Solutions to the Exercises of Chapter 6

6A. Derivatives

1. The solutions make use of the differentiation formula from Section 6.1 along with the sum and difference rules for derivatives from Section 5.5.

i. $f(x) = 3\sqrt{x} = 3x^{\frac{1}{2}}$. So $f'(x) = \dfrac{3}{2}x^{-\frac{1}{2}}$.

ii. $g(x) = \dfrac{4}{x^5} = 4x^{-5}$. So $g'(x) = -20x^{-6}$.

iii. $h(x) = -\dfrac{2}{\sqrt[3]{x}} = -2x^{-\frac{1}{3}}$. So $h'(x) = \dfrac{2}{3}x^{-\frac{4}{3}}$.

iv. $f(x) = \dfrac{5}{x^{100}} - 4x^{-\frac{1}{3}} = 5x^{-100} - 4x^{-\frac{1}{3}}$. So $f'(x) = -500x^{-101} + \dfrac{4}{3}x^{-\frac{4}{3}}$.

v. $g(x) = -2x^{\frac{1}{3}} + 3x^5 - 6$. So $g'(x) = -\dfrac{2}{3}x^{-\frac{2}{3}} + 15x^4$.

vi. $\dfrac{dy}{dx} = -\dfrac{2}{7}x^{-\frac{9}{7}} + 120x^3 - \dfrac{5}{12}x^{\frac{2}{3}}$.

6B. Antiderivatives and Definite Integrals

2. **i.** $F(x) = 2 \cdot \dfrac{1}{4}x^4 = \dfrac{1}{2}x^4$.

ii. $F(x) = 5 \cdot \dfrac{x^{\frac{4}{3}}}{\frac{4}{3}} = \dfrac{15}{4}x^{\frac{4}{3}}$.

iii. $F(x) = 3 \cdot \dfrac{x^6}{6} + \dfrac{1}{4} \cdot \dfrac{x^{\frac{9}{7}}}{\frac{9}{7}} = \dfrac{1}{2}x^6 + \dfrac{7}{36}x^{\frac{9}{7}}$.

iv. $F(x) = \dfrac{6}{5}x^5 - \dfrac{3}{8} \cdot \dfrac{x^{\frac{8}{3}}}{\frac{8}{3}} = \dfrac{6}{5}x^5 - \dfrac{9}{64}x^{\frac{8}{3}}$.

3. **i.** $\int_0^4 5x^2\,dx = \frac{5}{3}x^3 \,\Big|_0^4 = \frac{5}{3}4^3 - 0 = \frac{320}{3} = 106\frac{2}{3}$.

$\int_4^6 5x^2\,dx = \frac{5}{3}x^3 \,\Big|_4^6 = \frac{5}{3}\,(6^3 - 4^3) = \frac{5}{3}(216 - 64) = \frac{5}{3}(152) = \frac{760}{3} = 253\frac{1}{3}$.

ii. $\int_0^6 5x^2\,dx = \frac{5}{3}x^3 \,\Big|_0^6 = \frac{5}{3}\,(6^3) = \frac{5}{3}(216) = \frac{1080}{3} = 360$.

iii. $\int_0^5 3\sqrt{x}\,dx = \int_0^5 3x^{\frac{1}{2}}\,dx = 3 \cdot \frac{2}{3}x^{\frac{3}{2}} \,\Big|_0^5 = 2x^{\frac{3}{2}} \,\Big|_0^5 = 2{\cdot}5^{\frac{3}{2}} \approx 22.36$.

vi. $\int_2^4 (4x^3 + 2x^{\frac{1}{3}})\,dx = (x^4 + 2\cdot\frac{3}{4}x^{\frac{4}{3}}) \,\Big|_2^4 = (4^4 + \frac{3}{2}4^{\frac{4}{3}}) - (2^4 + \frac{3}{2}2^{\frac{4}{3}}) = 240 + \frac{3}{2}(4^{\frac{4}{3}} - 2^{\frac{4}{3}}) \approx 245.74$.

Note: In Exercises $4-8$ the student is asked to approximate definite integrals by using the method of Section 6.3. In each exercise there is an accuracy requirement up to a certain decimal place. In order to simplify the computations, the solutions below round off the computations to the desired accuracy right away. Greater accuracy could have been achieved by waiting to the end of the calculation before rounding off to the required degree of accuracy. However, the purpose of the exercises is the practice of the method rather than the degree of accuracy of the answer.

4. Let's start approximating $\int_0^{\frac{3}{4}} \frac{1}{1+x}\,dx$ by copying what was done in Section 6.3 for $\int_0^{\frac{1}{2}} \frac{1}{1+x}\,dx$. So $\int_0^{\frac{3}{4}} \frac{1}{1+x}\,dx \approx \int_0^{\frac{3}{4}} (1-x+x^2-x^3+x^4-x^5)\,dx$. Because $F(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6$ is an antiderivative of $1-x+x^2-x^3+x^4-x^5$, we get by the Fundamental Theorem of Calculus that

$$\begin{aligned}\int_0^{\frac{3}{4}} \frac{1}{1+x}\,dx &\approx F\left(\frac{3}{4}\right) - F(0) = F\left(\frac{3}{4}\right)\\ &= \frac{3}{4} - \frac{1}{2}\left(\frac{3}{4}\right)^2 + \frac{1}{3}\left(\frac{3}{4}\right)^3 - \frac{1}{4}\left(\frac{3}{4}\right)^4 + \frac{1}{5}\left(\frac{3}{4}\right)^5 - \frac{1}{6}\left(\frac{3}{4}\right)^6\\ &\approx 0.75 - 0.2813 + 0.1406 - 0.0791 + 0.0475 - 0.0297 = 0.5480.\end{aligned}$$

Is this answer accurate up to four decimal places? By the "rule of thumb" described in Section 6.3, we need to find the first term in the pattern $\frac{3}{4}, -\frac{1}{2}\left(\frac{3}{4}\right)^2, +\frac{1}{3}\left(\frac{3}{4}\right)^3, -\frac{1}{4}\left(\frac{3}{4}\right)^4, +\frac{1}{5}\left(\frac{3}{4}\right)^5, -\frac{1}{6}\left(\frac{3}{4}\right)^6, \ldots$, that rounds to zero and then keep adding/subtracting up to that term. Let's experiment: Because $+\frac{1}{9}\left(\frac{3}{4}\right)^9 \approx 0.0083$ does not round to zero, we have to go further; because $+\frac{1}{15}\left(\frac{3}{4}\right)^{15} \approx 0.0009$ does not round to zero, we have to go further still; the terms $+\frac{1}{21}\left(\frac{3}{4}\right)^{21}, -\frac{1}{22}\left(\frac{3}{4}\right)^{22}, +\frac{1}{23}\left(\frac{3}{4}\right)^{23}$ all round to ± 0.0001; but $-\frac{1}{24}\left(\frac{3}{4}\right)^{24} \approx 0.00004$ finally rounds to zero. According to the rule of thumb, the approximation process

$$\int_0^{\frac{3}{4}} \frac{1}{1+x}\,dx \approx \frac{3}{4} - \frac{1}{2}\left(\frac{3}{4}\right)^2 + \frac{1}{3}\left(\frac{3}{4}\right)^3 - \frac{1}{4}\left(\frac{3}{4}\right)^4 + \frac{1}{5}\left(\frac{3}{4}\right)^5 - \frac{1}{6}\left(\frac{3}{4}\right)^6 + \ldots$$

must be continued up to and including $+\frac{1}{23}\frac{1}{2}\left(\frac{3}{4}\right)^{23} \approx 0.0001$. In this way, we arrive at the approximation

$$\begin{aligned}\int_0^{\frac{3}{4}} \frac{1}{1+x}\,dx &\approx 0.75 - 0.2813 + 0.1406 - 0.0791 + 0.0475 - 0.0297 + 0.0191\\ &- 0.0125 + 0.0083 - 0.0056 + 0.0038 - 0.0026 + 0.0018 - 0.0013\\ &+ 0.0009 - 0.0006 + 0.0004 - 0.0003 + 0.0002 - 0.0002 + 0.0001\\ &- 0.0001 + 0.0001.\\ &= 0.5595.\end{aligned}$$

Recall that the rule of thumb is not completely precise (again due to roundoff error). The natural logarithm function (see Section 10.3) provides an antiderivative of the function $f(x) = \frac{1}{1+x}$. This fact can be used to show that the value of the integral that is accurate up to four decimal places is 0.5596.

Turn to $\int_0^{\frac{3}{4}} \frac{1}{1+x^2}\,dx$ next. We begin by substituting x^2 for x in the approximation $\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \ldots$, to get

$$\frac{1}{1+x^2} \approx 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12} - x^{14} + \ldots.$$

The fact that $|x^2| < 1$ when $|x| < 1$, tells us that this approximation is valid throughout the interval $\left[0, \frac{3}{4}\right]$. As in the previous case, we begin by using the first six terms. So

$$\int_0^{\frac{3}{4}} \frac{1}{1+x^2}\, dx \approx \int_0^{\frac{3}{4}} (1 - x^2 + x^4 - x^6 + x^8 - x^{10})\, dx.$$

Because $F(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11}$ is an antiderivative of $1 - x^2 + x^4 - x^6 + x^8 - x^{10}$, we get by the Fundamental Theorem of Calculus that

$$\begin{aligned} \int_0^{\frac{3}{4}} \frac{1}{1+x^2}\, dx &\approx F\left(\frac{3}{4}\right) - F(0) = F\left(\frac{3}{4}\right) \\ &= \frac{3}{4} - \frac{1}{3}\left(\frac{3}{4}\right)^3 + \frac{1}{5}\left(\frac{3}{4}\right)^5 - \frac{1}{7}\left(\frac{3}{4}\right)^7 + \frac{1}{9}\left(\frac{3}{4}\right)^9 - \frac{1}{11}\left(\frac{3}{4}\right)^{11} \\ &\approx 0.75 - 0.1406 + 0.0475 - 0.0191 + 0.0083 - 0.0038 \\ &= 0.6423. \end{aligned}$$

Is this accurate enough? Recall from the earlier analysis of $\int_0^{\frac{3}{4}} \frac{1}{1+x}\, dx$ that $+\frac{1}{23}\left(\frac{3}{4}\right)^{23}$ rounds to 0.0001 and that $-\frac{1}{24}\left(\frac{3}{4}\right)^{24}$ rounds to zero. It follows that the approximation

$$\int_0^{\frac{3}{4}} \frac{1}{1+x^2}\, dx \approx \frac{3}{4} - \frac{1}{3}\left(\frac{3}{4}\right)^3 + \frac{1}{5}\left(\frac{3}{4}\right)^5 - \frac{1}{7}\left(\frac{3}{4}\right)^7 + \frac{1}{9}\left(\frac{3}{4}\right)^9 - \frac{1}{11}\left(\frac{3}{4}\right)^{11} + \ \dots$$

must be continued up to and including the term $-\frac{1}{23}\left(\frac{3}{4}\right)^{23}$. Doing so gives us

$$\begin{aligned} \int_0^{\frac{3}{4}} \frac{1}{1+x^2}\, dx &\approx 0.75 - 0.1406 + 0.0475 - 0.0191 + 0.0083 - 0.0038 + 0.0018 \\ &- 0.0009 + 0.0004 - 0.0002 + 0.0001 - 0.0001 \\ &= 0.6434. \end{aligned}$$

The inverse tangent function (see Section 10.5) provides an antiderivative of the function $f(x) = \frac{1}{1+x^2}$. This fact can be used to show that the value of the integral accurate up to four decimal places is 0.6435.

Note: In both of the solutions above, the start was made with the first six terms of the approximation. Why six? We could have saved some energy by starting with fewer. But we need enough terms so as to be able to recognize the pattern in the flow of the terms. This is the essential element. It is needed in finding the first term that rounds to zero, and subsequently in computing the decimal expansion that satisfies the accuracy requirement.

5. We will compute with three decimal places and proceed as in the solution of Exercise 4. Let's start by using the first five terms of the approximation $\frac{x^{\frac{1}{2}}}{1+x} \approx x^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}} - x^{\frac{7}{2}} + x^{\frac{9}{2}} - x^{\frac{11}{2}} + \dots$. In view of the fact that $F(x) = \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}} - \frac{2}{9}x^{\frac{9}{2}} + \frac{2}{11}x^{\frac{11}{2}}$ is an antiderivative of

$x^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}} - x^{\frac{7}{2}} + x^{\frac{9}{2}}$, we get that

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{x^{\frac{1}{2}}}{1+x}\,dx &\approx F\left(\frac{1}{2}\right) = \frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}} - \frac{2}{5}\left(\frac{1}{2}\right)^{\frac{5}{2}} + \frac{2}{7}\left(\frac{1}{2}\right)^{\frac{7}{2}} - \frac{2}{9}\left(\frac{1}{2}\right)^{\frac{9}{2}} + \frac{2}{11}\left(\frac{1}{2}\right)^{\frac{11}{2}}\\ &= 0.236 - 0.071 + 0.025 - 0.010 + 0.004\\ &= 0.184.\end{aligned}$$

Is this the best approximation that our process can deliver? Checking the next several terms we see that $\frac{2}{13}\left(\frac{1}{2}\right)^{\frac{13}{2}} \approx 0.002$, $\frac{2}{15}\left(\frac{1}{2}\right)^{\frac{15}{2}} \approx 0.001$, and finally that $\frac{2}{17}\left(\frac{1}{2}\right)^{\frac{17}{2}} \approx 0$. So the approximation $0.184 - 0.002 + 0.001 = 0.183$ should be better. And it is. An antiderivative of $f(x) = \frac{x^{\frac{1}{2}}}{1+x}$ can be found by the "method of substitution" (see Section 13.3) followed by a polynomial division. This antiderivative (it involves the "inverse tangent") can be used to show that the value of the integral accurate up to three decimal places is 0.183.

6. We start with the first five terms of the approximation $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$. Observe that

$$F(x) = \frac{1}{2}x^2 - \frac{1}{4}\frac{x^4}{3!} + \frac{1}{6}\frac{x^6}{5!} - \frac{1}{8}\frac{x^8}{7!} + \frac{1}{10}\frac{x^{10}}{9!} = \frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \frac{x^{10}}{10!}$$

is an antiderivative of $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$. It follows that

$$\int_0^{\pi} \sin x\,dx \approx F(\pi) - F(0) = \frac{\pi^2}{2} - \frac{\pi^4}{4!} + \frac{\pi^6}{6!} - \frac{\pi^8}{8!} + \frac{\pi^{10}}{10!}.$$

The term $\frac{\pi^{10}}{10!} \approx 0.0258$ is not zero when rounded to four decimals, so we must go further. Because $\frac{\pi^{14}}{14!} \approx 0.0001$ and $\frac{\pi^{15}}{15!} \approx 0.00002$ this last term is the first to round to zero. It follows that the addition/subtraction process must be pursued up to and including the term $+\frac{\pi^{14}}{14!} \approx 0.0001$. Computing $\frac{\pi^2}{2} - \frac{\pi^4}{4!} + \dots + \frac{\pi^{14}}{14!}$ (each term to four decimal accuracy) we get

$$\begin{aligned}\int_0^{\pi} \sin x\,dx &\approx 4.9348 - 4.0587 + 1.3353 - 0.2353 + 0.0258 - 0.0019 + 0.0001\\ &= 2.0001.\end{aligned}$$

What about the actual value? We will see later in Section 8.6 that $-\cos x$ is an antiderivative of $\sin x$. It follows that $\int_0^{\pi} \sin x\,dx = -\cos\pi - (-\cos 0) = 1 + 1 = 2$.

Now on to $\int_0^1 \sin\sqrt{x}\,dx$. We begin by replacing x by $\sqrt{x} = x^{\frac{1}{2}}$ in the approximation $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$ to get the approximation

$$\sin x^{\frac{1}{2}} \approx x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3!} + \frac{x^{\frac{5}{2}}}{5!} - \frac{x^{\frac{7}{2}}}{7!} + \frac{x^{\frac{9}{2}}}{9!} - \cdots.$$

Taking only the first three terms this time, we get $\sin x^{\frac{1}{2}} \approx x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3!} + \frac{x^{\frac{5}{2}}}{5!}$. Because $F(x) = \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}\frac{x^{\frac{5}{2}}}{3!} + \frac{2}{7}\frac{x^{\frac{7}{2}}}{5!}$ is an antiderivative of $x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3!} + \frac{x^{\frac{5}{2}}}{5!}$, we see that

$$\begin{aligned}\int_0^1 \sin\sqrt{x}\,dx &\approx F(1) - F(0) = \frac{2}{3} - \frac{2}{5}\frac{1}{3!} + \frac{2}{7}\frac{1}{5!} \approx 0.6667 - 0.0667 + 0.0024\\ &= 0.6024.\end{aligned}$$

The next relevant term is $\frac{2}{9}\frac{x^{\frac{9}{2}}}{7!}$. Evaluating it at $x = 1$, we get $\frac{2}{9}\frac{1}{7!} \approx 0.00004$ which rounds to zero. This means that the process has already been carried out long enough and that 0.6024 is the required answer.

The precise answer can be obtained as follows. It can be shown that $2\sin x^{\frac{1}{2}} - 2x^{\frac{1}{2}}\cos x^{\frac{1}{2}}$ is an antiderivative of $\sin x^{\frac{1}{2}}$. It follows that the value of the integral is $2\sin 1 - 2\cos 1$. Check that this value rounded to four decimal places is equal to 0.6023.

7. Recall that $\binom{\frac{1}{2}}{1} = \frac{1}{2}$, $\binom{\frac{1}{2}}{2} = -\frac{1}{8}$, $\binom{\frac{1}{2}}{3} = \frac{1}{16}$, $\binom{\frac{1}{2}}{4} = -\frac{5}{128}$ and check that $\binom{\frac{1}{2}}{5} = \frac{105}{3840} = \frac{7}{256}$. The first six terms are more than sufficient to see what is going on. So will we work with the approximation

$$\sqrt{1+x} \;\approx\; 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5.$$

The area in question is given by $\int_0^5 \sqrt{1+x}\,dx$. The expectation is that we will get an approximation of this integral with the previous strategy. Take antiderivatives to get the terms

$$x,\; x+\frac{1}{4}x^2,\; x+\frac{1}{4}x^2-\frac{1}{24}x^3,\; x+\frac{1}{4}x^2-\frac{1}{24}x^3+\frac{1}{64}x^4,\;\; x+\frac{1}{4}x^2-\frac{1}{24}x^3+\frac{1}{64}x^4-\frac{1}{128}x^5$$

and finally, $x + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{64}x^4 - \frac{1}{128}x^5 + \frac{7}{6\cdot 256}x^6$. Evaluating them in succession at $x = 5$ gives us the sequence of numbers 5, 11.25, 6.04, 15.81, -8.60, and 62.61. Rather than closing in on a number (that approximates the integral), these numbers fluctuate. What has gone wrong? The problem is that the approximation

$$\sqrt{1+x} \;\approx\; 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \dots$$

is valid only for $|x| \le 1$ and *not* for $1 < x \le 5$. Therefore this approach cannot be used to evaluate $\int_0^5 \sqrt{1+x}\,dx$. (The integral can be evaluated by the methods of Chapter 13.)

8. Because the derivative of $f(x) = \frac{1}{3}x^3$ is $f'(x) = x^2$, it follows that the length of the arc in question is equal to $\int_{-1}^{1} \sqrt{1+x^4}\; dx$. Starting with the approximation

$$\sqrt{1+x} \;\approx\; 1 + \binom{\frac{1}{2}}{1}x + \binom{\frac{1}{2}}{2}x^2 + \binom{\frac{1}{2}}{3}x^3 + \binom{\frac{1}{2}}{4}x^4,$$

and replacing x by x^4, we get $\sqrt{1+x^4} \approx 1 + \binom{\frac{1}{2}}{1}x^4 + \binom{\frac{1}{2}}{2}x^8 + \binom{\frac{1}{2}}{3}x^{12} + \binom{\frac{1}{2}}{4}x^{16}$. Because $|x| \le 1$ in this problem, the difficulty encountered in Exercise 7 does not arise. By antidifferentiating term by term, we see that

$$F(x) \;=\; x + \binom{\frac{1}{2}}{1}\frac{1}{5}x^5 + \binom{\frac{1}{2}}{2}\frac{1}{9}x^9 + \binom{\frac{1}{2}}{3}\frac{1}{13}x^{13} + \binom{\frac{1}{2}}{4}\frac{1}{17}x^{17}$$

is an antiderivative of $1 + \binom{\frac{1}{2}}{1}x^4 + \binom{\frac{1}{2}}{2}x^8 + \binom{\frac{1}{2}}{3}x^{12} + \binom{\frac{1}{2}}{4}x^{16}$. As a consequence,

$$\begin{aligned}
\int_{-1}^{1} \sqrt{1+x^4}\,dx &\approx F(1)-F(-1)\\
&= \left(1+\binom{\frac{1}{2}}{1}\frac{1}{5}+\binom{\frac{1}{2}}{2}\frac{1}{9}+\binom{\frac{1}{2}}{3}\frac{1}{13}+\binom{\frac{1}{2}}{4}\frac{1}{17}\right)\\
&\quad -\left(-1-\binom{\frac{1}{2}}{1}\frac{1}{5}-\binom{\frac{1}{2}}{2}\frac{1}{9}-\binom{\frac{1}{2}}{3}\frac{1}{13}-\binom{\frac{1}{2}}{4}\frac{1}{17}\right)\\
&= 2+\frac{2}{5}\binom{\frac{1}{2}}{1}+\frac{2}{9}\binom{\frac{1}{2}}{2}+\frac{2}{13}\binom{\frac{1}{2}}{3}+\frac{2}{17}\binom{\frac{1}{2}}{4}\\
&= 2+\frac{2}{5}\frac{1}{2}-\frac{2}{9}\frac{1}{8}+\frac{2}{13}\frac{1}{16}-\frac{2}{17}\frac{5}{128}\\
&\approx 2+0.200-0.028+0.010-0.005\\
&= 2.177.
\end{aligned}$$

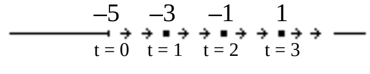
We use our rule of thumb to check for accuracy. A look at the pattern shows that the next term in the approximation of the integral is $\frac{2}{21}\binom{\frac{1}{2}}{5}$. Because $\binom{\frac{1}{2}}{5} = \frac{105}{3840}$, we get $\frac{2}{21}\binom{\frac{1}{2}}{5} \approx 0.003$. Since this does not round to zero (at the third decimal place), we go to $\frac{2}{25}\binom{\frac{1}{2}}{6}$. Check that $\binom{\frac{1}{2}}{6}$ $= -\frac{945}{46080}$ and that $\frac{2}{25}\binom{\frac{1}{2}}{6} \approx -0.002$. Repetitions of this calculation show that $\frac{2}{29}\binom{\frac{1}{2}}{7} \approx 0.001$, $\frac{2}{33}\binom{\frac{1}{2}}{8} \approx -0.001$, $\frac{2}{37}\binom{\frac{1}{2}}{9} \approx 0.001$, and $\frac{2}{41}\binom{\frac{1}{2}}{10} \approx -0.00045$. This last term, finally, rounds to zero at the third decimal place. The required approximation is

$$\begin{aligned}
\int_{-1}^{1} \sqrt{1+x^4}\,dx &\approx 2.177+0.003-0.002+0.001-0.001+0.001\\
&= 2.179.
\end{aligned}$$

Is this answer reasonable? To see that it is, refer to the graph of $y = x^3$ in Figure 5.25. convince yourself that the graph of $f(x) = \frac{1}{3}x^3$ is similar, but flatter. The length of the graph between the points $\left(-1, -\frac{1}{3}\right)$ and $\left(1, \frac{1}{3}\right)$ should therefore be roughly equal to the distance between these two points. Check that it is.

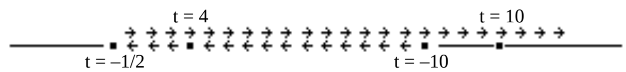
6C. Moving Points

9. Differentiating twice starting with $p(t) = 2t - 5$, we get $v(t) = 2$, and $a(t) = 0$. Because $p(0) = -5$, the point starts at $x = -5$. Because $v(t) = 2$, it moves to the right with a constant velocity of 2 units of distance per unit time. The net force on the particle at any



time is zero, because its acceleration is zero. Its motion is sketched above.

10. Differentiating twice starting with $p(t) = 2t^2 + 2t + 12$ we get $v(t) = 4t + 2$ and $a(t) = 4$. The initial position of the particle is $p(-10) = 200 - 20 + 12 = 192$. Because $v(-10) = -38$, it moves to the left with an initial speed of 38 units of distance per unit time. It continues to move to the left until it stops (notice the particle only stops once) at time $t = -\frac{1}{2}$ at the point $p(-\frac{1}{2}) = \frac{1}{2} - 1 + 12 = 11\frac{1}{2}$. After $t = -\frac{1}{2}$, however, $v(t)$ is positive so that the particle moves to the right. It continues to move to the right with greater and greater velocity. Because the acceleration is a positive constant, we can regard this motion to be the result of a constant force that pushes the particle to the right. Initially, this force slows the particle's velocity (which is to the left); then it stops the particle (at $t = -\frac{1}{2}$); thereafter it propels the particle to the right. The direction of the motion is sketched below. The three points singled out on

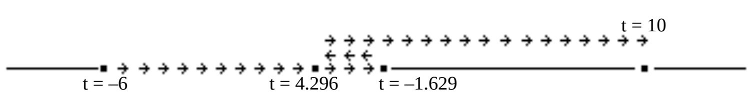


the axis are $p(-10) = 192$, $p(-\frac{1}{2}) = 11\frac{1}{2}$, $p(4) = 52$, and $p(10) = 232$.

11. Differentiating $p(t) = t^3 - 4t^2 - 21t$ twice, we get $v(t) = 3t^2 - 8t - 21$ and $a(t) = 6t - 8$. The initial position of the particle is $p(-6) = -216 - 144 - 126 = -486$ and its initial velocity is $v(-6) = 135$. So initially it moves to the right. Applying the quadratic formula to $3t^2 - 8t - 21 = 0$, tells us that it stops when

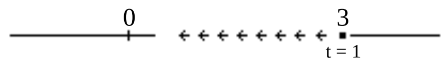
$$t = \frac{8 \pm \sqrt{64 + 4 \cdot 3 \cdot 21}}{6} = \frac{8 \pm 2\sqrt{16 + 63}}{6} = \frac{4 \pm \sqrt{79}}{3}.$$

So the particle stops at $t_1 = \frac{4-\sqrt{79}}{3} \approx -1.629$ and $t_2 = \frac{4+\sqrt{79}}{3} \approx 4.296$. The acceleration is $a(t) = 6t - 8 = 6(t - \frac{4}{3})$. So the acceleration is negative until $t = \frac{4}{3}$ and positive thereafter. View the motion in terms of the force that produces the acceleration. The force acts to the left until $t = \frac{4}{3}$ and to the right thereafter. Initially, at $t = -6$, the velocity is to the right and the force is to the left. So the particle slows down and (as we have seen) stops at $t_1 \approx -1.629$ at the point $p(t_1) \approx -19.27$. Since the force continues to act to the left, the particle now begins to move to the left. But after $t = \frac{4}{3}$ the force acts to the right. The particle slows down once more, until it stops for the second time at $t_2 \approx 4.296$ at the point $p(t_2) \approx -84.75$. Because the force is positive thereafter, the particle moves to the right forever after with greater and greater speed. For example, when $t = 10$, it is at the point $p(10) = 490$ on the axis moving with a velocity of $v(10) = 199$. The direction of the motion is sketched below. The points



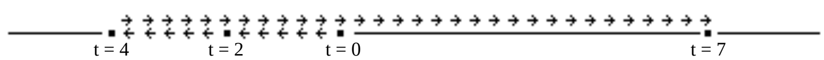
singled out on the axis are $p(-6) = -486$, $p(t_1) \approx p(-1.629) \approx -19.27$, $p(t_2) \approx p(4.296) \approx -84.75$ and $p(10) = 490$.

12. Differentiating $p(t) = \frac{3}{t} = 3t^{-1}$, we get $v(t) = -3t^{-2} = -\frac{3}{t^2}$ and $a(t) = 6t^{-3} = \frac{6}{t^3}$. The particle starts at $p(1) = 3$ with an initial velocity of $v(1) = -3$. So it is moving to the left. A look at $v(t)$ shows that it never stops. As time $t > 1$ continues to elapse, both $p(t)$ and $v(t)$ become smaller and smaller (with $p(t)$ always positive and $v(t)$ always negative). This means that the particle moves closer and closer to the origin and (hardly a surprise) becomes slower and slower in the process. The direction of the motion is sketched below. What is the interpretation in terms of an acting force?

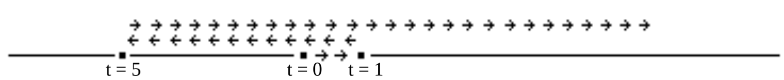


13. **a.** Antidifferentiating $a(t) = 6t - 12$, tells us that $v(t) = 3t^2 - 12t + C$ for some constant C. Because $v(0) = 0$, we see that $C = 0$. So $v(t) = 3t^2 - 12t$. Antidifferentiating again, we get $p(t) = t^3 - 6t^2 + D$ for some constant D. Because $p(0) = 0$, we see that $D = 0$. So $p(t) = t^3 - 6t^2$.

b. The particle starts at $p(0) = 0$ with a velocity of $v(0) = 0$ and a negative acceleration of $a(0) = -12$. Consequently, the particle begins by moving to the left. Its velocity $v(t) = 3t(t-4)$ is zero again at $t = 4$. The particle is now at $p(4) = -32$. It stops moving to the left at this point and starts moving to the right. Note that $v(t) \geq 0$ for $t > 4$. So the particle continues its motion to the right with ever increasing velocity. The analysis in terms of an acting force is in essence the same as that in Exercise 11. The motion is diagrammed above. The points $p(0) = 0, p(2) = -16, p(4) = -32$, and $p(7) = 49$ are singled out.



14. Proceeding as in Exercise 13, we get $v(t) = t^2 - 6t + 5$ and $p(t) = \frac{1}{3}t^3 - 3t^2 + 5t + 6$. Because $v(t) = (t-1)(t-5)$, the particle stops at $t = 1$ and $t = 5$. during the time interval from $t = 0$ to $t = 1, v(t)$ is positive, so that the particle moves to the right. Having stopped at $p(1) = 8\frac{1}{3}$, the particle moves to the left between $t = 1$ and $t = 5$ ($v(t)$ is negative). When $t = 5$, the particle stops again, this time at $p(5) = \frac{125}{3} - 75 + 25 + 6 = -2\frac{1}{3}$. Thereafter it moves to the right with ever increasing velocity. The diagram of the motion is sketched below. The points on the axis that are singled out are $p(0) = 5$, $p(1) = 8\frac{1}{3}$, and $p(5) = -2\frac{1}{3}$.



6D. Projectiles

Note: Air resistance will be ignored in all the exercises of this section, not just Exercise 17.

15. Plugging the given data into equation (6d), tells us that the maximal height is

$$\frac{1}{2g}v_0^2 \sin^2 \varphi_0 + y_0 = \frac{1}{2\cdot 32} \cdot 40^2(0.342)^2 + 5 = \frac{1}{64}(187.14) + 5 = 7.92 \text{ feet.}$$

By equation (6a), $x(t) = (40 \cdot \cos 20°)t$, so we get that the apple will arrive at Hooke's position after $t = \frac{35}{40\cos 20°} = \frac{35}{40(0.940)} = 0.931$ seconds. At this time, by equation (6b),

$$\begin{aligned} y(t) &= -16(0.931)^2 + 40(0.342)(0.931) + 5 = -13.87 + 12.74 + 5 \\ &= 3.9 \text{ feet.} \end{aligned}$$

So the apple will hit Hooke. By equation (6h), the speed with which it will hit Hooke is

$$\sqrt{40^2 + 32^2(0.931)^2 - 2(32)(40)(0.342)(0.931)} = 41 \text{ feet/sec.}$$

Note: Hooke was small, but it is safe to assume that he was not a midget (and taller than 3.9 feet). Hooke and Newton quarrelled over scientific matters for years. Newton closed one of his friendlier letters to Hooke with "if I have seen further it has only been by standing on the shoulders of giants." This has been widely interpreted as a nasty allusion to Hooke's smallish stature.

16. The dropping has an initial velocity of $v_0 = 20$ feet per second, an angle of departure $\varphi_0 = 30°$, and a starting height $y_0 = 50$. The maximal height reached by the dropping is gotten by using equation (6d):

$$\frac{1}{2g}v_0^2 \sin^2 \varphi_0 + y_0 = \frac{1}{2 \cdot 32} \cdot 20^2(0.50)^2 + 50 = \frac{1}{64}(100) + 50 = 51\frac{9}{16} \approx 51.6 \text{ feet.}$$

Let $t = 0$ be the instant the dropping is released and let $y(t)$ be the height of the dropping above the ground at any time $t \geq 0$. At what time t after it is released does the dropping reach its maximal height? When $y(t) = 51\frac{9}{16}$. Because $y(t) = -\frac{g}{2}t^2 + (v_0 \sin \varphi_0)t + y_0$ for any t (see equations (6b)), we must solve

$$-16t^2 + (20)(0.5)t + 50 = -16t^2 + 10t + 50 = 51\frac{9}{16}$$

for t. Solving $-16t^2 + 10t - 1\frac{9}{16} = -16t^2 + 10t - \frac{25}{16} = 0$ by the quadratic formula, we get

$$t = \frac{-10 \pm \sqrt{100 - (4)(-16)\left(-\frac{25}{16}\right)}}{-32} = \frac{-10 \pm \sqrt{100 - 100}}{-32} = \frac{5}{16} \approx 0.31 \text{ sec}.$$

To determine whether the dropping will hit the house, we first find the time t when the dropping is a horizontal distance of 30 feet from where it started. If the dropping hits the house, it will do so at this time. To find it, we set $x(t) = 30$ in $x(t) = (v_0 \cos \varphi_0)\, t$ and solve for t. Doing this, we get $20(\frac{\sqrt{3}}{2})t = 30$ and hence $t = \frac{3}{\sqrt{3}} = \sqrt{3} \approx 1.73$ sec. Plugging t into second equation of (6b), we see that the height of the dropping at this time will be $y(t) = -16(\sqrt{3})^2 + 10(\sqrt{3}) + 50 \approx 19.3$ feet. Because the wall of Newton's house is 20 feet high, the dropping will hit the wall. To get the velocity of "splatter" we plug $t = \sqrt{3}$ as well as the given data into equation (6h), to get $\sqrt{v_0^2 + g^2t^2 - 2g(v_0 \sin \varphi_0)t} \approx 52.9$ feet/sec.

17. In this problem $y_0 = 1.5$ meters and $\varphi_0 = 70°$. We are looking for v_0. Formula (6c) asserts that any point (x, y) on the trajectory of the arrow satisfies

$$y = \left(\frac{-g}{2v_0^2 \cos^2 \varphi_0}\right) x^2 + (\tan\ \varphi_0)\, x + y_0.$$

We are given that the point (25, 55) is on the trajectory. If we plug all of our information into this equation, we should be able to solve for v_0. Plugging in, we get

$$25 = \left(\frac{-9.8}{2v_0^2\cos^2 70°}\right)55^2 + (\tan 70°)\,55 + 1.5 \approx -\frac{41.89}{v_0^2}\,55^2 + (2.75)55 + 1.5.$$

Therefore, $\frac{41.89}{v_0^2}\,55^2 = -25 + 151.25 + 1.5 = 127.75$. So $v_0^2 \approx \frac{(41.89)(55^2)}{127.75}$ and hence $v_0 \approx 31$ meters per second.

18. Here $y_0 = 6$ feet and $v_0 = 120$ feet per second and we are looking for φ_0. Let's repeat the strategy used in Exercise 17. Plugging what we know into

$$y = \left(\frac{-g}{2v_0^2\cos^2\varphi_0}\right)x^2 + (\tan\varphi_0)\,x + y_0\,,$$

we get $62 = \left(\frac{-32}{2(120^2)\cos^2\varphi_0}\right)240^2 + (\tan\,\varphi_0)240 + 6$. Using the fact that $\frac{1}{\cos\varphi_0} = \sec\varphi_0$, we can rewrite this as $62 = -16(4)(\sec^2\varphi_0) + 240(\tan\,\varphi_0) + 6$. Because $\sec^2\varphi_0 = 1 + \tan^2\varphi_0$, we get $62 = -16(4)(\tan^2\varphi_0) + 240(\tan\,\varphi_0) - 16(4) + 6$. Therefore,

$$64(\tan^2\varphi_0) - 240(\tan\,\varphi_0) + 120 = 0\,.$$

By the quadratic formula, $\tan\,\varphi_0 = \frac{240\pm\sqrt{240^2-4(64)(120)}}{2\cdot 64}$. Because $240 = 16\cdot 15$, notice that $16^2 = 256$ is a factor of both terms under the radical. Therefore,

$$\tan\,\varphi_0 = \frac{240\pm 16\sqrt{15^2-120}}{2\cdot 64} = \frac{240\pm 16\sqrt{105}}{2\cdot 64} = \frac{15\pm\sqrt{105}}{8} \approx 3.16 \text{ or } 0.594.$$

By pushing "inverse tan" on your calculator, you will get either $\varphi_0 \approx 72.4°$ or $\varphi_0 \approx 30.7°$. Are there two different angles with which the arrow can be shot off so as to hit the target? Intuitively, if the arrow has the steeper trajectory, it will gain altitude earlier and will then descend towards the cauldron as the archer intends. With the flatter trajectory it would seem that the arrow will approach the cauldron from below and hit against its wall. To see more convincingly that this is the case, consider the steeper trajectory $\varphi_0 \approx 72.4°$. Refer to the discussion that develops equation (6d) and note that the arrow will reach its maximal height at time $t_1 = \frac{v_0\sin\varphi_0}{g} \approx \frac{(120)(0.953)}{32} \approx 3.57$ seconds. Thereafter, it will descend. By one of the equations in (6a), the arrow will reach its target $(240, 62)$ at time $t = \frac{240}{v_0\cos\varphi_0} \approx \frac{240}{(120)(0.302)} \approx$ 6.62 seconds. So the flaming arrow will hit the target on its descent.

19. The relevant data is $y_0 = 8$ feet, $v_0 = 22$ feet per second, and $\varphi_0 = 45°$. Let the shot be released at time $t = 0$. Suppose we can find the time t at which the bottom of the ball will be at a height of 10 feet above the floor on its descent. (Recall that the rim of the basket is 10 feet above the floor.) Then we can determine the horizontal distance that the ball has traveled during this time. If the shot is taken at this distance from the basket, the ball will go through the hoop (if, as we will assume, the ball travels in the correct direction). So the first question is what t gives us $y(t) = 10$? To get the answer, take $y(t) = 10$ in equation (6b) and solve for t. By substituting,

$$10 = -\frac{g}{2}t^2 + (v_0\sin\,\varphi_0)t + y_0 = -16t^2 + 22\cdot\frac{\sqrt{2}}{2}t + 8 = -16t^2 + 11\sqrt{2}t + 8\,.$$

By applying the quadratic formula to $16t^2 - 11\sqrt{2}t + 2 = 0$, we get

$$t = \frac{11\sqrt{2} \pm \sqrt{242 - 4 \cdot 16 \cdot 2}}{32} = \frac{11\sqrt{2} \pm \sqrt{114}}{32} \approx \frac{15.56 \pm 10.68}{32} \approx 0.820 \text{ or } 0.152\,.$$

At $t = 0.152$ seconds, the ball will be 10 feet above the floor on its ascent (by the argument already used in Exercise 18). So $t = 0.820$ seconds is the time of interest. By equation (6a), $x(0.820) \approx 22 \cdot \frac{\sqrt{2}}{2} \cdot 0.820 \approx 12.76$. So if the shot is taken at a distance of about $12\frac{3}{4}$ feet from basket, then the bottom of the ball will be 10 feet above the floor on its descent. To maximize the likelihood of scoring, the player should take his jump shot from about this distance.

20. We will determine the slope of the tangent line of the trajectory at the point of impact. The fact that the slope of a line is equal to the tangent of the angle that the line makes with the horizontal does the rest. (An explicit formula for the angle at impact, rather than the slope at impact requires the inverse tangent function which will be developed in Section 10.5). Equation (6c) tells us that the trajectory of the projectile is the parabola

$$y = \left(\frac{-g}{2v_0^2\cos^2\varphi_0}\right)x^2 + (\tan\varphi_0)x + y_0.$$

The slope of the tangent line at any point (x, y) on the trajectory is given by the derivative $\frac{dy}{dx} = (\frac{-g}{v_0^2\cos^2\varphi_0})x + \tan\varphi_0$. Equation (6a) tells us that $x = (v_0\cos\varphi_0)t_{\text{imp}}$ at the time of impact and by equation (6e), $t_{\text{imp}} = \frac{v_0\sin\varphi_0 + \sqrt{v_0^2\sin^2\varphi_0 + 2gy_0}}{g}$. Therefore at impact, $x = (v_0\cos\varphi_0)\frac{v_0\sin\varphi_0 + \sqrt{v_0^2\sin^2\varphi_0 + 2gy_0}}{g}$. So at the point of impact,

$$\begin{aligned}
\frac{dy}{dx} &= -\left(\frac{1}{v_0\cos\varphi_0}\right)\left(v_0\sin\varphi_0 + \sqrt{v_0^2\sin^2\varphi_0 + 2gy_0}\right) + \tan\varphi_0 \\
&= -\frac{\sin\varphi_0}{\cos\varphi_0} - \left(\frac{1}{v_0\cos\varphi_0}\right)\left(\sqrt{v_0^2\sin^2\varphi_0 + 2gy_0}\right) + \tan\varphi_0 \\
&= -\left(\frac{1}{v_0\cos\varphi_0}\right)\left(\sqrt{v_0^2\sin^2\varphi_0 + 2gy_0}\right).
\end{aligned}$$

There is another way of deriving this formula. Begin by drawing Figure 6.20 not at the time $t = 0$, but at time $t = t_{\text{imp}}$. Doing this will show that the slope just computed is also equal to $\frac{y'(t_{\text{imp}})}{x'(t_{\text{imp}})}$. By equations (6a) and (6b),

$$\frac{y'(t_{\text{imp}})}{x'(t_{\text{imp}})} = \frac{-g\,t_{\text{imp}} + v_0\sin\varphi_0}{v_0\cos\varphi_0} = -\frac{g\,t_{\text{imp}}}{v_0\cos\varphi_0} + \tan\varphi_0.$$

Substituting $t_{\text{imp}} = \frac{v_0\sin\varphi_0 + \sqrt{v_0^2\sin^2\varphi_0 + 2gy_0}}{g}$ into this expression provides the formula derived earlier.

6E. Ballistics

21. This is an application of the range equation

$$R = \frac{v_0^2}{2g}\sin(2\varphi_0) + \frac{v_0}{g}\sqrt{\frac{v_0^2}{4}\sin^2(2\varphi_0) + 2gy_0\cos^2\varphi_0}\ .$$

The data to be substituted is $v_0 = 1439$ feet per second, $\varphi_0 = 0°$, and $y_0 = 3.6$ feet (see Section 6.6). Because $\sin 0° = 0$ and $\cos 0° = 1$, we get $R = \frac{1439}{32}\sqrt{2(32)(3.6)} \approx 680$ feet. Figure 6.22 lists the range as 318 yards and hence 954 feet. So the observed range of the actual shot is considerable greater than the range predicted by the theory, even though the former occurred against air resistance and the latter assumes no air resistance. The inescapable conclusion is that there is a problem with the data in *The Artillerist's Manual*. Since the range would seem to be easy to measure, it is likely that the angle of departure and/or the muzzle velocity are inaccurate. Suppose, for instance, that the angle of departure and the muzzle velocity were in fact 0.2° and 1480 feet per second, instead of 0° and 1439 feet per second. What is the predicted range under those assumptions? This time we get

$$\begin{aligned} R &= \frac{1480^2}{2(32)}\sin 0.4° + \frac{1480}{32}\sqrt{\frac{1480^2}{4}\sin^2 0.4° + 2(32)(3.6)\cos^2 0.2°} \\ &\approx 238.93 + 46.25\sqrt{26.69 + 230.40} \approx 980 \text{ feet.} \end{aligned}$$

Now, as expected, the predicted range exceeds the actual range.

Correction: In the statement of Exercise 22 the word is "spherical" and not "sperical" and in the statement of Exercise 23 the word is "Figure" and not "Figuree."

22. We need to plug the data $v_0 = 1357$ feet per second, $\varphi_0 = 2°30'$, and $y_0 = 3.6$ feet into equation (6e). Doing so, we get

$$\begin{aligned} t_{\text{imp}} &= \frac{v_0\sin\varphi_0 + \sqrt{v_0^2\sin^2\varphi_0 + 2gy_0}}{g}. \\ &= \frac{1357\sin 2.5° + \sqrt{357^2\sin^2 2.5° + 2(32)(3.6)}}{32} \\ &\approx \frac{59.19 + \sqrt{3503.63 + 230.40}}{32} \approx \frac{59.19 + 61.11}{32} \\ &\approx 3.76 \approx 3.8 \text{ seconds.} \end{aligned}$$

To get the predicted range we plug the same data into the range equation (see the solution of Exercise 21) to get

$$\begin{aligned} R &= \frac{1357^2}{2(32)}\sin 5° + \frac{1357}{32}\sqrt{\frac{1357^2}{4}\sin^2 5° + 2(32)(3.6)\cos^2 2.5°} \\ &\approx 2507.70 + 42.41\sqrt{3496.97 + 229.96} \approx 5100 \text{ feet.} \end{aligned}$$

Figure 6.22 supplies the data $t_{\text{imp}} = 3$ seconds and $R = 840$ yards $= 2520$ feet. Again, air resistance and probable inaccuracies in the data explain the differences between the observed and predicted values. See the comments in the solution of Exercise 21.

23. This is another application of the range equation

$$R = \frac{v_0^2}{2g}\sin(2\varphi_0) + \frac{v_0}{g}\sqrt{\frac{v_0^2}{4}\sin^2(2\varphi_0) + 2gy_0\cos^2\varphi_0}\ ,$$

this time with $v_0 = 1486$ feet and φ_0 equal to $0°, 1°$, and $5°$. We will assume that the muzzle of the 12-pdr. field gun is the same $y_0 = 3.6$ feet from the ground as that of the 6-pdr. field gun. Taking $\varphi_0 = 0°$ we get

$$R \;=\; \frac{v_0}{g}\sqrt{2gy_0} \;=\; \frac{1486}{32}\sqrt{(64)(3.6)} \;\approx\; (46.44)(15.18) \;\approx 700 \text{ feet.}$$

With $\varphi_0 = 1°$ we get

$$\begin{aligned} R \;&=\; \frac{1486^2}{64}\sin 2° + \frac{1486}{32}\sqrt{\frac{1486^2}{4}\sin^2 2° + 64(3.6)\cos^2 1°} \\ &\approx\; 1204.14 + 46.44\sqrt{672.38 + 230.33} \;\approx\; 2600 \text{ feet.} \end{aligned}$$

Finally with $\varphi_0 = 5°$ we get

$$\begin{aligned} R \;&=\; \frac{1486^2}{64}\sin 10° + \frac{1486}{32}\sqrt{\frac{1486^2}{4}\sin^2 10° + 64(3.6)\cos^2 5°} \\ &\approx\; 5991.39 + 46.44\sqrt{16646.31 + 228.65} \;\approx\; 12{,}000 \text{ feet.} \end{aligned}$$

The observed ranges from Figure 6.22 are respectively, 347 yards, 662 yards, and 1663 yards, or 1041 feet, 1986 feet, and 4,989 feet. The discrepancies between the theoretical and observed distances are again explained by air resistance and possible inaccuracies of the data.

6F. Connections with Probability Theory

Correction: In the statement of Exercise 24 the inequality $v \le 900$ should be replaced by $v \le 890$ in the three places where it occurs. This is because in Figure 6.27 the interval under the curve from 0 to 0.6 corresponds to the range of velocities from 884 to 890, and not to those from 884 to 900. Incidentally, the "assumption" that the area under the entire graph is π is correct. (This follows from facts in Section 10.5.)

24. We will compute with an accuracy to three decimal places and start with the velocity range $880 \le v \le 884$. We need to compute the area under the graph of $y = \frac{1}{1+x^2}$. To do so, we will use the strategy in the solution of the of Exercise 4 (second problem) and the approximation

$$\frac{1}{1+x^2} \;\approx\; 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \dots\,.$$

established there. Because $F(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11}$ is an antiderivative of $1 - x^2 + x^4 - x^6 + x^8 - x^{10}$, we get

$$\begin{aligned}
\int_{-0.4}^{0} \frac{1}{1+x^2}dx &\approx F(0) - F(-0.4) = -F(-0.4) \\
&= -\left(-0.4 - \frac{1}{3}(-0.4)^3 + \frac{1}{5}(-0.4)^5 - \frac{1}{7}(-0.4)^7 + \frac{1}{9}(-0.4)^9 - \frac{1}{11}(-0.4)^{11}\right) \\
&\approx -(-0.400 + 0.021 - 0.002 + 0 - 0 + 0) \\
&\approx 0.38.
\end{aligned}$$

Therefore, the probability that a velocity measurement v falls in the range $880 \le v \le 884$ is approximately $\frac{0.38}{\pi} \approx 0.12$.

We turn to $884 \le v \le 890$ next. Proceeding as above, we get:

$$\begin{aligned}
\int_{0}^{0.6} \frac{1}{1+x^2}dx &\approx F(0.6) - F(0) = F(0.6) \\
&= 0.6 - \frac{1}{3}(0.6)^3 + \frac{1}{5}(0.6)^5 - \frac{1}{7}(0.6)^7 + \frac{1}{9}(0.6)^9 - \frac{1}{11}(0.6)^{11} \\
&\approx 0.600 - 0.072 + 0.016 - 0.004 + 0.001 - 0 \\
&\approx 0.54.
\end{aligned}$$

So the probability that a velocity measurement v falls in the range $884 \le v \le 890$ is approximately $\frac{0.54}{\pi} \approx 0.17$.

By combining the results just achieved, we see that the probability that a given observed velocity v falls into the range $880 \le v \le 890$ is approximately $0.12 + 0.17 = 0.29$. In the actual test only one of fifteen velocities fell into this range for a probability of $\frac{1}{15} = 0.06$. This suggests that the curve of Figure 6.27 might not provide a good probabilistic model for this problem. On the other hand, the 15 test firings of this analysis are too few to provide reliable conclusions. A more serious study of the probabilities in question would have required many more. Exercise 57 of Chapter 10 discusses an entire family of bell shaped curves that is often used in the determination of probabilities.