \text{ diopters} . \end{aligned}$$

So this is a 2.9 diopter lens.

47. By the Lens Maker's equation,

$$\frac{1}{0.38} = (n - 1) \left(\frac{1}{0.20} + \frac{1}{0.20} \right) = (n - 1)(10).$$

So $n - 1 \approx 0.26$, and $n \approx 1.26$.

48. The answer to the first question follows from the formula $\frac{1}{s_i} = -\frac{1}{s_0} + \frac{1}{f}$. By substituting $\frac{1}{f} = 20$ and $s_0 = 0.55$, we get $\frac{1}{s_i} = -\frac{1}{0.55} + 20$, so that $\frac{1}{s_i} \approx -1.82 + 20 = 18.18$. Hence the image is $s_i = 0.055$ meters, or 5.5 centimeters, from the lens. The fact that s_i turned out to be positive means that the image is real. A look at the way the objective lens in Figure 9.43 functions, tells us that if y' is the size of the image, then

$$\frac{0.16}{0.55} = \tan \varphi = \frac{y'}{0.055}.$$

So $y' = 0.016$, and hence the image is 1.6 centimeters high.

49. The magnification m is approximately $m \approx 1 + \frac{1}{4f}$. Because $\frac{1}{f} = 40$, we get $m \approx 1 + \frac{1}{4} \cdot 40 = 11$. Thus, the image of the ant is $11 \cdot 7 = 77$ millimeters or 7.7 centimeters long.

50. The magnification is $\frac{F}{f}$ where F and f are the focal lengths of the objective lens and the eyepiece respectively. Because $F = 4$ meters and $\frac{1}{f} = 20$ diopters, we get a magnification of 80.

51. The length of the telescope must exceed the focal length F of the objective. See Figure 9.43. Because $200 = \frac{F}{f} = F \left(\frac{1}{f} \right) = 40F$, we get $F = 5$ meters. So the length of the telescope will exceed 5 meters.

9I. The Speed of Light

Correction: Lines 5, 6. Replace "Mirror 1 is free to rotate about an axis perpendicular to its plane." by "Mirror 1 is free to rotate about an axis perpendicular to the plane of the page." Line 7 should read "the reflection 'of' the focal point, not "reflection 'on' the focal point."

9J. The Phenomenon of Refraction in Astronomy

Correction: On page 296, left column, line 2. Insert after sine: (see Exercise 2C of Chapter 2). Same column, line 2 from the bottom: Replace 1.00028 by 1.000283, and in the right column, line 3, delete $58.372941 \tan z_{\text{app}} \approx$.

52. For $z_{\text{app}} = 20^\circ$, $\rho \approx (58.37) \tan 20^\circ \approx 21''$ by the formula, and $\rho = 21''$ from the table. For $z_{\text{app}} = 60^\circ$,

$$\rho \approx (58.37) \tan 60^\circ \approx 101''$$

by the formula. This again agrees with the value from the table. For $z_{\text{app}} = 80^\circ$,

$$\rho \approx (58.37) \tan 80^\circ \approx 331''.$$

This is 12 seconds more than the value from the table. Because $z_{\text{true}} = z_{\text{app}} + \rho$, we get the true zenith distances of $20^\circ + \left(\frac{21}{3600}\right)^\circ = 20.006^\circ$, $60^\circ + \left(\frac{101}{3600}\right)^\circ = 60.03^\circ$, and $80^\circ + \left(\frac{331}{3600}\right)^\circ = 80.09^\circ$. Observe that the effects of refraction for these apparent zenith distances are small.

- 53.** The apparent zenith position of the top of the Sun's disc is $90^\circ - 29'$ or $89^\circ 31'$. It follows from the table that the true zenith position of the top of the Sun's disc is $89^\circ 31' + \left(\frac{1760}{60}\right)' = 89^\circ 31' + 29' 20'' = 90^\circ 0' 20'' > 90^\circ$. The true zenith position of the bottom of the Sun's disc is $90^\circ + \left(\frac{2123}{60}\right)' = 90^\circ 35' 23''$. So the true vertical diameter of the Sun is $90^\circ 35' 23'' - 90^\circ 0' 20''$ or about $35'$.

Note: It turns out that the perceived angular diameter of the Sun varies. There are tables for its angular size for every day of the year in the *Astronomical Almanac*. The apparent 'semi-diameter' varies from $15' 45''$ to $16' 17''$.