Solutions to the Exercises of Chapter 8

8A. Domains of Functions

1. For $\sqrt{7-x}$ to make sense, we need $7-x \geq 0$ or $7 \geq x$. So the domain of $f(x)$ is $\{x \mid x \leq 7\}$. For $\sqrt{x+5}$ to make sense, $x+5 \geq 0$. So the domain of $g(x)$ is $\{x \mid x \geq -5\}$. For $h(x)$ to make sense, both $f(x)$ and $g(x)$ must make sense. So the domain of $h(x)$ is $\{x \mid -5 \leq x \leq 7\}$.

2. For $\sqrt{3x-4}$ to be defined, we must have $3x-4 \geq 0$. So $x \geq \frac{4}{3}$ and hence the domain of $f(x)$ is $\{x \mid x \geq \frac{4}{3}\}$. For $\sqrt{2x-3}$ to make sense, $2x-3$ needs to be greater than or equal to 0. So $2x-3 \geq 0$ and hence the domain of $g(x)$ is $\{x \mid x \geq \frac{3}{2}\}$. Note that for $x \geq \frac{3}{2}$, both $f(x)$ and $g(x)$ are defined. For $k(x)$ to make sense, we also need $g(x) \neq 0$. So the domain of $k(x)$ is $\{x \mid x > \frac{3}{2}\}$.

3. In the case of $f(x)$, we need both $x+6 \geq 0$ and $3 \geq \sqrt{x+6}$. So $x \geq -6$ and $9 \geq x+6$. Hence $x \geq -6$ and $3 \geq x$. So the domain of $f(x)$ is $\{x \mid -6 \leq x \leq 3\}$. For $g(x)$ to make sense, we have to have $\frac{x-5}{x+3} \geq 0$. So either $x+3 > 0$ and $x-5 \geq 0$; or $x+3 < 0$ and $x-5 \leq 0$. So $x \geq 5$ or $x < -3$. Hence the domain of $g(x)$ is $\{x \mid x < -3 \text{ or } x \geq 5\}$.

8B. Evaluations of Limits

4. $\lim\limits_{x\to 2} (x^2+1)(x^2+4x) = (5)(12) = 60.$

5. $\lim\limits_{x\to 1} \frac{x-2}{x^2+4x-3} = \frac{-1}{1+4-3} = -\frac{1}{2}.$

6. $\lim\limits_{x\to 4} \sqrt{x+\sqrt{x}} = \sqrt{4+2} = \sqrt{6}.$

7. $\lim\limits_{x\to 3} \frac{x^2-x+12}{x+3} = \frac{9-3+12}{3+3} = \frac{18}{6} = 3.$

8. Notice that in the limit $\lim\limits_{x\to -3} \frac{x^2-x+12}{x+3}$, the numerator goes to 24 while the denominator goes to 0. So the ratio becomes larger and larger. Because it does not close in on a finite number, there is no limit. Check that $\lim\limits_{x\to -3^-} \frac{x^2-x+12}{x+3} = -\infty$ and $\lim\limits_{x\to -3^+} \frac{x^2-x+12}{x+3} = +\infty$.

9. This is a limit of "$\frac{0}{0}$" type. So we are looking for a cancellation. Because $x^2-x-12 = (x+3)(x-4)$, we see that $\lim\limits_{x\to -3} \frac{x^2-x-12}{x+3} = \lim\limits_{x\to -3} \frac{(x+3)(x-4)}{x+3} = \lim\limits_{x\to -3}(x-4) = -7$. Check that L'Hospital's rule gives the same answer.

10. This is another limit of "$\frac{0}{0}$" type. It is solved with a cancellation: $\lim\limits_{t\to -1} \frac{t^3-t}{t^2-1} = \lim\limits_{t\to -1} \frac{t(t^2-1)}{(t^2-1)} = \lim\limits_{t\to -1} t = -1$. Check that L'Hospital's rule gives the same answer.

11. This is a limit of "$\frac{0}{0}$" type. By L'Hospital's rule, $\lim\limits_{x\to 1} \frac{x^3-1}{x^2-1} = \lim\limits_{x\to 1} \frac{3x^2}{2x} = \frac{3}{2}$. Check that this limit can also be solved with a cancellation.

12. This limit is also a "$\frac{0}{0}$" limit that can be solved by a cancellation:

$$\lim_{h\to 0}\frac{(h-5)^2-25}{h}=\lim_{h\to 0}\frac{h^2-10h+25-25}{h}=\lim_{h\to 0}\frac{h(h-10)}{h}=\lim_{h\to 0}(h-10)=-10\,.$$

Show that L'Hospital's rule gives the same thing. For which function $f(x)$ is this limit equal to $f'(-5)$?

13. By rationalizing, factoring, and canceling,

$$\frac{x^2-81}{\sqrt{x}-3}=\frac{x^2-81}{\sqrt{x}-3}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3}=\frac{x^2-81}{x-9}(\sqrt{x}+3)=(x+9)(\sqrt{x}+3)\,.$$

So $\lim_{x\to 9}\frac{x^2-81}{\sqrt{x}-3}=\lim_{x\to 9}(x+9)(\sqrt{x}+3)=108$. Check that L'Hospital's rule gives the same result.

14. By rationalizing, $\frac{x}{\sqrt{1+3x}-1}=\frac{x}{\sqrt{1+3x}-1}\cdot\frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}=\frac{x(\sqrt{1+3x}+1)}{(1+3x)-1}=\frac{\sqrt{1+3x}+1}{3}$. So $\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}=\lim_{x\to 0}\frac{\sqrt{1+3x}+1}{3}=\frac{2}{3}$. What do you get with L'Hospital's rule?

15. By rationalizing, $\frac{4-\sqrt{s}}{s-16}=\frac{4-\sqrt{s}}{s-16}\cdot\frac{4+\sqrt{s}}{4+\sqrt{s}}=\frac{16-s}{(s-16)(4+\sqrt{s})}=\frac{-1}{4+\sqrt{s}}$. So $\lim_{s\to 16}\frac{4-\sqrt{s}}{s-16}=\lim_{s\to 16}\frac{-1}{4+\sqrt{s}}=-\frac{1}{8}$. What answer does L'Hospital's rule provide?

16. The limit $\lim_{x\to 2}\frac{|x-2|}{x-2}$ does not exist. To see this, check that $\lim_{x\to 2^-}\frac{|x-2|}{x-2}=\lim_{x\to 2^-}\frac{-(x-2)}{x-2}=-1$. This is so, because for $x<2$, we get $x-2<0$, so that $|x-2|=-(x-2)$. Show in a similar way that $\lim_{x\to 2^+}\frac{|x-2|}{x-2}=1$. Since the limit from the left is not equal to the limit from the right, $\lim_{x\to 2}\frac{|x-2|}{x-2}$ does not exist.

17. Because $\lim_{x\to a}f(x)=5$ and $\lim_{x\to a}g(x)=3$, we see that $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{5}{3}$. Similarly, $\lim_{x\to a}\frac{2f(x)}{g(x)-f(x)}=\frac{2\cdot 5}{3-5}=\frac{10}{-2}=-5$.

8C. Continuity

18. For $\frac{x^4+17}{6x^2+x-1}$ to make sense we need only for $6x^2+x-1$ to be non-zero. By the quadratic formula, $6x^2+x-1=0$ when $x=\frac{-1\pm\sqrt{1+24}}{12}=\frac{-1\pm 5}{12}=-\frac{1}{2}$ or $\frac{1}{3}$. So the domain of $G(x)=\frac{x^4+17}{6x^2+x-1}$ is $\{x\mid x\neq-\frac{1}{2}\text{ and }x\neq\frac{1}{3}\}$. For $\sqrt{x+1}$ to make sense, we need $x+1\geq 0$, or $x\geq-1$. For $\frac{1}{\sqrt{x+1}}$ to make sense we need $x>-1$. So the domain of $H(x)=\frac{1}{\sqrt{x+1}}$ is $\{x\mid x>-1\}$. That $G(x)$ is continuous on its domain follows from the third Remark in Section 8.2. That $H(x)$ is continuous on its domain follows from a combination of the second and third Remarks of Section 8.2 and the fact (see Section 8.5) that the composite of two continuous functions is continuous.

19. That the functions $cx+1$ and cx^2-1 are continuous for any constant c follows from the third Remark in Section 8.2. For the function $f(x)$ to be continuous, its graph must be in one connected piece. In view of what was already said, this will be so precisely if the graphs of $cx+1$ and cx^2-1 meet when $x=3$. So we need to have $3c+1=9c-1$. So $6c=2$ and $c=\frac{1}{3}$.

20. For (i), we need to check that the Continuity Criterion is satisfied for $c = 5$. Because $f(5) = 1 + \sqrt{5^2 - 9} = 1 + \sqrt{16} = 5$, we know that $f(5)$ makes sense. In view of the fact that $\lim_{x\to c} f(x) = \lim_{x\to 5} (1 + \sqrt{x^2 - 9}) = 1 + \sqrt{16} = 5 = f(5)$, we now know that the Continuity Criterion is satisfied. So $f(x)$ is continuous at $c = 5$. To show that $g(x) = \frac{x+1}{2x^2-1}$ is continuous at $c = 4$, we need to check the Continuity Criterion for $c = 4$. Because $g(4) = \frac{5}{2\cdot 16-1} = \frac{5}{31}$, $g(x)$ is defined at $x = 4$. Since $\lim_{x\to 4} g(x) = \lim_{x\to 4} \frac{x+1}{2x^2-1} = \frac{5}{31} = g(4)$, the criterion is met. So $g(x)$ is continuous at $c = 4$.

21. **i.** We know from the third Remark of Section 8.2 that the function $\frac{x^2-1}{x+1}$ is continuous except when $x = -1$. For $x = -1$, the function $\frac{x^2-1}{x+1}$ is not defined so its graph has a gap. We need to see whether defining $f(-1) = 6$ closes the gap. Is this the case? It will be only if $\lim_{x\to -1} \frac{x^2-1}{x+1} = 6$. However $\lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1} \frac{(x+1)(x-1)}{x+1} = \lim_{x\to -1}(x-1) = -2$. So the function $f(x)$ is not continuous for $x = -1$. Hence it is not continuous on its domain.

ii. As in (i), the function $\frac{x^2-2x-8}{x-4}$ is continuous, except when $x = 4$ where it is not defined. If $\lim_{x\to 4} \frac{x^2-2x-8}{x-4} = 6$, then the definition $f(4) = 6$ will close the gap in the graph. Because $\lim_{x\to 4} \frac{x^2-2x-8}{x-4} = \lim_{x\to 4} \frac{(x-4)(x+2)}{x-4} = \lim_{x\to 4}(x+2) = 6$, the gap is indeed closed. So $f(x)$ is continuous at $x = 4$ and hence for all x. Why are $f(x)$ and $g(x) = x + 2$ exactly the same function?

22. Let $f(x) = 2x^3 + x^2 + 2$. Because $f(-2) = 2(-2)^3 + (-2)^2 + 2 = -16 + 4 + 2 = -10 < 0$ and $f(-1) = 2(-1)^3 + (-1)^2 + 2 = -2 + 1 + 2 = 1 > 0$, it follows from the Intermediate Value Theorem that there exists some x in $(-2, -1)$ such that $f(x) = 2x^3 + x^2 + 2 = 0$.

23. If m and M are the minimum and maximum values of f on $[-1, 1]$, then $m \leq 3 < 4 \leq M$. So by the Intermediate Value Theorem, there is, for any number v with $3 \leq v \leq 4$, a number r between -1 and 1 such that $f(r) = v$. Taking $v = \pi$ gives us the r we need.

8D. Tangent Lines

24. Because $g'(x) = -3x^2$, the slope of the tangent to the graph at the point $(0, 1)$ is $g'(0) = 0$. By the point-slope form of the equation of a line, the equation of the tangent line is $y-1 = 0(x-0)$ or $y = 1$.

25. Because $h'(x) = -1(2x-1)^{-2}(2) = \frac{-2}{(2x-1)^2}$, we see that $h'(-1) = \frac{-2}{(-3)^2} = -\frac{2}{9}$. By the point-slope form, the equation of the tangent line is $y + \frac{1}{3} = -\frac{2}{9}(x+1)$ or $y = -\frac{2}{9}x - \frac{5}{9}$.

26. Because $y' = \frac{1(x-3)-x(1)}{(x-3)^2} = \frac{-3}{(x-3)^2}$, we see that the slope of the tangent line is $\frac{-3}{(6-3)^2} = \frac{-3}{9} = -\frac{1}{3}$. By the point-slope form of the equation of a line, we get that the equation of the tangent is $y - 2 = -\frac{1}{3}(x-6)$ or $y = -\frac{1}{3}x + 4$.

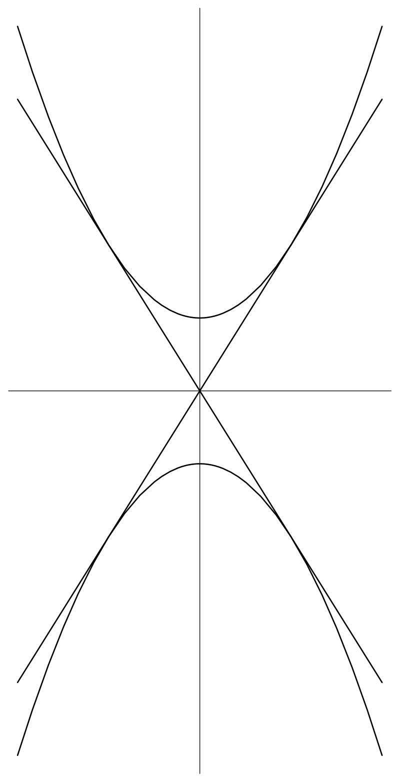
27. Converting the equation $x-2y = 1$ into slope-intercept form, we get $2y = x-1$ or $y = \frac{1}{2}x - \frac{1}{2}$. So $\frac{1}{2}$ is the slope of the line. Next, we need the point on the graph of $f(x) = x^2 - 1$ with the property that the tangent at that point has slope $\frac{1}{2}$. Because $f'(x) = 2x$, this occurs when

$x = \frac{1}{4}$. So the point is $\left(\frac{1}{4}, f(\frac{1}{4})\right) = \left(\frac{1}{4}, -\frac{15}{16}\right)$. The equation we are looking for is that of the line through $\left(\frac{1}{4}, -\frac{15}{16}\right)$ with slope $\frac{1}{2}$. By the point-slope form of the equation of a line we get $y - \left(-\frac{15}{16}\right) = \frac{1}{2}(x - \frac{1}{4})$ or $y + \frac{15}{16} = \frac{1}{2}x - \frac{1}{8}$ or, finally, $y = \frac{1}{2}x - \frac{17}{16}$.

28. For the graph of $f(x) = 2x^3 - 3x^2 - 6x + 87$ to have a horizontal tangent, we need to have $f'(x) = 6x^2 - 6x - 6 = 0$. By the quadratic formula, $6(x^2 - x - 1) = 0$ for $x = \frac{1 \pm \sqrt{5}}{2}$.

29. For $y = 6x^3 + 5x - 3$ to have a tangent line of slope 4, the derivative $y' = 18x^2 + 5$ must be equal to 4. But $4 = 18x^2 + 5$ implies that $x^2 = -\frac{1}{18}$ and this is impossible.

30. Start with the graph of $y = x^2$ and then observe that the relevant diagram is shown below:



Let $y = mx + b$ be one of the two lines and let (x_1, y_1) and (x_2, y_2) be the two points of tangency, respectively, on the graphs of $f(x) = x^2 + 1$ and $g(x) = -x^2 - 1$. Observe that $f'(x_1) = m = g'(x_2)$ and hence that $2x_1 = m = -2x_2$. So $x_2 = -x_1$. Therefore, $y_2 = -x_2^2 - 1 = -(-x_1)^2 - 1 = -x_1^2 - 1 = -y_1$. Because $y_1 = mx_1 + b$ and $y_2 = mx_2 + b$, we get $y_1 = mx_1 + b$ and $-y_1 = -mx_1 + b$, and hence that $2b = 0$ and $b = 0$. Because $m = 2x_1$, and (x_1, y_1) lies on the graphs of both $y = mx$ and $y = x^2 + 1$, we get $x_1^2 + 1 = y_1 = mx_1 = 2x_1^2$. So $x_1^2 = 1$ and hence $x_1 = \pm 1$. When $x_1 = 1$, we get $y_1 = 1^2 + 1 = 2, x_2 = -1$ and $y_2 = -(-1)^2 - 1 = -2$. So the two points are $(1, 2)$ and $(-1, -2)$. These are the points in the diagram. When $x_1 = -1$, we get $y_1 = (-1)^2 + 1 = 2, x_2 = 1$, and $y_2 = -1^2 - 1 = -2$. So the other two points are $(-1, 2)$ and $(1, -2)$.

31. A reformulation of the question is this: For what point on the graph of $y = \frac{1}{10}x^2$ will the tangent line hit the point (10, 5)? Let this point be (x_1, y_1). Because the slope of the tangent line is $\frac{1}{5}x_1$ and the point (x_1, y_1) lies on it, we see that the equation of the tangent is $y - y_1 = \frac{1}{5}x_1(x - x_1)$. Since $(10, 5)$ must be on this line, $5 - y_1 = \frac{1}{5}x_1(10 - x_1)$. Since (x_1, y_1) is also on the parabola, $y_1 = \frac{1}{10}x_1^2$. Therefore, $5 - \frac{1}{10}x_1^2 = \frac{1}{5}x_1(10 - x_1)$. Multiplying by 10 gives $50 - x_1^2 = 20x_1 - 2x_1^2$. So $x_1^2 - 20x_1 + 50 = 0$, and by the quadratic formula, $x_1 = 10 \pm 5\sqrt{2}$. Because the car is to the left of the point (10, 5), we need to take $x_1 = 10 - 5\sqrt{2}$. When this is the x-coordinate of the headlights, the headlights will beam in on (10, 5).

8E. About Derivatives

32. Recall that $f'(x) = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$. Denoting Δx by h this becomes $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$. The pattern of (i) suggests that $x = -5$. So $f'(-5) = \lim\limits_{h \to 0} \frac{f(-5+h)-f(-5)}{h}$. Taking $f(x) = x^2$, we see $f(-5+h) = (h-5)^2$ and $f(-5) = 25$. So the limit in (i) is the derivative of $f(x) = x^2$ at $x = -5$. Because (iii) can be rewritten as $\lim\limits_{h \to 0} \frac{\sqrt{1+h}-1}{h}$, we see that this limit is the derivative of $f(x) = \sqrt{x}$ at $x = 1$. Only (ii) remains. A reading of Section 8.3 informs us that $f'(c) = \lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c}$. Taking $c = 1$, we get $f'(1) = \lim\limits_{x \to 1} \frac{f(x)-f(1)}{x-1}$. It follows that $\lim\limits_{x \to 1} \frac{x^9-1}{x-1}$ is the derivative of $f(x) = x^9$ at $x = 1$.

33. **i.** The domain of $f(x) = x - \frac{2}{x}$ is $\{x \mid x \neq 0\}$. Now to the computation of $f'(x)$:

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left(x + \Delta x - \frac{2}{x+\Delta x}\right) - \left(x - \frac{2}{x}\right)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\Delta x + \frac{2}{x} - \frac{2}{x+\Delta x}\right] = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\Delta x + \frac{2(x+\Delta x) - 2x}{x(x+\Delta x)}\right] \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\Delta x + \frac{2\Delta x}{x(x+\Delta x)}\right] = \lim_{\Delta x \to 0}\left[1 + \frac{2}{x(x+\Delta x)}\right] \\
&= 1 + \frac{2}{x^2}.
\end{aligned}$$

The domain of $f'(x)$ is $\{x \mid x \neq 0\}$. It coincides with that of $f(x)$.

ii. The domain of $f(x) = \sqrt{6-x}$ is $\{x \mid x \leq 6\}$. The derivative is obtained by rationalizing:

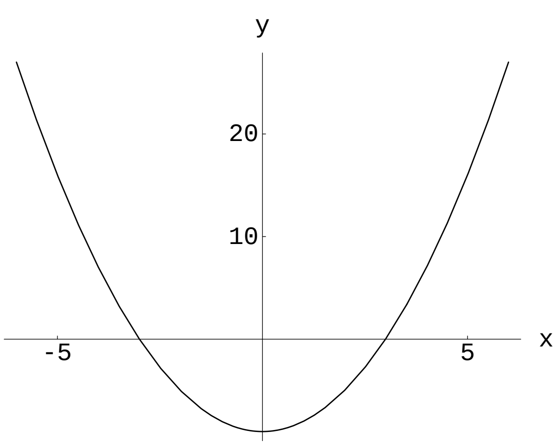
$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{6-(x+\Delta x)} - \sqrt{6-x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\left(\sqrt{6-(x+\Delta x)} - \sqrt{6-x}\right)\left(\sqrt{6-(x+\Delta x)} + \sqrt{6-x}\right)}{\Delta x\left(\sqrt{6-(x+\Delta x)} + \sqrt{6-x}\right)} \\
&= \lim_{\Delta x \to 0} \frac{6-(x+\Delta x)-(6-x)}{\Delta x\left(\sqrt{6-(x+\Delta x)} + \sqrt{6-x}\right)} \\
&= \lim_{\Delta x \to 0} \frac{-1}{\sqrt{6-(x+\Delta x)} + \sqrt{6-x}} = -\frac{1}{2\sqrt{6-x}}.
\end{aligned}$$

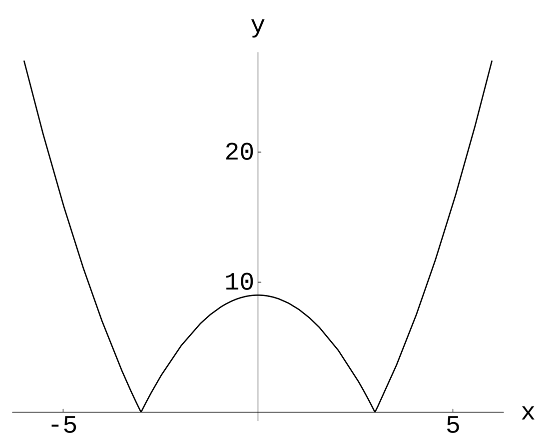
The domain of $f'(x)$ is $\{x \mid x < 6\}$.

34. The facts to remember are these: If $f'(x) > 0$ for all x in an interval I, then $f(x)$ is increasing over I; and if $f'(x) < 0$ for all x in I, then $f(x)$ is decreasing over I. If $f'(x) = 0$, then the graph of f has a horizontal tangent. Going from left to right: We see that the function whose derivative has Graph a is increasing, then suddenly decreasing, then suddenly increasing, and then suddenly decreasing again. This is the pattern of Graph ii. The function whose derivative has Graph b is increasing, then has a horizontal tangent, then is decreasing, has another horizontal tangent, then increases until it has another horizontal tangent, and it is decreasing thereafter. This is the pattern of Graph iv. Similar considerations match Graph c with Graph iii and Graph d with Graph i.

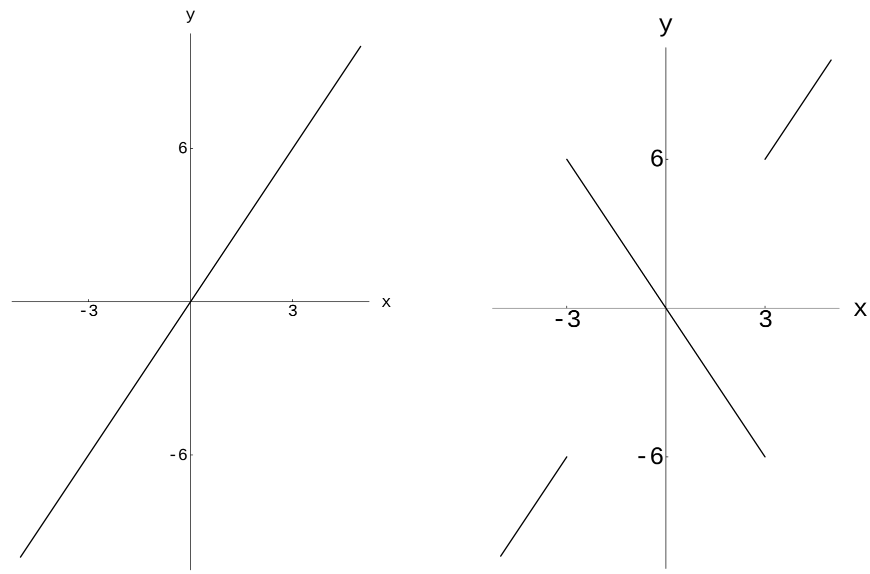
35. Because $\left(\frac{f}{g}\right)'(3)$ is the derivative of the quotient of $\frac{f(x)}{g(x)}$ evaluated at $x = 3$, it is computed as follows: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$; and at $x = 3$, $\left(\frac{f}{g}\right)'(3) = \frac{f'(3)g(3)-f(3)g'(3)}{g(3)^2} = \frac{(-6)(2)-(4)(5)}{4} = -8$. By the chain rule, $(f(g(x)))' = f'(g(x)) \cdot g'(x)$. Evaluating this at $x = 3$, we get $f'(g(3)) \cdot g'(3) = f'(2) \cdot 5 = (-3)(5) = -15$.

36. The graph of $f(x) = x^2 - 9$ is sketched below. Because $|\ |$ makes everything positive, the graph of $g(x) = |x^2 - 9|$ is obtained by rotating the portion of the graph of $f(x) = x^2 - 9$ that is below the x-axis upward as shown. So the graph of $g(x) = |x^2 - 9|$ has sharp corners at $x = -3$ and $x = 3$. So $g(x)$ is not differentiable at $x = -3$ and $x = 3$.





The functions $f(x)$ and $g(x)$ coincide except when $-3 \leq x \leq 3$. So $g'(x) = f'(x) = 2x$ except when $-3 \leq x \leq 3$. For $-3 \leq x \leq 3, g(x) = -f(x)$. So $g'(x) = -f'(x) = -2x$. The graphs of $f'(x)$, and $g'(x)$ are sketched below:



37. The discussion in Section 8.5A tells us that $y = cx^2 + 1$ is differentiable for all x no matter what c is. By the same discussion, $y = \sqrt{x} + d$ is differentiable for all $x \geq 0$ no matter what d is. So the question is this: For which c and d do the graphs of $y = cx^2 + 1$ and $y = \sqrt{x} + d$ fit together in such a way that the graph of $f(x)$ that results is smooth at $x = 4$? Because the graphs need to connect when $x = 4$ (Why?), we need to have $c \cdot 4^2 + 1 = \sqrt{4} + d$, or $16c + 1 = d + 2$. Because the connection needs to be smooth (no corner), the derivative of $y = cx^2 + 1$ at $x = 4$ needs to be equal to the derivative of $y = \sqrt{x} + d$ at $x = 4$. So $2c \cdot 4 + 0 = \frac{1}{2}(4)^{-\frac{1}{2}} + 0$, and hence $8c = \frac{1}{4}$. So $c = \frac{1}{32}$, and by the earlier equation, $d = 16c - 1 = \frac{16}{32} - 1 = -\frac{1}{2}$.

38. This is done by rationalizing: $\lim\limits_{x \to a} \frac{f(x)-f(a)}{\sqrt{x}-\sqrt{a}} = \lim\limits_{x \to a} \frac{(f(x)-f(a))(\sqrt{x}+\sqrt{a})}{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})} = \lim\limits_{x \to a}(\sqrt{x}+\sqrt{a})\left(\frac{f(x)-f(a)}{x-a}\right) = 2\sqrt{a}\, f'(a)$.

8F. Rates of Change

39. For $[4, 7]$ this is $\frac{62{,}000-4{,}000}{7-4} = \frac{58{,}000}{3} = 19{,}333$ bacteria per minute. For $[7, 9]$ it is $\frac{154{,}000-62{,}000}{9-7} = \frac{92{,}000}{2} = 46{,}000$ bacteria per minute.

40. For $[0, 5]$ this is $\frac{35-70}{5-0} = \frac{-35}{5} = -7$ milligrams per day, and for $[0, 10]$ it is $\frac{17.5-70}{10-0} = \frac{-52.5}{10} = -5.25$ milligrams per day. The minus sign means that the amount is decreasing at these rates.

41. If the formula $T(x) = 500 - ax$ is to hold for all x with $0 \leq x \leq 80$, it must hold for $x = 20$. So $499 = T(20) = 500 - a(20)$. So $-20a = -1$ and hence $a = \frac{1}{20}$. Therefore $T(x) = 500 - \frac{x}{20}$. We now get that $T(30) = 500 - \frac{30}{20} = 498.5$ and $T(35) = 500 - \frac{35}{20} = 498.25$. So $T(30) - T(35) = 0.25$ and hence the temperature is decreasing at a rate of $\frac{0.25}{5} = \frac{1}{20}$ degrees per inch over $[30, 35]$. The rate of change of the temperature at any x is $T'(x) = -\frac{1}{20}$ degrees per inch. So $T'(30)$ is also equal to $-\frac{1}{20}$ degrees per inch.

42. Because $V(0) = 3000$ and $V(25) = 0$, the average rate at which the tank drained was $\frac{3000}{25} = 120$ gallons/minute. After 10 minutes there were $V(10) = 3000\left(1 - \frac{10}{25}\right)^2 = 1080$ gallons in the tank, and after 20 minutes there were $V(20) = 3000\left(1 - \frac{20}{25}\right)^2 = 120$ gallons. During the time interval $[10, 20]$, the average rate at which the water drained was $\frac{V(20)-V(10)}{20-10} = \frac{120-1080}{20-10} = -96$ gallons/minute. The minus means that the volume of water in the tank was decreasing. Note that $V'(t) = 6000\left(1 - \frac{t}{25}\right)\left(-\frac{1}{25}\right) = -240\left(1 - \frac{t}{25}\right)$. So at $t = 10$ and $t = 20$, the rates were $V'(10) = -144$ gallons/minute, and $V'(20) = -48$ gallons/minute, respectively.

43. The volume is equal to $V = V(x) = x^3$. When x changes from 3 to 4, the average change in the volume is $\frac{V(4)-V(3)}{4-3} = \frac{4^3-3^3}{1} = 37$. When x changes from 3 to 3.1, this average change is $\frac{V(3.1)-V(3)}{3.1-3} = \frac{(3.1)^3-3^3}{0.1} = 27.91$, and when x changes from 3 to 3.01, the average change is $\frac{V(3.01)-V(3)}{0.01} = \frac{(3.01)^3-3^3}{0.01} = 27.09$. The rate of change of V when $x = 3$ is $V'(3)$. Because $V'(x) = 3x^2$, this is $V'(3) = 3 \cdot 3^2 = 27$. Because the surface area of the cube is $6x^2$, $V'(x) = 3x^2$ is one-half the surface area.

44. The area of a circle of radius r is $A = \pi r^2$. It follows that the answers to (i), (ii), and (iii) are, respectively, $\frac{A(3)-A(2)}{3-2} = \frac{9\pi-4\pi}{1} = 5\pi \approx 15.71$, $\frac{A(2.5)-A(2)}{2.5-2} = \frac{\pi(2.5)^2-\pi 2^2}{0.5} \approx 14.14$, and $\frac{A(2.1)-A(2)}{2.1-2} = \frac{\pi(2.1)^2-\pi 2^2}{0.1} \approx 12.88$. Note finally that $A'(x) = 2\pi r$ is the circumference of the circle of radius r, and that $A'(2) \approx 12.57$.

45.
- **i.** Because $P = \frac{800}{V}$ as a function of V, this is $\frac{P(250)-P(200)}{250-200} = \frac{3.2-4}{50} = -0.016$ pounds/in^3. The $-$ means that the pressure is decreasing (as V increases).
- **ii.** Because $V = \frac{800}{P}$, we see that $\frac{dV}{dP} = -\frac{800}{P^2} = -800\,\frac{1}{P^2}$.

Correction: In the statement of Exercise 46, change "If the bodies are moving, find the rate of change of F relative to r." to "If the distance between the bodies is changing, find the rate of change of F relative to r. Note that if one of the bodies is in a circular orbit around the other (so the distance between them is fixed) then F is a constant and the derivative is zero."

46. This is $\frac{dF}{dr} = \frac{-2GmM}{r^3}$.

8G. Differentiating Functions

47. $\frac{dF}{dx} = 3(16x)^2 \cdot 16 = 12{,}288x^2$.

48. $G'(x) = 2x(2x-7) + (x^2+1)2 = 4x^2 - 14x + 2x^2 + 2 = 6x^2 - 14x + 2$.

49. $f'(u) = \frac{d}{du}\frac{a-u^2}{1+u^2} = \frac{-2u(1+u^2)-(a-u^2)2u}{(1+u^2)^2} = \frac{-2u(1+u^2+a-u^2)}{(1+u^2)^2} = \frac{-2u(1+a)}{(1+u^2)^2}$.

50. $\frac{ds}{dt} = \frac{d}{dt}\left[t^{\frac{1}{3}}(t+2)\right] = \frac{1}{3}t^{-\frac{2}{3}}(t+2) + t^{\frac{1}{3}} = \frac{1}{3t^{\frac{2}{3}}}(t+2) + t^{\frac{1}{3}} = \frac{t+2+3t}{3t^{\frac{2}{3}}} = \frac{4t+2}{3t^{\frac{2}{3}}}$.

51. $\frac{dy}{dx} = \frac{d}{dx}(x^4+x^2+1)^{-1} = -(x^4+x^2+1)^{-2}(4x^3+2x)$.

52. $\frac{dy}{dx} = \frac{d}{dx}(2x^3+4x^5)^6(7x^8+9x^{10})^{11}$

$= [6(2x^3+4x^5)^5(6x^2+20x^4)](7x^8+9x^{10})^{11} + (2x^3+4x^5)^6 11(7x^8+9x^{10})^{10}(56x^7+90x^9)\,.$

53. Because $y = \frac{x}{\sqrt{9-4x}} = x(9-4x)^{-\frac{1}{2}}$, we get $\frac{dy}{dx} = (9-4x)^{-\frac{1}{2}} + x\left(-\frac{1}{2}\right)(9-4x)^{-\frac{3}{2}}(-4) = \frac{(9-4x)+2x}{(9-4x)^{\frac{3}{2}}} = \frac{9-2x}{(9-4x)^{\frac{3}{2}}}\,.$

54. $F'(x) = \frac{[5(x^2+4x+6)^4(2x+4)](x^3+4x^5)^{\frac{1}{2}} - (x^2+4x+6)^5[\frac{1}{2}(x^3+4x^5)^{-\frac{1}{2}}(3x^2+20x^4)]}{x^3+4x^5}.$

55. Because $s(t) = \sqrt[4]{\frac{t^3+1}{t^3-1}} = \left(\frac{t^3+1}{t^3-1}\right)^{\frac{1}{4}}$, we find that

$s'(t) = \frac{1}{4}\left(\frac{t^3+1}{t^3-1}\right)^{-\frac{3}{4}}\left[\frac{3t^2(t^3-1)-(t^3+1)3t^2}{(t^3-1)^2}\right] = \frac{1}{4}\left(\frac{t^3-1}{t^3+1}\right)^{\frac{3}{4}}\left(\frac{-6t^2}{(t^3-1)^2}\right) = -\frac{3}{2}\left(\frac{t^3-1}{t^3+1}\right)^{\frac{3}{4}}\frac{t^2}{(t^3-1)^2}\,.$

56. Because $y = (x^2+2x+3)^2$, we get $\frac{dy}{dx} = 2(x^2+2x+3)(2x+2)$. On the other hand, $\frac{dy}{du} = 2u$ and $\frac{du}{dx} = 2x+2$. So $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 2u(2x+2) = 2(x^2+2x+3)(2x+2)$, as before. Finally, $\frac{dy}{dx}\Big|_{x=1} = 2(1+2+3)(2+2) = (2)(6)(4) = 48.$

57. **i.** The y-coordinate of the point on the circle above x is $y = \sqrt{r^2-x^2}$. Because the volume of a cylinder equals area of circular base $\times$ height, we get $V(x) = (\pi x^2)(2y) = 2\pi x^2\sqrt{r^2-x^2}$.

ii. We need to compute $V'(x)$. This is

$$\begin{aligned} V'(x) &= 2\pi\left(2x(r^2-x^2)^{\frac{1}{2}} + x^2\frac{1}{2}(r^2-x^2)^{-\frac{1}{2}}(-2x)\right) \\ &= 2\pi\left(2x(r^2-x^2)^{\frac{1}{2}} - \frac{x^3}{(r^2-x^2)^{\frac{1}{2}}}\right) = 2\pi\left(\frac{2x(r^2-x^2)-x^3}{(r^2-x^2)^{\frac{1}{2}}}\right) \\ &= 2\pi x\left(\frac{2r^2-3x^2}{(r^2-x^2)^{\frac{1}{2}}}\right). \end{aligned}$$

Since neither $x=0$ nor $x=r$ provides a maximum (because $V(x)=0$ in either case), the remaining possibility occurs when $3x^2 = 2r^2$, or $x = \sqrt{\frac{2}{3}}r$. When $x < \sqrt{\frac{2}{3}}r$, then $x^2 < \frac{2}{3}r^2$, so $3x^2 < 2r^2$, and $V'(x) > 0$. Similarly, when $x > \sqrt{\frac{2}{3}}r$, then $V'(x) < 0$. It follows that $V(x)$ is increasing when $x < \sqrt{\frac{2}{3}}r$ and decreasing when $x > \sqrt{\frac{2}{3}}r$. So $x = \sqrt{\frac{2}{3}}r$ gives us the maximum volume.

iii. Because $V\left(\sqrt{\frac{2}{3}}r\right) = 2\pi\frac{2}{3}r^2\sqrt{r^2-\frac{2}{3}r^2} = \frac{4}{3}\pi r^2\sqrt{\frac{1}{3}r^2} = \frac{4}{3}\pi r^3\sqrt{\frac{1}{3}}$, this is the maximum volume that an inscribed cylinder has. So the ratio is $\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi r^3\sqrt{\frac{1}{3}}} = \sqrt{3}$.

Note: A related problem was considered by Archimedes. The sphere of radius r just fits into the cylinder with base the circle of radius r and height $2r$. Archimedes had derived the expression $\frac{4}{3}\pi r^3$ for the volume of a sphere of radius r, so he knew that the ratio of the volume of the cylinder to that of the sphere was $\frac{(\pi r^2)(2r)}{\frac{4}{3}\pi r^3} = \frac{2}{\frac{4}{3}} = \frac{3}{2}$. Archimedes was evidently very proud of this achievement. According to the eye-witness report of the Roman statesman Cicero, the fraction $\frac{3}{2}$ and a figure of the cylinder and the inscribed sphere were etched on Archimedes's tomb. (Unfortunately, the tomb appears not to exist anymore.)

58. **i.** $\lim_{x\to -3} \frac{x^2-x-12}{x+3} = \lim_{x\to -3} \frac{2x-1}{1} = -7$.

ii. $\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \lim_{x\to 1} \frac{3x^2}{2x} = \frac{3}{2}$.

iii. $\lim_{x\to 9} \frac{x^2-81}{\sqrt{x}-3} = \lim_{x\to 9} \frac{x^2-81}{x^{\frac{1}{2}}-3} = \lim_{x\to 9} \frac{2x}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim_{x\to 9} 4x \cdot x^{\frac{1}{2}} = (4)(9)(3) = 108.$

59. By the Mean Value Theorem we know that there is a number c between 0 and 9 such that $f'(c) = \frac{f(9)-f(0)}{9-0} = \frac{12}{9} = \frac{4}{3}$. Because $f'(x) = 1 + \frac{1}{2\sqrt{x}}$, we need to solve $\frac{4}{3} = 1 + \frac{1}{2\sqrt{x}}$ for x. Doing so, we get $\frac{1}{3} = \frac{1}{2\sqrt{x}}$, so $2\sqrt{x} = 3$, and hence $x = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

60. **i.** $F'(x) = f(x)^3 + x \cdot 3f(x)^2 \cdot f'(x)$.

ii. $G'(x) = \frac{3}{2}f(x)^{-\frac{1}{2}}f'(x) + f(x) + xf'(x)$.

iii. $H'(x) = 4f(x)^{-2} + 4x(-2)(f(x))^{-3}f'(x) = \frac{4}{f(x)^2} - \frac{8xf'(x)}{f(x)^3}$.

8H. Calculus of Trigonometric Functions

61. Because $y = \sin(x^{-1})$, we get $y' = \cos(x^{-1}) \cdot (-x^{-2}) = -\frac{\cos x^{-1}}{x^2}$.

62. Because $\sin^2(\cos 4x) = [\sin(\cos 4x)]^2$, we get
$y' = 2[\sin(\cos(4x))] \cdot \cos(\cos 4x) \cdot (-\sin 4x) \cdot 4 = -8(\sin(\cos 4x))(\cos(\cos 4x))(\sin 4x)$.

63. $y' = \frac{[(2\sin x)(\cos x)]\cos x - (\sin^2 x)(-\sin x)}{\cos^2 x} = \frac{2\sin x \cos^2 x + \sin^3 x}{\cos^2 x}$.

64. Because $y = x\sin(x^{-1})$, we get $y' = \sin(x^{-1}) + x \cdot \cos(x^{-1}) \cdot (-x^{-2}) = \sin(x^{-1}) - x^{-1}\cos(x^{-1})$.

65. $y' = \sec^2(3x) \cdot 3 = 3\sec^2(3x)$.

66. $y' = -5(\cos\sqrt{x^2+1})^{-6}(-\sin\sqrt{x^2+1})(\frac{1}{2})(x^2+1)^{-\frac{1}{2}}(2x) = \frac{5x\sin\sqrt{x^2+1}}{(x^2+1)^{\frac{1}{2}}\cos^6(\sqrt{x^2+1})}$.

67. $y' = 6(1+\sec^3 x)^5(3\sec^2 x)(\sec x \tan x) = 18\tan x(\sec^3 x)(1+\sec^3 x)^5$.

68. $y' = \sec^2(x^2) \cdot (2x) + 2\tan x \sec^2 x = 2x\sec^2(x^2) + 2\tan x \sec^2 x$.

69. $y' = \frac{1}{2}(1+2\tan x)^{-\frac{1}{2}}(2\sec^2 x) = \frac{\sec^2 x}{\sqrt{1+2\tan x}}$.

70. $\lim_{\theta\to\frac{\pi}{3}} \frac{\cos\theta - 0.5}{\theta - \frac{\pi}{3}} = \lim_{\theta\to\frac{\pi}{3}} \frac{\cos\theta - \cos\frac{\pi}{3}}{\theta - \frac{\pi}{3}} = \left(\frac{d}{dx}\cos\theta\right)|_{\theta=\frac{\pi}{3}} = -\sin\theta|_{\theta=\frac{\pi}{3}} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$.

71. Because $y' = \sec^2 x$, the slope of the tangent line is $\sec^2\frac{\pi}{3} = 4$. So its equation is $y - \sqrt{3} = 4(x - \frac{\pi}{3})$ or $y = 4x - \frac{4\pi}{3} + \sqrt{3}$.

8I. Increase and Decrease of Functions

72. Because $f'(x) = 3x^2 - 3$, the critical numbers are obtained by solving $3x^2 - 3 = 0$ for x. Doing so, we get $x = \pm 1$.

73. $F'(x) = \frac{4}{5}x^{-\frac{1}{5}}(x-4)^2 + x^{\frac{4}{5}}(2(x-4)) = \frac{\frac{4}{5}(x-4)^2+x(2x-8)}{x^{\frac{1}{5}}} = \frac{\frac{4}{5}(x-4)^2+2x(x-4)}{x^{\frac{1}{5}}} = \frac{(x-4)[\frac{4}{5}(x-4)+2x]}{\sqrt[5]{x}} = \frac{(x-4)(\frac{14}{5}x-\frac{16}{5})}{\sqrt[5]{x}}$. It follows that the critical numbers are $0, 4$, and $\frac{16}{14} = \frac{8}{7}$.

74. Note that $T'(x) = 2x(2x-1)^{\frac{2}{3}} + x^2\frac{2}{3}(2x-1)^{-\frac{1}{3}}(2) = 2x(2x-1)^{\frac{2}{3}} + \frac{4x^2}{3(2x-1)^{\frac{1}{3}}} = \frac{6x(2x-1)+4x^2}{3(2x-1)^{\frac{1}{3}}} = \frac{16x^2-6x}{3(2x-1)^{\frac{1}{3}}} = \frac{16x(x-\frac{6}{16})}{3(2x-1)^{\frac{1}{3}}}$. So the critical numbers are $\frac{1}{2}, 0$, and $\frac{6}{16} = \frac{3}{8}$.

Note: The instructions for Exercises 75-78 and 81-84 should be more carefully worded to ask for "the values of the variable x at which f has a local minimum or a local maximum" rather than for "the local maximum and minimum values of f." Observe also that it is the understanding in this text that a local maximum or minimum of a function cannot occur at an endpoint of the domain of the function. However, the absolute maximum and minimum values can occur at such endpoints.

75. Because $f'(x) = 3x^2 - 4x + 1$, the critical numbers are $\frac{4\pm\sqrt{16-4\cdot 3}}{6} = \frac{4\pm 2}{6} = \frac{1}{3}$ and 1. Take $0, \frac{1}{2}$, and 2 as test points. Since $f'(0) = 1, f'(\frac{1}{2}) = \frac{3}{4} - 1 = -\frac{1}{4}$, and $f'(2) = 5$, we find that f is increasing over the intervals $(-\infty, \frac{1}{3})$ and $(1, \infty)$ and decreasing over $(\frac{1}{3}, 1)$. It follows that f has a local maximum value at $\frac{1}{3}$ and a local minimum value at 1.

76. Check that $f'(x) = 4x^3 - 12x^2 - 16x = 4x(x^2 - 3x - 4) = 4x(x-4)(x+1)$. So the critical numbers are $-1, 0$ and 4. Take $-2, -\frac{1}{2}, 1$, and 5 to be the test points. Check that $f'(-2) = (-8)(-6)(-1) = -48$; $f'(-\frac{1}{2}) = -2(-\frac{9}{2})(\frac{1}{2}) = \frac{9}{2}$, $f'(1) = 4(-3)(2) = -24$, and $f'(5) = 20(1)6 = 120$. It follows that f is increasing over $(-1, 0)$ and $(4, \infty)$, and decreasing over $(-\infty, -1)$ and (0, 4). So f has local minima at -1 and 4 and a local maximum at 0.

77. Observe first that $f(x)$ is defined only when $1 \geq x^2$ or for $-1 \leq x \leq 1$. Note that $f'(x) = (1-x^2)^{\frac{1}{2}} + x\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = (1-x^2)^{\frac{1}{2}} - \frac{x^2}{(1-x^2)^{\frac{1}{2}}} = \frac{1-x^2-x^2}{(1-x^2)^{\frac{1}{2}}} = \frac{1-2x^2}{(1-x^2)^{\frac{1}{2}}}$. It follows that the critical numbers are ± 1 and $\pm\frac{1}{\sqrt{2}}$, so they are in increasing order: $-1, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$, and 1. Because $\frac{1}{\sqrt{2}} \approx 0.71$ and $-1 \leq x \leq 1$, we take $-0.8, 0$, and 0.8 as test points. Check that $f'(-0.8) < 0$, $f'(0) > 0$, and $f'(0.8) < 0$. So f is decreasing over $(-1, -\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, 1)$ and increasing over $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Notice that f has a local minimum at $-\frac{1}{\sqrt{2}}$ and a local maximum at $\frac{1}{\sqrt{2}}$.

78. For $f(x)$ to be defined we need $x \geq x^2$. So $1 \geq x$ if $x > 0$ and $1 \leq x$ if $x < 0$. Observing that the second alternative is impossible, we see that the domain of f consists of the interval $0 \leq x \leq 1$. Check that $f'(x) = (x-x^2)^{\frac{1}{2}} + x\frac{1}{2}(x-x^2)^{-\frac{1}{2}}(1-2x) = (x-x^2)^{\frac{1}{2}} + \frac{x(1-2x)}{2(x-x^2)^{\frac{1}{2}}} = \frac{2(x-x^2)+x(1-2x)}{2(x-x^2)^{\frac{1}{2}}} = \frac{-4x^2+3x}{2(x-x^2)^{\frac{1}{2}}} = \frac{-4x(x-\frac{3}{4})}{2(x-x^2)^{\frac{1}{2}}}$. So the critical numbers are $0, \frac{3}{4}$, and 1. Because $0 \leq x \leq 1$, we only need the test points $\frac{1}{2}$ and $\frac{4}{5}$. Check that $f'(\frac{1}{2}) > 0$ and $f'(\frac{4}{5}) < 0$. Therefore f is increasing over $(0, \frac{3}{4})$ and decreasing over $(\frac{3}{4}, 1)$. Hence f has a local maximum at $\frac{3}{4}$.

79. Check that $f'(x) = 1 - \frac{1}{x^2}$. When $x > 1$, $\frac{1}{x^2} < 1$, so $-\frac{1}{x^2} > -1$ and hence $f'(x) = 1 - \frac{1}{x^2} > 0$. So f is increasing for $x > 1$. Because $f'(1) = 0$, the graph of f has a horizontal tangent at the point (1, 2). It follows that f is increasing over $[1, \infty)$. The verification of the inequality follows from the definition of increasing function.

Correction: The inequality in Exercise 80 should have two $<$ in place of the two $\leq$.

80. First observe that the inequality $\frac{\sin\beta}{\sin\alpha} < \frac{\beta}{\alpha}$ for $0 < \alpha < \beta < \frac{\pi}{2}$ is equivalent to $\frac{\sin\alpha}{\alpha} > \frac{\sin\beta}{\beta}$ for $0 < \alpha < \beta < \frac{\pi}{2}$. So we must show that $f(x) = \frac{\sin x}{x}$ is a decreasing function for $0 < x < \frac{\pi}{2}$. This will follow from the fact that $f'(x) < 0$ for $0 < x < \frac{\pi}{2}$. Because $f(x) = (\sin x)x^{-1}$, we get that

$$\begin{aligned} f'(x) &= (\cos x)x^{-1} + (\sin x)(-x^{-2}) = \frac{\cos x}{x} - \frac{\sin x}{x^2} = \frac{x\cos x - \sin x}{x^2} \\ &= -\left(\frac{\sin x - x\cos x}{x^2}\right). \end{aligned}$$

It remains to show that $\sin x - x\cos x > 0$ for $0 < x < \frac{\pi}{2}$. Consider the function $g(x) = \sin x - x\cos x$ with x in $\left[0, \frac{\pi}{2}\right]$. Check that $g'(x) = \cos x - \cos x - x(-\sin x) = x\sin x$. So $g'(x) > 0$ for x in $(0, \frac{\pi}{2})$ and hence $g(x)$ is increasing over $(0, \frac{\pi}{2})$. Because $g(0) = 0$, this means that $g(x) > 0$ for $0 < x < \frac{\pi}{2}$.

The inequality $\frac{\beta}{\alpha} < \frac{\tan\beta}{\tan\alpha}$ for $0 < \alpha < \beta < \frac{\pi}{2}$ is equivalent to $\frac{\tan\alpha}{\alpha} < \frac{\tan\beta}{\beta}$ for $0 < \alpha < \beta < \frac{\pi}{2}$. So we need to show that $f(x) = \frac{\tan x}{x}$ is an increasing function. Because $f(x) = (\tan x)x^{-1}$, we get $f'(x) = (\sec^2 x)x^{-1} + (\tan x)(-x^{-2}) = \frac{x\sec^2 x - \tan x}{x^2}$. It remains to verify that $g(x) = x\sec^2 x - \tan x > 0$ for $0 < x < \frac{\pi}{2}$. Check that $g'(x) = \sec^2 x + x(2\sec x)(\sec x\tan x) - \sec^2 x = 2x\sec^2 x\tan x > 0$. So $g(x)$ is increasing. But $g(0) = 0$ and therefore $x\sec^2 x - \tan x = g(x) > 0$ for $0 < x < \frac{\pi}{2}$. So $f'(x) > 0$ for $0 < x < \frac{\pi}{2}$, and $f(x) = \frac{\tan x}{x}$ is increasing as asserted.

81. Check that $f'(x) = 1 - 2\cos x$. So the critical points x are those with $1 - 2\cos x = 0$, or $\cos x = \frac{1}{2}$. A look at Figure 4.25 tells us that there are exactly two such x in $[0, 2\pi]$. By Section 1.4 and Example 4.11 they are $x = \frac{\pi}{3}, \frac{5\pi}{3}$. Take $\frac{\pi}{4}, \pi$, and $\frac{7\pi}{4}$ as test points. By Section 1.4 and Example 4.11, $f'(\frac{\pi}{4}) < 0, f'(\pi) > 0$ and $f'(\frac{7\pi}{4}) = f'(-\frac{\pi}{4}) = f'(\frac{\pi}{4}) < 0$. So $f(x)$ is decreasing over $(0, \frac{\pi}{3})$, increasing over $(\frac{\pi}{3}, \frac{5\pi}{3})$, and decreasing over $(\frac{5\pi}{3}, 2\pi)$. There is a local minimum at $\frac{\pi}{3}$ and a local maximum at $\frac{5\pi}{3}$.

82. Check that $f'(x) = \sin x + x\cos x - \sin x = x\cos x$. Because $-\pi \leq x \leq \pi$, the critical numbers are $-\frac{\pi}{2}, 0$, and $\frac{\pi}{2}$. Take $-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}$, and $\frac{3\pi}{4}$ as test points. Because $f'(-\frac{3\pi}{4}) > 0$, $f'(-\frac{\pi}{4}) < 0$, $f'(\frac{\pi}{4}) > 0$, and $f'(\frac{3\pi}{4}) < 0$, the function $f(x)$ is increasing on $\left(-\pi, -\frac{\pi}{2}\right)$, decreasing on $\left(-\frac{\pi}{2}, 0\right)$, increasing on $\left(0, \frac{\pi}{2}\right)$, and decreasing on $\left(\frac{\pi}{2}, \pi\right)$. There are local maxima at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and there is a local minimum at 0.

83. Refer to Figure 4.26 of Section 4.4 and notice that f is not defined when $x = -\frac{\pi}{2}$ and $\frac{\pi}{2}$. Check that $f'(x) = 2\sec^2 x - 2\tan x\sec^2 x = 2\sec^2 x(1 - \tan x) = \frac{2(1-\tan x)}{\cos^2 x}$. So the critical points occur when $1 - \tan x = 0$ and $\cos x = 0$. Because $-\pi \leq x \leq \pi$, we get by consulting Chapters 1.4 and 4.4 that the critical numbers are $-\frac{3\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{4}$, and $\frac{\pi}{2}$. Take $\frac{-4\pi}{5}, \frac{-3\pi}{5}, 0, \frac{2\pi}{5}$, and $\frac{4\pi}{5}$ as test points. Evaluate $\tan x$ at $-0.8\pi, -0.6\pi, 0, 0.4\pi$ and 0.8π with a calculator and conclude that $f'(\frac{-4\pi}{5}) > 0, f'(\frac{-3\pi}{5}) < 0, f'(0) > 0, f'(\frac{2\pi}{5}) < 0$, and $f'(\frac{4\pi}{5}) > 0$. Thus we see that f is increasing over $\left(-\pi, -\frac{3\pi}{4}\right)$, decreasing over $\left(-\frac{3\pi}{4}, -\frac{\pi}{2}\right)$, increasing over $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$, decreasing over $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, and increasing over $(\frac{\pi}{2}, \pi)$. There are local maxima at $-\frac{3\pi}{4}$ and $\frac{\pi}{4}$. Because f is not defined at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, there are no local minima.

84. The derivative is $g'(x) = \sin x + \cos x$. A comparison of Figures 4.24 and 4.25 shows that there is only a single x with $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ that satisfies $\sin x = -\cos x$. Observe that $x = -\frac{\pi}{4}$ satisfies this equality and that this is the only critical number. Take $-\frac{\pi}{3}$ and 0 as test points to get that $g(x)$ is decreasing on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$ and increasing on $\left(-\frac{\pi}{4}, \frac{\pi}{2}\right)$. So f has a local minimum at $-\frac{\pi}{4}$.

85. Check that $f'(x) = 2(x+1)$. Evaluating f at the critical number -1 and at the endpoints $-2, 5$, we get $f(-2) = 2,\ f(-1) = 1$, and $f(5) = 37$. So the maximum value of f is $f(5) = 37$ and the minimum value is $f(-1) = 1$.

86. The derivative is $f'(x) = 3x^2 - 12 = 3(x^2 - 4)$. So the critical numbers are ± 2. Evaluating f at the critical numbers and also at -3 and 5, we get $f(-3) = 10,\ f(-2) = 17,\ f(2) = -15$, and $f(5) = 66$. So the maximum value is $f(5) = 66$ and the minimum value is $f(2) = -15$.

87. The derivative is $f'(x) = 12x^2 - 30x + 12 = 6(2x^2 - 5x + 2)$. By the quadratic formula, the critical numbers are $\frac{1}{2}$ and 2. Evaluating f at the required points, we get $f(0) = 7,\ f(\frac{1}{2}) = 9.75,\ f(2) = 3$, and $f(3) = 16$. So the maximum value of f is $f(3) = 16$ and the minimum value is $f(2) = 3$.

88. The derivative is $f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$. So the critical numbers are $-1, 0$, and 1. Evaluating f at the required points, we get $f(-2) = -57,\ f(-1) = 1,\ f(0) = -1,\ f(1) = -3$, and $f(2) = 55$. So the maximum value is $f(2) = 55$ and the minimum value is $f(-2) = -57$.

89. Check that the derivative is $f'(x) = \frac{-x}{\sqrt{9-x^2}}$. So the only critical number in $[-1, 2]$ is 0. Evaluating the function at $x = -1, 0$ and 2, we get $f(-1) = \sqrt{8},\ f(0) = 3$, and $f(2) = \sqrt{5}$. So the maximum value is $f(0) = 3$ and the minimum value is $f(2) = \sqrt{5}$.

90. The upper right corner of the rectangle is the point $(x, \frac{b}{a}\sqrt{a^2 - x^2})$ with $x > 0$. The area of the rectangle is equal to $A(x) = (2x)\left(2\frac{b}{a}\sqrt{a^2 - x^2}\right) = 4\frac{b}{a}x(a^2 - x^2)^{\frac{1}{2}}$. The domain of A is $[0, a]$. We are looking for the value of x for which the function $A(x)$ attains its maximum value. Differentiating $A(x)$, we get $A'(x) = 4\frac{b}{a}\left[(a^2 - x^2)^{\frac{1}{2}} + x \cdot \frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x)\right]$. By taking common denominators, we get $A'(x) = \frac{4b}{a}\left[\frac{a^2 - x^2 - x^2}{(a^2 - x^2)^{\frac{1}{2}}}\right] = \frac{4b(a^2 - 2x^2)}{a(a^2 - x^2)^{\frac{1}{2}}}$. The value $x = a$ can be ignored because $A(x) = 0$ in this case. Notice that $A'(x) = 0$ precisely when $x = \frac{a}{\sqrt{2}}$. When $x < \frac{a}{\sqrt{2}}$, then $x^2 < \frac{a^2}{2}$. So $2x^2 < a^2$ and hence $A'(x) > 0$. When $x > \frac{a}{\sqrt{2}}$, then $x^2 > \dfrac{a^2}{2}$. So $2x^2 > a^2$, and this time $A'(x) < 0$. It follows that $A(x)$ is increasing to the left of $x = \frac{a}{\sqrt{2}}$ and decreasing to the right. Therefore $x = \frac{a}{\sqrt{2}}$ gives us the maximum we are looking for. The dimensions of the maximal rectangle are: the base is $2 \cdot \frac{a}{\sqrt{2}} =$

$\sqrt{2}a$ and the height is $2\frac{b}{a}\sqrt{a^2 - \frac{a}{2}^2} = 2\frac{b}{a}\sqrt{\frac{a^2}{2}} = \frac{2}{\sqrt{2}}b = \sqrt{2}b$. Its area is $2ab$.

8J. More Problems from the Books of L'Hospital and Agnesi

91. Let $d = AB$ and $x = AE$. The function $f(x) = x^2(d-x)^2$, where $0 \le x \le d$, has to be maximized. By the product rule,

$$f'(x) = 2x(d-x)^2 + x^2 2(d-x)(-1) = 2x(d-x)[d-x-x] = 2x(d-x)(d-2x).$$

For $x = 0$ or $x = d$, the product $x^2(d-x)^2 = 0$ is not the maximum we are looking for. So only $x = \frac{d}{2}$ remains. So E has to be placed at the midpoint of AB. It remains to check the sign of $f'(x)$ and to confirm that $x = \frac{d}{2}$ actually results in a maximum. For $x < \frac{d}{2}$, we have $2x < d$ and hence $f'(x) > 0$. For $x > \frac{d}{2}, 2x > d$ and this time $f'(x) < 0$. Observe therefore that $f(x)$ increases to the left of $x = \frac{d}{2}$ and decreases to the right. So $x = \frac{d}{2}$ gives us the maximum we are looking for.

92. The ratio expressed as a function of x is $R(x) = \frac{AE \times EB}{CE \times EF} = \frac{(a+x)(b-x+c)}{x(b-x)}$. Note that we must have $0 < x < b$, so that this is the domain of R. By the product and quotient rules,

$$R'(x) = \frac{[(b-x+c) + (a+x)(-1)]x(b-x) - [(a+x)(b-x+c)](b-2x)}{[x(b-x)]^2}.$$

Letting $N(x)$ be the numerator, we get:

$$\begin{aligned}
N(x) &= (b-x+c)[x(b-x) - (a+x)(b-2x)] - (a+x)x(b-x) \\
&= (b-x+c)[-x^2 + bx + 2x^2 - bx + 2ax - ab] - (b-x)(x^2 + ax) \\
&= (b-x+c)[x^2 + 2ax - ab] + (b-x)(-x^2 - ax) \\
&= (b-x)[x^2 + 2ax - ab] + (b-x)(-x^2 - ax) + c[x^2 + 2ax - ab] \\
&= (b-x)[x^2 + 2ax - ab - x^2 - ax] + c(x^2 + 2ax - ab) \\
&= (b-x)(ax - ab) + c(x^2 + 2ax - ab) \\
&= (c-a)x^2 + (2ac + ab + ab)x - abc - ab^2 \\
&= (c-a)x^2 + 2a(b+c)x - ab(b+c)
\end{aligned}$$

as asserted in the hint. Because $0 < x < b$, the denominator $x^2(b-x)^2$ of $R'(x)$ is always positive. So it remains to find the numbers x with $0 < x < b$ for which the numerator

$$N(x) = (c-a)x^2 + 2a(b+c)x - ab(b+c)$$

of $R'(x)$ is equal to zero.

i) Suppose $c = a$. Then $N(x) = 2a(b+c)x - ab(b+c)$ is zero only when $2x = b$ or $x = \frac{b}{2}$.

Suppose $c \neq a$. By the quadratic formula,

$$\begin{aligned} x &= \frac{-2a(b+c) \pm \sqrt{4a^2(b+c)^2 + 4(c-a)ab(b+c)}}{2(c-a)} \\ &= \frac{-a(b+c) \pm \sqrt{a(b+c)[a(b+c) + b(c-a)]}}{c-a} \\ &= \frac{-a(b+c) \pm \sqrt{a(b+c)c(a+b)}}{c-a}. \end{aligned}$$

ii) Suppose $c > a$. Because $x > 0$,

$$x_1 = \frac{-a(b+c) + \sqrt{ac(b+c)(a+b)}}{c-a}$$

is the only possibility. A comparison of the terms $a(b+c)$ and $\sqrt{ac(b+c)(a+b)}$ (square them both and use $c > a$) tells us that $x_1 > 0$ as required. But is $x_1 < b$? That this is so follows by reversing the following chain of inequalities:

$$\begin{gathered} \tfrac{-a(b+c)+\sqrt{ac(b+c)(a+b)}}{c-a} < b \\ -a(b+c) + \sqrt{ac(b+c)(a+b)} < b(c-a) \\ \sqrt{ac(b+c)(a+b)} < bc - ab + ab + ac \\ ac(b+c)(a+b) < c^2(a+b)^2 \\ a(b+c) < c(a+b) \\ ab + ac < ca + cb \\ a < c. \end{gathered}$$

So $x_1 = \frac{-a(b+c)+\sqrt{ac(b+c)(a+b)}}{c-a}$ is the solution we are looking for.

iii) Finally, suppose $c < a$. In this case $x = \frac{-a(b+c)-\sqrt{ac(b+c)(a+b)}}{c-a}$ is greater than b and must be ruled out. This is so, because $\frac{-a(b+c)-\sqrt{ac(b+c)(a+b)}}{c-a} \leq b$ implies (since $c - a < 0$) that $-a(b+c) - \sqrt{ac(b+c)(a+b)} \geq b(c-a)$ and hence that

$$-\sqrt{ac(b+c)(a+b)} \geq bc - ba + ab + ac = c(a+b).$$

This is not possible because $c(a+b) > 0$. So again,

$$x_1 = \frac{-a(b+c) + \sqrt{ac(b+c)(a+b)}}{c-a}$$

is the only possibility. It is not hard to show that $0 < x_1 < b$.

Are we finished? Not quite! We do not as yet know that $R(x) = \frac{AE \times EB}{CE \times EF}$ actually has a minimum at the point we found. Suppose $c > a$. Then the numerator

$$N(x) = (c-a)x^2 + 2a(b+c)x - ab(b+c)$$

of $R'(x)$ is a parabola that opens upward. We saw in (ii) that $(c-a)x^2+2a(b+c)x-ab(b+c)$ has one root on the negative part of the x-axis and another between 0 and b on the positive part. Because the parabola has y-intercept $-ab(b+c)$ it follows that the graph of the parabola lies below the x-axis from $x=0$ to x_1 and above the x-axis from x_1 to b. This implies that $R'(x)<0$ for $0<x<x_1$ and $R'(x)>0$ for $x_1<x<b$. Therefore, $R(x)$ is decreasing to the left of x_1 and increasing to the right of x_1. So $R(x)$ has a minimum at x_1 as required. The cases $c<a$ and $c=a$ are handled in a similar way.

93. To compute the length L of QH we will use the Pythagorean theorem. If L is to be expressed as a function of x, we need to express QD in terms of x. By similar triangles, $\frac{QD}{DC}=\frac{CB}{BH}$, so $QD=DC\cdot\frac{CB}{BH}=AB\cdot\frac{AD}{BH}=\frac{bd}{x}$ as required. Observe that

$$L=\sqrt{\left(\frac{bd}{x}+d\right)^2+(b+x)^2}=\sqrt{d^2\left(\frac{b+x}{x}\right)^2+(b+x)^2}=(b+x)\sqrt{\frac{d^2}{x^2}+1}\,.$$

So $f(x)=L^2=(b+x)^2\left(1+\frac{d^2}{x^2}\right)$, where $x>0$. Differentiating, we get

$$\begin{aligned} f'(x) &= 2(b+x)\left(1+\frac{d^2}{x^2}\right)+(b+x)^2(-2d^2x^{-3}) \\ &= 2(b+x)\left(\frac{x^3+xd^2-(b+x)d^2}{x^3}\right)=\frac{2(x+b)(x^3-d^2b)}{x^3}. \end{aligned}$$

So $f'(x)=0$ only when $x=(d^2b)^{\frac{1}{3}}=\sqrt[3]{d^2b}$. The fact that $f'(x)$ is not defined at $x=0$ can be ignored. (Why?) That $x=\sqrt[3]{d^2b}$ gives us the QH of minimal length can be confirmed by noticing that $f'(x)<0$ when $x<\sqrt[3]{d^2b}$ and that $f'(x)>0$ when $x>\sqrt[3]{d^2b}$. For $x=(d^2b)^{\frac{1}{3}}=d^{\frac{2}{3}}b^{\frac{1}{3}}$, we get

$$L=(b+(bd^2)^{\frac{1}{3}})\sqrt{1+\frac{d^2}{(bd^2)^{\frac{2}{3}}}}\,.$$

This is the shortest that L can be.

8K. Newton's Method for Solving Equations

94. Setting $f(x)=\frac{1}{2}x^2-1=0$, we get $\frac{1}{2}x^2=1$ and hence $x^2=2$. So $x=\pm\sqrt{2}$ are the roots of $\frac{1}{2}x^2-1$. Let's see what Newton's Method gives us. Note that $f'(x)=x$. Starting with $c_1=2$, we get

$$\begin{aligned} c_2 &= 2-\frac{f(2)}{f'(2)}=2-\frac{\frac{1}{2}\cdot 2^2-1}{2}=2-\frac{1}{2}=\frac{3}{2} \\ c_3 &= \frac{3}{2}-\frac{f(\frac{3}{2})}{f'(\frac{3}{2})}=\frac{3}{2}-\frac{\frac{1}{2}(\frac{3}{2})^2-1}{\frac{3}{2}}=\frac{3}{2}-\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{9}{4}+\frac{2}{3}=1.4167. \\ c_4 &= 1.4167-\frac{f(1.4167)}{f'(1.4167)}=1.4167-0.0025=1.4142. \end{aligned}$$

Checking (with a calculator) that $\sqrt{2} = 1.414213562\ldots$ we see that Newton's method has already closed in on the root $\sqrt{2}$ to within the required four decimal place accuracy. This should mean that $c_5 = 1.4142$ rounded to four decimal places. Let's check. Because $c_4 = 1.4142$,

$$c_5 = 1.4142 - \frac{f(1.4142)}{f'(1.4142)} = 1.414213562\ldots$$

So c_5 turns out to be an approximation of $\sqrt{2}$ that is accurate not only up to four but, in fact, up to nine decimal places.

95. We get $f'(x) = 3x^2 + 2x - 7$. Starting with $c_1 = 3$ gives us

$$\begin{aligned}
c_2 &= 3 - \frac{f(3)}{f'(3)} = 3 - \frac{8}{26} = 2.6923 \\
c_4 &= 2.6923 - \frac{f(2.6923)}{f'(2.6923)} = 2.6923 - \frac{0.9175}{20.1300} = 2.6467 \\
c_5 &= 2.6467 - \frac{f(2.6467)}{f'(2.6467)} = 2.6467 - \frac{0.0183}{19.3085} = 2.6458 \\
c_6 &= 2.6458 - \frac{f(2.6458)}{f'(2.6458)} = 2.6458 - \frac{0.0009}{19.2924} = 2.64575335
\end{aligned}$$

This agrees with c_5 when rounded off. So the process is finished.

Refer to Exercises 3F of Chapter 3. From the fact that $f(-1) = -1 + 1 + 7 - 7 = 0$, it follows that $x + 1$ divides $x^3 + x^2 - 7x - 7$. Doing the division $x + 1\,\overline{|x^3 + x^2 - 7x - 7}$ we get that $x^3 + x^2 - 7x - 7 = (x + 1)(x^2 - 7)$. So c_6 must be an approximation of $\sqrt{7}$. (Why?) Because $\sqrt{7} \approx 2.645751311$, this is indeed so.