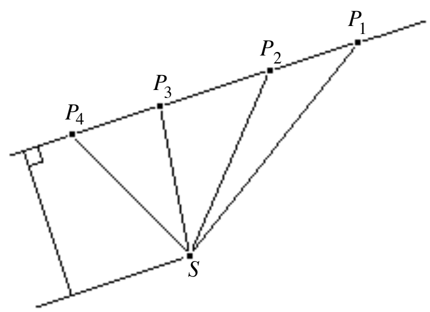
## Solutions to the Exercises of Chapter 7

### 7A. Equal Areas in Equal Times

**Correction:** In the statement of Exercise 1, it should be "centripetal" and not "certripetal."

**1.** Let $t$ be any fixed time interval. The diagram shows the object $P$ as having travelled from $P_1$ to $P_2$ and from $P_3$ to $P_4$ in the same time $t$. Because the velocity is constant, the bases $P_1P_2$ and $P_3P_4$ of the triangles $\Delta P_1P_2S$ and $\Delta P_3P_4S$ are equal. The diagram also shows that the heights of the two triangles (relative to these bases) are equal as well. It follows that




$\Delta P_1P_2S$ and $\Delta P_3P_4S$ have equal areas. So $SP$ sweeps out equal areas in equal time.

It is the purpose of Exercises 2 to 5 to provide a numerical illustration of Newton's "equal areas in equal time" argument in the situation of the orbit of the Earth. The area that the Earth sweeps out (Exercise 3) is compared (in Exercise 5) to the approximation given by Newton's triangles (Exercise 4). Notice that the statement of Exercise 3 fails to mention that $\kappa$ should be computed in the unit $\mathrm{AU}^2/\mathrm{day}$. The reader should refer to the section "Additional Exercises for Chapter 7" for a revised and more realistic version of these exercises. There, the Earth's orbit is taken to be the ellipse that it is. The solutions are no more difficult than those of the exercises below.

**2.** Since $\Delta t = 1$ day and $t_1 = 61$ days, there are 61 triangles inscribed in the sector $SPQ$. So in reference to Section 7.1, $n = 61$.

**3.** Kepler's constant for the orbit of any planet is equal to $\kappa = \frac{A_t}{t}$ where $A_t$ is the area traced out during time $t$. With $t = 365.2422$ days, we get $\kappa = \frac{\pi}{365.2422} \approx 0.0086\,\mathrm{AU}^2$ per day. The areas of the sectors $SPP_1$ and $SPQ$ are approximately $0.0086\,\mathrm{AU}^2$ and $61(0.0086) \approx 0.5247\,\mathrm{AU}^2$ respectively. (This approximation is gotten by using a more precisely computed $\kappa$ and then rounding off.) In reference to Section 7.1, $A_1 =$ area sector $SPQ \approx 0.5247\,\mathrm{AU}^2$.

**4.** By a fact established in Section 3.2, the area of the circular sector $SPP_1$ is $\frac{1}{2}(a-e)^2\theta$. By Exercise 3, we get that $\frac{1}{2}(0.9800)^2\theta \approx 0.0086$. So $\theta \approx \frac{2(0.0086)}{0.9604} \approx 0.0179$ radians. Take $PP_1$ to be the base of the triangle $\Delta SPP_1$ of Figure 7.25 and let $h$ be the corresponding height.

Since $\sin\frac{\theta}{2} = \frac{\frac{1}{2}PP_1}{a-e}$ and $\cos\frac{\theta}{2} = \frac{h}{a-e}$, we get that the area of $\Delta SPP_1$ is equal to

$$\frac{1}{2}(PP_1)h = \left((a-e)\sin\frac{\theta}{2}\right)\left((a-e)\cos\frac{\theta}{2}\right) = \frac{1}{2}(a-e)^2\sin\theta\,.$$

For the last equality use the addition formula for the sine (refer to Exercise 6(iii) of Chapter 2 and take $\alpha = \beta = \frac{\theta}{2}$). With $\theta \approx 0.0179$ radians, we get

$$\Delta A = \text{area }\Delta SPP_1 \approx \frac{1}{2}(0.9800^2)(0.0179) = \frac{1}{2}(0.9604)(0.0179) \approx 0.0086\,\text{AU}^2.$$

**5.** Combining the results of Exercises 2 and 4, we get $n(\Delta A) \approx 61(0.0086) \approx 0.5247\,\text{AU}^2$. (This last approximation is gotten by using a more precisely computed $\Delta A$ and rounding off.) Exercise 3 informed us that $A_1 = 0.5247\,\text{AU}^2$. So $A_1 = n(\Delta A)$ up to four decimal place accuracy in $\text{AU}^2$.

The conclusion of Exercise 5 tells us that even with the relatively large $\Delta t$ of one day, Newton's approximation $A_1 \approx n(\Delta A)$ of the area of the elliptical sector by inscribed triangles is accurate to within a fraction of an $\text{AU}^2$. The conversion of $\text{AU}^2$ into $\text{miles}^2$ (as suggested in the statement of Exercise 5) provides no additional insight because the accuracy of the approximation is limited by the accuracy of the given data, for example, $e = a\varepsilon = 0.0167$ AU. If we were to work with greater accuracy, for example $\varepsilon = 0.01671022$, then differences between $A_1$ and $n(\Delta A)$ would undoubtedly become visible.

**6.** Consider an elliptical orbit with semimajor axis $a$ and semiminor axis $b$ and let $T$ be the period. Because the area of the ellipse is $ab\pi$, Kepler's constant is equal to $\kappa = \frac{ab\pi}{T}$. The semiminor axis $b$ can be computed by recalling that $b = \sqrt{a^2 - e^2}$ where the linear eccentricity $e$ is equal to $a\varepsilon$ with $\varepsilon$ the astronomical eccentricity. For Mercury's orbit, $a = 0.3871\,\text{AU}, T = 0.2408$ years, and $\varepsilon = 0.2056$. So $e = (0.3871)(0.2056) = 0.0796\,\text{AU}$ and $b = \sqrt{a^2 - e^2} = \sqrt{0.1435} = 0.3788$ AU. Therefore, $\kappa = \frac{ab\pi}{T} = 1.9131\,\text{AU}^2$ per year. For Jupiter, $a = 5.2028$ AU, $T = 11.8622$ years, and $\varepsilon = 0.0484$. So $e = (5.2028)(0.0484) = 0.2518$ AU and $b = \sqrt{a^2 - e^2} = \sqrt{27.0057} = 5.1967$ AU. So for Jupiter, $\kappa = \frac{ab\pi}{T} = 7.1606\,\text{AU}^2$ per year. A look at the "Average speed" column of Table 4.2 tells us that Mercury's average velocity is greater than that of Jupiter. However, because Jupiter is much farther from the Sun, it sweeps out more area per unit time.

**7.** Because $\frac{A(t)}{t} = \kappa$, we get $A(t) = \kappa t$.

### 7B. Computations Related to the Inverse Square Law

**8.** Note that $\sin\theta = \frac{QT}{OQ} = QT$ and that $\cos\theta = \frac{OP}{OR} = \frac{1}{1+QR}$. Following the hint, we get

$\sec\theta = 1 + QR$, and hence $QR = \sec\theta - 1$. So

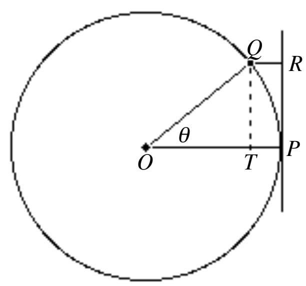
$$\begin{aligned}\frac{QR}{QT^2} &= \frac{1}{\sin^2\theta}(\sec\theta - 1) = \frac{1}{\sin^2\theta}(\sec\theta - 1)\,\frac{\sec\theta+1}{\sec\theta+1} \\ &= \frac{\tan^2\theta}{\sin^2\theta(\sec\theta+1)} = \frac{1}{\cos^2\theta\left(\frac{1}{\cos\theta}+1\right)} = \left(\frac{1}{1+\cos\theta}\right)\left(\frac{1}{\cos\theta}\right).\end{aligned}$$

Pushing $\theta$ to zero, we see that $\frac{QR}{QT^2}$ goes to $\frac{1}{2}$. Much more direct than the flow suggested by the hint is the computation,

$$\frac{QR}{QT^2} = \frac{1}{\sin^2\theta}\left(\frac{1}{\cos\theta} - 1\right) = \left(\frac{1}{1-\cos^2\theta}\right)\left(\frac{1-\cos\theta}{\cos\theta}\right) = \left(\frac{1}{1+\cos\theta}\right)\left(\frac{1}{\cos\theta}\right).$$

Applying Newton's formula to the case of a circle of radius 1, take $a = b = 1$ to also get $\frac{1}{2}$.

A closer look at Figure 7.27 (the figure that accompanies this problem) shows that it is not what the discussion in Section 7.2 requires. In particular, in Figure 7.10 the segment $QR$ is parallel to $SP$, but in Figure 7.27 it is not. Therefore in the context of Newton's Inverse Square Law, the limit computed above is the wrong limit. The correct figure is supplied




above and the correct limit computation is

$$\frac{QR}{QT^2} = \frac{1-\cos\theta}{\sin^2\theta} = \frac{1-\cos\theta}{1-\cos^2\theta} = \frac{1}{1+\cos\theta}.$$

Pushing $\theta$ to zero, provides the required $\frac{1}{2}$.

**9.** Solve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for $y$ to get $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$. The foci of this ellipse are the two points $(\pm e, 0)$. The $y$-coordinates of the two points on the ellipse with $x$-coordinate $e$ are $y = \pm\frac{b}{a}\sqrt{a^2 - e^2} = \pm\frac{b^2}{a}$. It follows that the length of the segment in question is $\frac{2b^2}{a}$.

## 7C. The Satellites of Jupiter

**10.** Newton's theory (see Section 7.6) predicts Kepler's third law for Jupiter, namely that the constant $\frac{a^3}{T^2}$, where $a$ is the semimajor axis and $T$ the period of the elliptical orbit, is the same for Jupiter's satellites. Let's check this:

$$\begin{aligned}
\text{Satellite 1}: &\quad \frac{5.578^3}{42.48^2} \approx \frac{173.55}{1804.55} \approx 0.096\\
\text{Satellite 2}: &\quad \frac{8.876^3}{89.30^2} \approx \frac{699.28}{7974.49} \approx 0.088\\
\text{Satellite 3}: &\quad \frac{14.159^3}{172^2} \approx \frac{2838.56}{29584} \approx 0.096\\
\text{Satellite 4}: &\quad \frac{24.903^3}{402.09^2} \approx \frac{15443.83}{161676.37} \approx 0.096
\end{aligned}$$

So the prediction is confirmed, except for the number 0.088 which is "out of tune." Why? Page 14 of

Newton's *A Treatise of the System of the World*, Dawsons of Pall Mall, London, 1969, second edition, reprinted with an introduction by B. Cohen,

lists Flamsteed's observations of the periods of the four satellites, in terms of days, hours, minutes, and seconds, as $1^d\ 18^h\ 28'\ 36''$, $3^d\ 17^h\ 17'\ 54''$, $7^d\ 3^h\ 59'\ 36''$, and $16^d\ 18^h\ 5'\ 13''$. This is the data - converted to 42.48, 89.30, 172 (171.99 is more accurate), and 402.09 hours - that was taken for the exercise. However, in the table on page 13 of the same volume, the period of Jupiter's second satellite is given as $3^d\ 13^h\ 17'\ 54''$. This is equal to 85.30 hours. Because $\frac{8.876^3}{85.30^2} \approx \frac{699.28}{7276.09} \approx 0.096$, it seems that the entry on page 14 is a misprint.

In his analysis of the satellites of Jupiter, Newton assumes that the semimajor axes of their orbits are equal (at least approximately) to their maximal distances from Jupiter. The potential problem with this assumption is that if the astronomical eccentricity of the orbit is large, then the difference between the semimajor axis and the maximal distance is large. Why? But the fact is, see

http://sse.jpl.nasa.gov/features/planets/jupiter/jupiter.html

that the orbits of the four satellites (now named Io, Europa, Ganymede, and Callisto) are all close to being circles. Their astronomical eccentricities range from 0.002 to 0.009. The average distances of the satellites from Jupiter range from 670,000 to 1,890,000 kilometers. One last question. Given that he had determined $G$, could Cavendish have deduced the mass of Jupiter from Newton's data?

### 7D. Systems of Units

**Correction:** Interchange **Mass:** and **Force:** in both places where they occur in bold print in the left column of page 201.

**11.** For the radius of the Earth, 3950 miles $\times$ 1.61 $\frac{\text{km}}{\text{mile}} \approx$ 6360 kilometers. For that of the Moon, 1080 miles $\times$ 1.61 $\frac{\text{km}}{\text{mile}} \approx$ 1740 kilometers.

**12.** Since 1 slug weighs 32.17 pounds, we see that $\frac{1}{32.17} \approx 0.031$ slugs weigh 1 pound. So 1.3 pounds corresponds to a mass of $(1.3)(0.031) \approx 0.040$ slugs.

**13.** Converting meters to feet and kilograms to slugs, we get that

$$\begin{aligned} G &\approx 6.67 \times 10^{-11} \cdot \frac{3.28^3}{0.07} \approx (6.67 \times 10^{-11})(504) \\ &\approx 3360 \times 10^{-11} \approx 3.36 \times 10^{-8} \frac{\text{feet}^3}{\text{slug} \cdot \text{sec}^2} . \end{aligned}$$

## 7E. Applying Newton's Formulas to the Moon

**14.** The formula that applies is $M = \frac{4\pi^2 a^3}{GT^2}$. To get the mass $M$ of the Earth, we need to substitute data about the Moon's orbit. Turning to Section 7.4, we will take $a = 240{,}000$ miles and $T = 27.32$ days. In order to use $G = 6.67 \times 10^{-11} \frac{\text{meters}^3}{\text{kilograms} \cdot \text{sec}^2}$, we need to convert $a$ to meters and $T$ to seconds. Doing this, we get $a = 3.86 \times 10^8$ meters and $T = 2.36 \times 10^6$ seconds. So

$$M \approx \frac{4\pi^2 (3.86 \times 10^8)^3}{(6.67 \times 10^{-11})(2.36 \times 10^6)^2} \approx \frac{2270.50 \times 10^{24}}{37.15 \times 10} \approx 6.11 \times 10^{24} \text{ kilograms.}$$

Since the Earth is approximately a sphere of radius 3950 miles or 6360 kilometers, its volume $V$ is approximately,

$$V \approx \frac{4}{3}\pi r^3 \approx \frac{4}{3}\pi (6.36 \times 10^6)^3 \approx 1080 \times 10^{18} \approx 1.08 \times 10^{21} \text{ meters}^3 .$$

Therefore the average density of the Earth is $\frac{M}{V} \approx 5.66 \times 10^3$ kilograms per cubic meter.

**15.** The Moon is approximately a sphere of radius 1740 kilometers. Therefore, its volume is $\frac{4}{3}\pi r^3 \approx \frac{4}{3}\pi (1.74 \times 10^6)^3 \approx 2.21 \times 10^{19}$ cubic meters. Assuming that the Moon has the same average density of $5.66 \times 10^3$ kilograms per cubic meter as the Earth, we get the estimate of $12.5 \times 10^{22}$ kilograms for its mass.

More accurately than determined in Exercise 14, the mass of the Earth is $5.97 \times 10^{24}$ kilograms and its average density is $5.52 \times 10^3$ kilograms per cubic meter. See

http://www.jpl.nasa.gov/earth/earth_fast_facts.html .

The assumption in Exercise 15 that the density of the Moon is the same as that of the Earth turns out to be wrong. Precise values for the average density and mass of the Moon are $3.34 \times 10^3$ kilograms per cubic meter and $7.35 \times 10^{22}$ kilograms, respectively. See

http://sse.jpl.nasa.gov/features/planets/moon/moon.html

Notice that the Earth is denser than the Moon by a factor of $\frac{5.52 \times 10^3}{3.34 \times 10^3} = 1.65$. Incidentally, Newton thought that the Moon was denser than the Earth by a factor of about 1.5. See *A Treatise of the System of the World*, pages 88 – 90.

**16.** Let $d$ be the average distance from the center of the Earth to the center of the Moon and let $x$ be the distance from the center of the Earth to the barycenter **B**. By the Law of the Lever,

$$(6 \times 10^{24})(x) = (7.4 \times 10^{22})(d - x).$$

Using $d = 240{,}000$ miles, we get

$$(6.0 \times 10^{24} + 0.074 \times 10^{24})(x) = (7.4 \times 10^{22})(2.4 \times 10^{5})$$

and hence $x = \frac{17.76 \times 10^{27}}{6.07 \times 10^{24}} \approx 2900$ miles. Because the radius of the Earth is 3950 miles, the barycenter lies approximately 1000 miles below the surface of the Earth.

**Note:** In using both kilograms and miles, Exercise 16 "mixes" units. This is normally a very bad idea. An expensive illustration of this occured when a NASA probe designed to go into orbit around Mars was lost. Incredibly, one of the teams responsible for the probe was working with metric units and another with American units. In Exercise 16 a "loss" was averted because kilograms cancelled out.

**17.** This is a repetition for the Moon of the calculation in Section 7.6 that provided the estimate $g \approx 10$ meters/sec$^2$ for the Earth's gravitational acceleration. Taking $M = 7.4 \times 10^{22}$ kilograms and $r = 1.74 \times 10^6$ meters for the mass and radius of the Moon respectively, we get

$$G\frac{M}{r^2} \approx (6.67 \times 10^{-11})\frac{7.4 \times 10^{22}}{(1.74 \times 10^6)^2} \approx 16.3 \times 10^{-1} \approx 1.63\,\text{meters/sec}^2$$
$$\approx 5.35\,\text{feet/sec}^2.$$

**18.** By Exercise 12, the basketball has a mass of 0.040 slugs. Because weight = mass × gravitational acceleration, we see (using the conclusion of Exercise 17) that the basketball weighs $(0.040)(5.35) = 0.21$ pounds on the Moon (compared to about 1.3 pounds on Earth).

**19.** The computations are the same as those in Example 15 of Section 6.4. Since $a(t) = -5.35$, we get $v(t) = -5.35t$ and $y(t) = -2.68t^2 + 177$. It remains to solve the equation $2.68t^2 = 177$ for $t$. Doing so, we get $t = 8.1$ seconds. The fall took 3.3 seconds on Earth.

**20.** Denote the forces of the Sun and the Earth on the Moon by $F_S$ and $F_E$ respectively. In each case, we will use the formula $F = G\frac{mM}{r^2}$ with $m = 7.4 \times 10^{22}$ kilograms the mass of the Moon. Recall that the distance from the Earth to the Sun is $93 \times 10^6$ miles or $(93)(1.61) \times 10^6$ kilometers = $1.50 \times 10^{11}$ meters. To estimate $F_S$, substitute $M = 2.0 \times 10^{30}$ kilograms to obtain

$$F_S \approx 6.67 \times 10^{-11}\frac{(7.4 \times 10^{22})(2.0 \times 10^{30})}{(1.5 \times 10^{11})^2} \approx 4.4 \times 10^{20} \text{ newtons}.$$

Substituting $M = 6.0 \times 10^{24}$ kilograms and $r = 3.8 \times 10^8$ meters, we get

$$F_E \approx 6.67 \times 10^{-11}\frac{(7.4 \times 10^{22})(6.0 \times 10^{24})}{(3.8 \times 10^8)^2} \approx 2.1 \times 10^{20} \text{ newtons}.$$

So $F_S$ is about twice as strong as $F_E$. To understand why the Moon does not fly off in the direction of the Sun, start by thinking of the Moon and Earth independently, both in orbit around the Sun. This is the essential situation. The gravitational force of the Earth causes the Moon to loop around the Earth as it proceeds around the Sun.

## 7F. Computing Masses and Forces

**21.** Since mass is proportional to volume (density being the constant of proportionality), we get that the Earth, when shrunk to the size of a basketball, would have a mass of about

$$\frac{1}{(5\times 10^7)^3}\cdot 6\times 10^{24} \approx 48 \text{ kilograms} \approx 3.4 \text{ slugs.}$$

Because a basketball has a mass of about 0.04 slugs (see the solution of Exercise 12), this heavier "basketball" is $\frac{3.4}{0.04} \approx 85$ times more massive. Note by way of comparison that a bowling ball has a mass of about 0.5 slugs.

**22.** Assume that Sputnik's orbit was a circle centered at the center of the Earth. Because the radius of the Earth is 3950 miles, the radius of the orbit was $3950 + 560 = 4510$ miles. If the data is consistent, then the constant $\frac{a^3}{T^2}$ for the orbits of Sputnik and the Moon must be the same (because both are satellites of the Earth). Let's check this. For the Moon, $a = 240{,}000$ miles and $T = 27.32$ days; and for Sputnik, $a = 4510$ miles and $T = 95$ minutes $= 0.066$ days. Therefore,

$$\frac{a^3}{T^2} \approx \frac{240{,}000^3}{27.32^2} \approx 1.85\times 10^{13} \text{ miles}^3/\text{day}^2 \quad \text{and}$$

$$\frac{a^3}{T^2} \approx \frac{4510^3}{0.066^2} \approx 2.11\times 10^{13} \text{ miles}^3/\text{day}^2$$

for the Moon and Sputnik respectively. Given that we are working with approximations, this seems close enough. What about Sputnik's speed of 18,000 miles per hour? Because the length of one orbit is $2\pi(4510) \approx 28{,}300$ miles and the period 95 minutes $= 1.58$ hours, Sputnik had a speed of $\frac{28{,}300}{1.58} \approx 17{,}900$ miles per hour. This is also consistent with the report. To compute the mass $M$ of the Earth, we plug Sputnik's data into $M = \frac{4\pi^2a^3}{GT^2}$. Using the conversions 4510 miles $= 2.38\times 10^7$ feet and 95 minutes $= 5.70\times 10^3$ seconds, as well as the fact that $G = 3.36\times 10^{-8}$ in American units (see Exercise 13), we get

$$M = \frac{4\pi^2a^3}{GT^2} \approx \frac{4\pi^2(2.38\times 10^7)^3}{(3.36\times 10^{-8})(5.70\times 10^3)^2} \approx 4.88\times 10^{23} \text{ slugs.}$$

Because 1 slug $= 14.59$ kilograms, this corresponds to $7.12\times 10^{24}$ kilograms. This seems reasonably close to the $6\times 10^{24}$ kilograms derived in Exercise 14.

**Corrections:** Sputnik was launched on October 4th in the year 1957, not on October 5th as stated in the exercise. It turns out, see the data supplied in Exercise 39 of the Additional Exercises, that Sputnik's orbit was not a circle. It orbited from about 370 to 1500 miles above the surface of the Earth.

**23.** If $M$ is the mass of the planet, $m$ the mass of the probe, and $r$ the distance between them, then the force with which the planet attracts the probe is given both by $F = \frac{8\kappa^2 m}{L}\frac{1}{r^2}$ and $F = G\frac{Mm}{r^2}$, where $\kappa$ and $L$ are, respectively, Kepler's constant and the latus rectum of the hyperbolic trajectory. (See the **Overview of the Formulas.)** Therefore, $M = \frac{8\kappa^2}{GL}$. In the

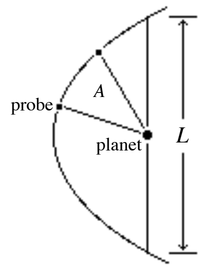



figure above, $A$ is the area traced out by the probe in a certain time $t$. By measuring the distance between the probe and the planet repeatedly, $A$ and $L$ can be estimated. So $\kappa$ can be estimated, and knowing $m$ and $G$ does the rest.

**24.** By Exercise 37 of Chapter 4, the semimajor and semiminor axes of Halley are respectively $a = 17.94\,\text{AU}$ and $b = 4.56\,\text{AU}$. Plugging these data into the formula of Exercise 9, we find that $L = \frac{2b^2}{a} = 2.32$ AU. To estimate the mass $M$ of the Sun, we use the formula $M = \frac{4\pi^2 a^3}{GT^2}$ once more, this time with Halley's orbital data. Because

$$1\,\text{AU} = 9.3 \times 10^7 \text{ miles} = 1.50 \times 10^8 \text{ kilometers} = 1.50 \times 10^{11} \text{ meters},$$

we get $a = 17.94(1.50 \times 10^{11}) = 2.69 \times 10^{12}$ meters. Also, $T = 76$ years $= 2.4 \times 10^9$ seconds. So

$$M = \frac{4\pi^2 a^3}{GT^2} \approx \frac{4\pi^2(2.69 \times 10^{12})^3}{(6.67 \times 10^{-11})(2.4 \times 10^9)^2} \approx 20 \times 10^{29} = 2 \times 10^{30} \text{ kilograms}.$$

This is in agreement with the result obtained in Section 7.6.

**25.** A mass of 0.25 kilograms weighs $(0.25)(9.80) = 2.45$ newtons. Because the "orbit" is a circle, the formula $F_P = \frac{4\pi^2 a^3 m}{T^2}\frac{1}{r_P^2}$ applies with $a = r_P = 0.80$ meters. The fact that there are 3 revolutions per second tells us that $T = 0.33$ seconds. Because $m = 0.25$ kilograms, it follows that

$$F_P \approx \frac{4\pi^2 a^3 m}{T^2}\frac{1}{r_P^2} \approx \frac{4\pi^2(0.80^3)(0.25)}{0.33^2}\frac{1}{0.80^2} \approx 72.5$$

newtons (because everything is in M.K.S.). Notice that 72.5 newtons is equal to about $0.22(72.5) = 16.0$ pounds. The object attached to the string has a mass of about $0.07(0.25) = 0.018$ slugs. It weighs approximately $0.018(32.2) = 0.58$ pounds.

Question: Should gravity have been considered in this problem? Have a look at "For the Instructor" to see what impact it has.

## 7G. A Speculation of Newton

**26.** (The text failed to number this Exercise.) By Exercise 14, the average density of the Earth is $5.66 \times 10^3$ kilograms per cubic meter. So the density is $(5.66 \times 10^3)\frac{0.07}{3.28^3} = 11.23$ slugs per cubic foot. Because the volume of a sphere of radius $r$ is $\frac{4}{3}\pi r^3$, it follows that each sphere has a mass of $\frac{4}{3}\pi\left(\frac{1}{2}\right)^3 \cdot 11.23 \approx 5.88$ slugs. To compute the forces we use the formula $F = G\frac{Mm}{r^2}$ with $G = 3.36 \times 10^{-8}\frac{\text{feet}^3}{\text{slug}\cdot\text{sec}^2}$. (Use the conclusion of Exercise 13). Note that the force is greatest when the spheres are closest, in other words touch each other, and weakest at the start of the motion, when their centers are 1.02 feet apart (as in Figure 7.28). In this last case, the force of attraction is

$$F \approx G\frac{5.88^2}{1.02^2} \approx (3.36 \times 10^{-8})\frac{5.88^2}{1.02^2} \approx 1.12 \times 10^{-6} \text{ pounds.}$$

The same computation shows that when the spheres touch, the attractive force is

$$F \approx G\frac{5.88^2}{1.00^2} \approx (3.36 \times 10^{8})\frac{5.88^2}{1.00^2} \approx 1.16 \times 10^{-6} \text{ pounds.}$$

Suppose that a constant force of $1.16 \times 10^{-6}$ pounds pulls the sphere on the right to the left. By $F = ma$, we get $-1.16 \times 10^{-6} = 5.88a$. So the acceleration $a$ is approximately equal to

$$a = -\frac{1.16 \times 10^{-6}}{5.88} = -0.197 \times 10^{-6} = -1.97 \times 10^{-7} \text{ feet/sec}^2.$$

Suppose that the force begins to act at time $t = 0$ and that the initial velocity of the center of the sphere is 0. Let $v(t)$ and $x(t)$ be the velocity and $x$-coordinate of the center of the sphere at any time $t \geq 0$. Because $v(t) = at$,

$$x(t) = \frac{a}{2}t^2 + x(0) = -(0.985 \times 10^{-7})t^2 + 0.51.$$

The sphere on the right will touch the $y$-axis when $x(t) = 0.50$. At what time will this occur? Solving $0.50 = -(0.985 \times 10^{-7})t^2 + 0.51$ for $t$, we get $t^2 = \frac{0.01}{0.985\times 10^{-7}} = 10.2 \times 10^4$. Therefore, $t \approx 320$ seconds. So the sphere will touch the vertical axis of Figure 7.28 after about $5\frac{1}{3}$ minutes. For the weaker force of $-1.12 \times 10^{-12}$ pounds, we get in the same way that $t^2 = \frac{0.01}{0.95\times 10^{-7}} = 10.5 \times 10^4$, and therefore that $t \approx 324$ seconds. This is again approximately $5\frac{1}{3}$ minutes. Because of the symmetry of the situation, the analysis just carried out also applies to the sphere on the left. As the spheres move from the position specified in Figure 7.28 to the terminal position where they touch each other at the vertical axis, the attractive force on each sphere will increase from its minimum of $1.12 \times 10^{-6}$ pounds to its maximum of $1.16 \times 10^{-6}$ pounds. It follows that the time of the motion of each sphere will fall between the two times computed above. So the spheres will meet in about $5\frac{1}{3}$ minutes. In particular, Newton was wrong when he speculated that the spheres would not come together "in less than a month's time." Why couldn't Newton simply have carried out the above computation?