## Solutions to the Exercises of Chapter 5

### 5A. Lines and Their Equations

**1.** The slope is $\frac{-3-2}{2-(-6)} = \frac{-5}{8}$. Since $(2,-3)$ is a point on the line, $y-(-3) = \frac{-5}{8}(x-2)$ is an equation of the line in point-slope form. This simplifies to $y = -\frac{5}{8}x - \frac{17}{4}$.

**2.** Using the slope-intercept form of the equation of a line, we get $y = -3x + 4$.

**3.** The point-slope form of the equation is $y-(-2) = \frac{1}{2}(x-3)$. In simplified form it is $y = \frac{1}{2}x - \frac{7}{2}$.

**4.** Rewriting the equation as $7y = -2x - 2$ and then as $y = \frac{-2}{7}x - \frac{2}{7}$ puts it into slope-intercept form. So the slope is $\frac{-2}{7}$ and the $y$-intercept is $\frac{-2}{7}$.

### 5B. Computing Slopes of Tangents

**5.** Take $Q = (2+\Delta x, 4+\Delta y)$ on the graph. So $4+\Delta y = (2+\Delta x)^2 = 4 + 4\Delta x + (\Delta x)^2$ and hence $\Delta y = 4\Delta x + (\Delta x)^2 = \Delta x(4+\Delta x)$. Dividing both sides by $\Delta x$, gives $\frac{\Delta y}{\Delta x} = 4 + \Delta x$ and hence $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = 4$. So $m_P = 4$.

**6.** Take $(2+\Delta x, 8+\Delta y)$ on the graph. So

$$8+\Delta y = (2+\Delta x)^3 = 2^3 + 3{\cdot}2^2\Delta x + 3{\cdot}2(\Delta x)^2 + (\Delta x)^3, \text{and}$$

$$\Delta y = 3{\cdot}2^2\Delta x + 3{\cdot}2(\Delta x)^2 + (\Delta x)^3 = \Delta x(3{\cdot}2^2 + 3{\cdot}2\Delta x + (\Delta x)^2).$$

Dividing both sides by $\Delta x$, gives $\frac{\Delta y}{\Delta x} = 3{\cdot}2^2 + 3\cdot 2\Delta x + (\Delta x)^2 = 12 + 6\Delta x + (\Delta x)^2$ and hence $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = 12$. So $m_P = 12$.

**7.** Take $(x+\Delta x, y+\Delta y)$ on the graph. So $y+\Delta y = \frac{1}{x+\Delta x}$ and hence $\Delta y = \frac{1}{x+\Delta x} - y$. Taking common denominators, we get $\Delta y = \frac{1-yx-y\Delta x}{x+\Delta x}$. Because $y = \frac{1}{x}$, it follows that $\Delta y = \frac{-y\Delta x}{x+\Delta x}$. So $\frac{\Delta y}{\Delta x} = \frac{-y}{x+\Delta x}$. Pushing $\Delta x$ to zero, we see that $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \frac{-y}{x}$. So $m_P$ is equal to $\frac{-y}{x}$.

**Note:** The hint supplied for the solution of Exercise 8 is not relevant.

**8.** Take $(x+\Delta x, y+\Delta y)$ the graph. So $(y+\Delta y)^3 = x + \Delta x$, and after multiplying out, $y^3 + 3y^2\Delta y + 3y\Delta y^2 + (\Delta y)^3 = x + \Delta x$. Since $(x,y)$ is on the graph, $y^3 = x$ and hence $3y^2\Delta y + 3y\Delta y^2 + (\Delta y)^3 = \Delta x$. Factoring out a $\Delta y$ and dividing by $\Delta x$, now gives us that $\Delta y(3y^2 + 3y\Delta y + (\Delta y)^2) = \Delta x$ and $\frac{\Delta y}{\Delta x}(3y^2 + 3y\Delta y + (\Delta y)^2) = 1$. So $\frac{\Delta y}{\Delta x} = \frac{1}{3y^2+3y\Delta y+(\Delta y)^2}$. Now push $\Delta x$ to zero. In the process $\Delta y$ goes to zero, and therefore, $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \frac{1}{3y^2}$. Since $y^3 = x, y = x^{\frac{1}{3}}$, and $y^2 = x^{\frac{2}{3}}$. So $m_P$ is equal to $\frac{1}{3y^2}$ or $\frac{1}{3}x^{-\frac{2}{3}}$.

**9.** Take $(x+\Delta x, y+\Delta y)$ on the ellipse. So $\frac{(x+\Delta x)^2}{5^2} + \frac{(y+\Delta y)^2}{4^2} = 1$. Therefore,

$$\frac{x^2 + 2x\Delta x + (\Delta x)^2}{5^2} + \frac{y^2 + 2y\Delta y + (\Delta y)^2}{4^2} = 1,$$

and hence $\frac{x^2}{5^2}+\frac{2x\Delta x+(\Delta x)^2}{5^2}+\frac{2y\Delta y+(\Delta y)^2}{4^2}+\frac{y^2}{4^2}=1$. Since $\frac{x^2}{5^2}+\frac{y^2}{4^2}=1$, $\frac{2x\Delta x+(\Delta x)^2}{5^2}+\frac{2y\Delta y+(\Delta y)^2}{4^2}=0$. So $\frac{2y\Delta y+(\Delta y)^2}{4^2}=-\frac{2x\Delta x+(\Delta x)^2}{5^2}$ and hence $\Delta y(2y+\Delta y)=-\frac{4^2}{5^2}\Delta x(2x+\Delta x)$. Therefore, $\frac{\Delta y}{\Delta x}=-\frac{4^2}{5^2}\frac{2x+\Delta x}{2y+\Delta y}$. Since $\Delta y$ goes to 0 when $\Delta x$ is pushed to 0, we now find that $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=-\frac{4^2}{5^2}\frac{2x}{2y}=-\frac{4^2}{5^2}\frac{x}{y}$. So $m_P=-\frac{4^2}{5^2}\frac{x}{y}$.

## 5C. Derivatives

**10.** Note that

$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{(x+\Delta x)^3-x^3}{\Delta x}=\frac{x^3+3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3-x^3}{\Delta x}=\frac{3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3}{\Delta x}=3x^2+3x\Delta x+(\Delta x)^2$.

Pushing $\Delta x$ to zero, tells us that $f'(x)=3x^2$.

**11.** Observe that $\frac{h(x+\Delta x)-h(x)}{\Delta x}=\frac{\frac{1}{(x+\Delta x)^2}-\frac{1}{x^2}}{\Delta x}$. Working with the numerator, we get

$$\frac{1}{(x+\Delta x)^2}-\frac{1}{x^2}=\frac{x^2-(x+\Delta x)^2}{(x+\Delta x)^2x^2}=\frac{x^2-x^2-2x\Delta x-(\Delta x)^2}{(x+\Delta x)^2x^2}$$
$$=\frac{-2x\Delta x-(\Delta x)^2}{(x+\Delta x)^2x^2}=\frac{\Delta x(-2x-\Delta x)}{(x+\Delta x)^2x^2}.$$

It follows that $\frac{\frac{1}{(x+\Delta x)^2}-\frac{1}{x^2}}{\Delta x}=\frac{1}{\Delta x}\frac{\Delta x(-2x-\Delta x)}{(x+\Delta x)^2x^2}=\frac{-2x-\Delta x}{(x+\Delta x)^2x^2}$. Pushing $\Delta x$ to zero, gives us

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{-2x}{x^2\cdot x^2}=-2\frac{1}{x^3}=-2x^{-3}.$$

**12.** **i.** $f'(x)=3x^2$. The slope in question is $f'(-2)=3(-2)^2=12$.

**ii.** $g(x)=x^{\frac{1}{3}}$. So $g'(x)=\frac{1}{3}x^{-\frac{2}{3}}$. The slope is $g'(-3)=\frac{1}{3}(-3)^{-\frac{2}{3}}=\frac{1}{3}\frac{1}{3^{\frac{2}{3}}}=\frac{1}{3^{\frac{5}{3}}}$.

**iii.** $f(x)=\frac{1}{x}=x^{-1}$. So $f'(x)=-x^{-2}=-\frac{1}{x^2}$. The slope is $f'(-\frac{1}{3})=-\frac{1}{\frac{1}{9}}=-9$.

**iv.** $f(x)=\frac{1}{x^2}=x^{-2}$. So $f'(x)=-2x^{-3}=-2\frac{1}{x^3}$. The slope is $f'(-2)=-2\frac{1}{-8}=\frac{1}{4}$.

**13.** **i.** $f'(x)=0$.

**ii.** $\frac{dy}{dx}=4$.

**iii.** $f'(x)=14x-5$.

**iv.** $y=2x^{\frac{1}{3}}+\pi x^3$. So $\frac{dy}{dx}=\frac{2}{3}x^{-\frac{2}{3}}+3\pi x^2$.

**v.** $g(x)=3x^{-1}+3x-6$. So $g'(x)=-3x^{-2}+3$.

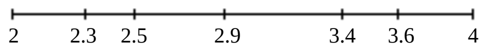
**vi.** $f(x)=2x^3+3x+4-x^{-2}$. So $f'(x)=6x^2+3+2x^{-3}$.

**14.** **i.** $\frac{dy}{dx}=-2x+8$. So $\frac{dy}{dx}=0$, when $x=4$. Therefore the point in question is (4, 16).

**ii.** Since $y=-x^2+8x=-x(x-8)$, the parabola crosses the $x-$axis at 0 and again at 8. In reference to Archimedes's theorem note that the area of the inscribed triangle is $\frac{1}{2}\cdot 8\cdot 16=64$. The area of the parabolic section is, therefore, $\frac{4}{3}\cdot 64=\frac{256}{3}$.

## 5D. Definite Integrals

**15.** Inserting the given points on the $x-$axis we have



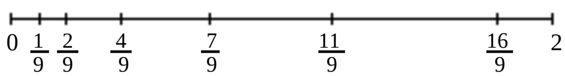

Proceeding from left to right and adding as in Examples 5.9 or 5.10, we get the following

$$\frac{1}{2}(0.3)+\frac{1}{2.3}(0.2)+\frac{1}{2.5}(0.4)+\frac{1}{2.9}(0.5)+\frac{1}{3.4}(0.2)+\frac{1}{3.6}(0.4)=$$

$$0.150+0.087+0.160+0.172+0.059+0.111=0.739.$$

This is a rough approximation of the area under the graph of $y=\frac{1}{x}$ over the interval from 2 to 4 on the $x-$axis. The actual value of this area can be shown to be 0.693 (up to three decimal accuracy). This computation involves the logarithm function that will be studied in Chapter 10.3.

**16.** Inserting the given points we get




Proceeding from left to right and adding, we get:

$$0\cdot\frac{1}{9}+\sqrt{\frac{1}{9}}\cdot\frac{1}{9}+\sqrt{\frac{2}{9}}\cdot\frac{2}{9}+\sqrt{\frac{4}{9}}\cdot\frac{3}{9}+\sqrt{\frac{7}{9}}\cdot\frac{4}{9}+\sqrt{\frac{11}{9}}\cdot\frac{5}{9}+\sqrt{\frac{16}{9}}\cdot\frac{2}{9}$$

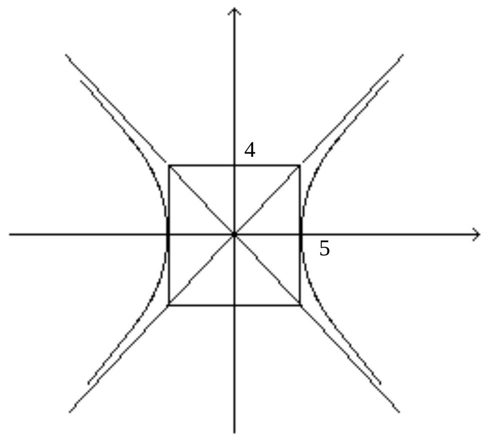
$$=\frac{1}{27}(1+2\sqrt{2}+6+4\sqrt{7}+5\sqrt{11}+8)=1.67.$$

This is a rough approximation of the area under the graph of $y=\sqrt{x}$ from 0 to 2. The actual value is

$$\int_0^2\sqrt{x}\;dx=\int_0^2 x^{\frac{1}{2}}dx=\frac{2}{3}x^{\frac{3}{2}}\Big|_0^2=\frac{2}{3}\left(\sqrt{2}\right)^3=1.89.$$

**17.** In each case, the symbol represents the area under the curve in question as obtained by the "addition of small rectangles" process described in the text. For $\int_0^3 x^2dx$ the meaning of the symbol is illustrated by drawing the graph of $y=x^2$ from 0 to 3 and inscribing under it narrow rectangles as in Firgure 5.14(c). For $\int_3^{12}\sqrt{x}\;dx$, the graph is different but the task is identical.

**18.** The graph of the hyperbola $\frac{x^2}{5^2}-\frac{y^2}{4^2}=1$ is obtained by the process explained in Section 5.1. It is sketched below. Solving for $y$, we get $y=\pm\frac{4}{5}\sqrt{x^2-5^2}$. So the graph of $y=\frac{4}{5}\sqrt{x^2-5^2}$




is the upper portion (both parts) of this graph. The definite integral $\int_7^{10} \frac{4}{5}\sqrt{x^2 - 5^2}\, dx$ is the area under the upper part of the hyperbola that falls between 7 and 10 (as obtained by the process of "adding small rectangles").

**5E. The Tractrix**

**Correction:** In Figure 5.40, delete $B$ and then replace $(x, y)$ by $B(x, y)$.

**19.** The slope of the string is $-\frac{z}{x}$. Note that it is negative because the string slopes downward from left to right. By Pythagoras's theorem, $z = \sqrt{a^2 - x^2}$. So the slope is $m_x = \frac{-\sqrt{a^2-x^2}}{x}$. Since the string is always tangent to the curve, the slope is also given as $f'(x)$. It follows that $f'(x) = \frac{-\sqrt{a^2-x^2}}{x}$. Recall that Perrault asked Leibniz about the exact nature of this curve. The answer is: the graph of $y = f(x)$. So the answer involves finding an antiderivative of $\frac{-\sqrt{a^2-x^2}}{x}$. This is not easy. See Section 10.4.

**20.** The formula for the length $L$ of a curve from a point (a, c) to a point (b, d), where $a \leq b$, is

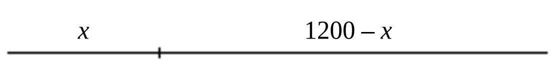
$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Since $f'(x) = \frac{-\sqrt{a^2-x^2}}{x}$ and the arc runs from (c, d) to (a, 0), where $0 < c \leq a$, we get

$$\begin{aligned} L &= \int_c^a \sqrt{1 + \frac{a^2-x^2}{x^2}}\, dx = \int_c^a \sqrt{\frac{x^2+a^2-x^2}{x^2}}\, dx \\ &= \int_c^a \sqrt{\frac{a^2}{x^2}}\, dx = \int_c^a \frac{a}{x}\, dx. \end{aligned}$$

**5F. Maximum and Minimum Values**

**21.** In view of the diagram



the function that needs to be analyzed is

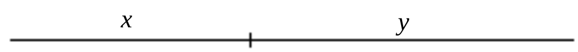
$$\begin{aligned} f(x) = x^2(1200-x)^2 &= x^2(1200^2 - 2\cdot 1200x + x^2) \\ &= 1200^2x^2 - 2\cdot 1200x^3 + x^4. \end{aligned}$$

Using the rules already established, we get $f'(x) = 2\cdot 1200^2 x - 6\cdot 1200x^2 + 4x^3$. Set $f'(x) = 0$ and solve for $x$. Because $x = 0$ is not the answer to the question, we can take $x \neq 0$. After canceling $4x$, we get $x^2 - 3(600)x + 2(600)^2 = 0$. By the quadratic formula,

$$x = \frac{3(600) \pm \sqrt{9(600)^2 - 4\cdot 2(600)^2}}{2} = \frac{3(600) \pm 600}{2} = 1200 \text{ or } 600\,.$$

Because $x \neq 1200$, the maximum occurs when $x = 600$. So to achieve the maximum, both pieces have the same length of 600 units. This answer is the same as that of Example 5.20. Is this surprising?

**22.** Subdivide the segment into two pieces $x$ and $y$ as shown:




Since the product $xy = 300$, the sum of the lengths is $f(x) = x + \frac{300}{x} = x + 300x^{-1}$. So $f'(x) = 1 - 300x^{-2} = \frac{x^2-300}{x^2}$. So the minimum must occur at $x = \sqrt{300}$. This means that the smallest length that the segment can have is $\sqrt{300} + \frac{300}{\sqrt{300}} = 2\sqrt{300}$.

**23.** Let the two sides of the rectangle be $x$ and $y$. Then $2x + 2y = 1000$, so $x + y = 500$, and $y = 500 - x$. The area is $f(x) = xy = x(500 - x) = -x^2 + 500x$. So $f'(x) = 500 - 2x$. It follows that $x = 250$ and $y = 250$ give the required dimensions.

**24.** Let $(x, y)$ be a random point on the parabola. The distance between $(x, y)$ and $(3, 1)$ is

$$\begin{aligned} d &= \sqrt{(x-3)^2 + (y-1)^2} = \sqrt{(x-3)^2 + (x^2+1-1)^2} \\ &= \sqrt{(x^2 - 6x + 9) + (x^2)^2} = \sqrt{x^4 + x^2 - 6x + 9}. \end{aligned}$$

The point $(x, y)$ for which the distance $d$ is a minimum is the same point for which the distance squared $d^2 = x^4 + x^2 - 6x + 9$ is a minimum. It remains, therefore, to analyze the function $f(x) = x^4 + x^2 - 6x + 9$. Note that $f'(x) = 4x^3 + 2x - 6$. This polynomial has $x = 1$ as a root. So $x - 1$ divides it. By polynomial division, $4x^3 + 2x - 6 = (x-1)(4x^2 + 4x + 6)$. Use of the quadratic formula shows that $4x^2 + 4x + 6$ is never zero. So $x = 1$ is the only 0 of $f'(x)$. The point on the parabola corresponding to $x = 1$ is (1, 2).

**25.** The circular base of the cylinder has radius $y$ and hence area $\pi y^2$. Because the volume of a cylinder is base $\times$ height, it follows that the volume is

$$V(x) = \pi y^2 x = \pi(3-x)^2 x = \pi(9 - 6x + x^2)x = \pi(x^3 - 6x^2 + 9x).$$

Observe that $V'(x) = \pi(3x^2 - 12x + 9) = 3\pi(x^2 - 4x + 3) = 3\pi(x-1)(x-3)$. So the maximum is achieved for $x = 1$ or $x = 3$. Since $x = 3$ gives $V(3) = 0$, it follows that $V(1) = \pi(2)^2 = 4\pi$ is the largest volume the cylinder can have.

## 5G. Areas and Definite Integrals

**26.** By the Fundamental Theorem of Calculus:

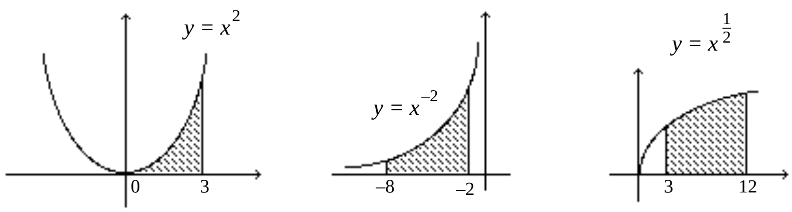
$$\int_0^3 x^2\,dx = \tfrac{1}{3}x^3\Big|_0^3 = \tfrac{27}{3} - 0 = 9\,.$$

$$\int_{-8}^{-2} \tfrac{1}{x^2}\,dx = \int_{-8}^{-2} x^{-2}\,dx = -x^{-1}\Big|_{-8}^{-2} = -(-2)^{-1} - (-(-8)^{-1}) = \tfrac{1}{2} - \tfrac{1}{8} = \tfrac{3}{8}\,.$$

$$\int_3^{12} \sqrt{x}\,dx = \int_3^{12} x^{\frac{1}{2}}\,dx = \tfrac{2}{3}x^{\frac{3}{2}}\Big|_3^{12} = \tfrac{2}{3}\left((\sqrt{12})^3 - (\sqrt{3}^3)\right)$$

$$= \tfrac{2}{3}\left(12\sqrt{12} - 3\sqrt{3}\right) = \tfrac{2}{3}\left(24\sqrt{3} - 3\sqrt{3}\right) = \tfrac{2}{3}\left(21\sqrt{3}\right) = 14\sqrt{3}\,.$$

The area represented by the first integral is that under the curve $y = x^2$ and above the segment from 0 to 3 on the $x-$axis; the area represented by the second integral is that under




the curve $y = \frac{1}{x^2}$ and above the segment from $-8$ to $-2$ on the $x-$axis; and similarly for the third integral.

**27.** Solving $x^2 + y^2 = 4$ for $y$ gives $y = \pm\sqrt{4 - x^2}$. So the graph of $y = \sqrt{4 - x^2}$ is the upper half of the circle and the graph of $y = -\sqrt{4 - x^2}$ is the lower half. Convince yourself that the definite integral

$$\int_0^2 \sqrt{4 - x^2}\,dx$$

represents the area of one quarter of the circle of radius 2. So $\int_0^2 \sqrt{4 - x^2}\,dx = \frac{1}{4}\pi(2)^2 = \pi$. In the same way, $\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{1}{4}\pi a^2$.

**28.** Since $\int_0^5 \frac{5}{2}\sqrt{5^2 - x^2}\,dx = \frac{5}{2}\cdot\int_0^5 \sqrt{5^2 - x^2}\,dx$, we get by using Exercise 27, that $\int_0^5 \frac{5}{2}\sqrt{5^2 - x^2}\,dx = \frac{5}{2}\cdot\frac{1}{4}\pi 5^2 = \frac{125}{8}\pi$.

**29.** As above, $\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \frac{b}{a}\cdot\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{b}{a}\cdot\frac{1}{4}\pi a^2 = \frac{1}{4}\pi ab$. Recall that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse. By solving this equation for $y$ and considering positive $y$ only, check that the graph of $y = \frac{b}{a}\sqrt{a^2 - x^2}$ is the upper half of the ellipse. So the definite integral $\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$ is the area of the upper right quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

**30.** **i.** The slope of the segment is $\frac{\frac{8}{3} - 0}{4 - 0} = \frac{8}{3}\frac{1}{4} = \frac{2}{3}$. The slope of the tangent at any point $(x, y)$ on the parabola is $\frac{1}{3}x$. If this tangent is to be parallel to the segment, then $\frac{1}{3}x = \frac{2}{3}$. So $x = 2$ and $Q = \left(2, \frac{2}{3}\right)$.

**ii.** The slope of the line through the point $Q$ perpendicular to $OP$ is $-\frac{1}{\frac{2}{3}} = -\frac{3}{2}$. The point-slope form of the line in question is $y - \frac{2}{3} = -\frac{3}{2}(x-2)$.

**iii.** Take the segment $OP$ as base of the triangle $\Delta OQP$. The length of the segment is

$$\sqrt{4^2 + \left(\tfrac{8}{3}\right)^2} = \sqrt{16 + \tfrac{64}{9}} = \tfrac{1}{3}\sqrt{144+64} = \tfrac{1}{3}\sqrt{208} = \tfrac{2}{3}\sqrt{52} = \tfrac{4}{3}\sqrt{13}\,.$$

Let $T$ be the point of intersection between the line of (ii) and the segment $OP$. Then $QT$ is the height of the triangle $\Delta OQP$. Note that $T$ is the intersection of the lines $y = \frac{2}{3}x$ and $y - \frac{2}{3} = -\frac{3}{2}(x-2)$. Solving $\frac{2}{3}x - \frac{2}{3} = -\frac{3}{2}(x-2)$ for $x$, we get $\frac{2}{3}x - \frac{2}{3} = -\frac{3}{2}x + 3$, or $\frac{13}{6}x = \frac{11}{3}$. So $x = \frac{22}{13}$ and $T = \left(\frac{22}{13}, \frac{44}{39}\right)$. The length of $QT$ is

$$\sqrt{\left(2 - \frac{22}{13}\right)^2 + \left(\frac{2}{3} - \frac{44}{39}\right)^2} = \sqrt{\left(\frac{4}{13}\right)^2 + \left(\frac{18}{3\cdot 13}\right)^2} = \sqrt{\frac{16\cdot 3^2 + 9^2\cdot 2^2}{3^2\cdot 13^2}}$$

$$= \sqrt{\frac{16 + 3^2\cdot 2^2}{13^2}} = \sqrt{\frac{52}{13^2}} = \sqrt{\frac{4}{13}} = \frac{2}{\sqrt{13}}\,.$$

The area of the triangle $\Delta OQP$ is $\frac{1}{2}\cdot\frac{4}{3}\sqrt{13}\cdot\frac{2}{\sqrt{13}} = \frac{4}{3}$.

**iv.** By Archimedes's theorem, the area of the parabolic section $OPQ$ is $\frac{4}{3}\cdot\frac{4}{3} = \frac{16}{9}$. Substracting this from the triangular area under the segment $OP$ gives

$$\frac{1}{2}\cdot 4\cdot\frac{8}{3} - \frac{16}{9} = \frac{48}{9} - \frac{16}{9} = \frac{32}{9}\,.$$

for the area under the parabola from 0 to 4.

**v.** By the Fundamental Theorem of Calculus, this area is also given by

$$\int_0^4 \frac{1}{6}x^2\,dx = \frac{1}{6}\cdot\frac{1}{3}x^3\,\Big|_0^4 = \frac{64}{18} = \frac{32}{9}\,.$$

## 5H. Definite Integrals as Areas, Volumes, and Lengths of Curves

**31.** Notice that the rotation of this triangular region produces the required cone. The equation of the line through 0 and $(h, r)$ is $y = \frac{r}{h}x$. So

$$V = \int_0^h \pi f(x)^2\,dx = \int_0^h \pi\frac{r^2}{h^2}x^2\,dx = \frac{1}{3}\pi\frac{r^2}{h^2}x^3\,\Big|_0^h = \frac{1}{3}\pi r^2 h\,.$$

**32.** The volume is given by

$$V = \int_0^3 \pi f(x)^2\,dx = \int_0^3 \pi x\,dx = \pi\left(\frac{1}{2}x^2\,\Big|_0^3\right) = \frac{9}{2}\pi\,.$$

**33.** The length of the arc on the parabola $y = x^2$ from the point $(2,4)$ to the point $(5,25)$ is

$$L = \int_2^5 \sqrt{1+f'(x)^2}\ dx = \int_2^5 \sqrt{1+4x^2}\ dx\ .$$

Refer to the last part of Section 5.1 to see that the graph of $y^2 - 4x^2 = 1$ is a hyperbola. Solving this equation for $y$ shows that the upper half of this hyperbola is the graph of the function $f(x) = \sqrt{1+4x^2}$. It follows that the area under the upper half of the hyperbola from 2 to 5 is also equal to $\int_2^5 \sqrt{1+4x^2}\ dx$ .

**34.** Solve $\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$ for $y$ to see that the upper half of the ellipse is the graph of $f(x) = \frac{4}{5}\sqrt{5^2 - x^2}$ .

**i.** $A = \int_{-5}^{5} f(x)\,dx = \int_{-5}^{5} \frac{4}{5}\sqrt{5^2-x^2}\,dx.$

**ii.** $V = \int_{-5}^{5} \pi f(x)^2\,dx = \int_{-5}^{5} \frac{16\pi}{25}(5^2 - x^2)\,dx.$

**iii.** Since $f'(x)$ is the slope of the tangent at the point $(x,y)$, we see from Exercise 9 that

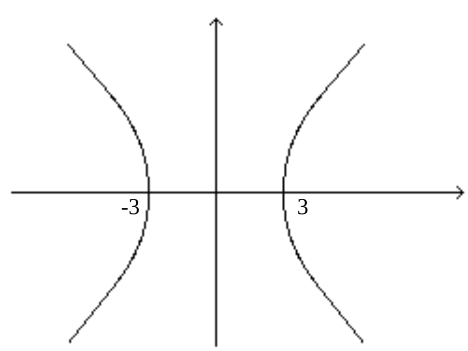
$$f'(x) = -\frac{4^2}{5^2}\frac{x}{y} = -\frac{4^2}{5^2}\frac{x}{\frac{4}{5}\sqrt{5^2-x^2}} = -\frac{4x}{5\sqrt{5^2-x^2}}\ .$$

It follows that the length $L$ of the arc is given by

$$L = \int_1^3 \sqrt{1+f'(x)^2}\ dx = \int_1^3 \sqrt{1+\frac{16x^2}{25(5^2-x^2)}}\ dx.$$

**Correction:** In the statement of Exercise 35(v) the $y$-coordinates of the two points are incorrectly listed. They should be $\frac{2}{3}\sqrt{7}$ and $\frac{2}{3}\sqrt{40}$ respectively.

**35.** Consider the hyperbola $\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1$. So $\frac{x^2}{3^2} = 1 + \frac{y^2}{2^2}$. Therefore, $\frac{x^2}{3^2} \geq 1$ and $x^2 \geq 3^2$. It follows from the considerations towards the end of Section 5.1 that the general shape of this hyperbola is




**i.** To apply Leibniz's tangent method let $(x+\Delta x, y+\Delta y)$ be a point on the graph that lies on the same part of the curve as $P$. Notice that

$$\begin{aligned}
\tfrac{(x+\Delta x)^2}{3^2} - \tfrac{(y+\Delta y)^2}{2^2} &= 1\\
\tfrac{x^2+2x\Delta x+(\Delta x)^2}{3^2} - \tfrac{y^2+2y\Delta y+(\Delta y)^2}{2^2} &= 1\\
\tfrac{x^2}{3^2} - \tfrac{y^2}{2^2} + \tfrac{2x\Delta x+(\Delta x)^2}{3^2} - \tfrac{2y\Delta y+(\Delta y)^2}{2^2} &= 1\\
\tfrac{2x\Delta x+(\Delta x)^2}{3^2} - \tfrac{2y\Delta y+(\Delta y)^2}{2^2} &= 0\\
\tfrac{\Delta x(2x+\Delta x)}{3^2} &= \tfrac{\Delta y(2y+\Delta y)}{2^2}\,.
\end{aligned}$$

Divide both sides by $\Delta x$ to get

$$\frac{\Delta y}{\Delta x}\cdot\frac{2y+\Delta y}{2^2} = \frac{2x+\Delta x}{3^2}\,.$$

So $\frac{\Delta y}{\Delta x} = \frac{4}{9}\frac{2x+\Delta x}{2y+\Delta y}$. Pushing $\Delta x$ to zero, gives $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \frac{4x}{9y}$. This is the slope at any point $P=(x,y)$ on the hyperbola. Note that we need to have $y\neq 0$.

**ii.** To find a function whose graph is precisely the upper half of the hyperbola, solve $\frac{x^2}{3^2}-\frac{y^2}{2^2}=1$ for $y$. Since $\frac{y^2}{2^2}=\frac{x^2}{3^2}-1=\frac{x^2-3^2}{3^2}$, we get $y^2=\frac{2^2}{3^2}(x^2-3^2)$ and $y=\pm\frac{2}{3}\sqrt{x^2-3^2}$. Note that $y=f(x)=\frac{2}{3}\sqrt{x^2-3^2}$ is the required function. By part (i), the derivative is equal to $f'(x)=\frac{4}{9}\frac{x}{f(x)}=\frac{4}{9}\frac{x}{\frac{2}{3}\sqrt{x^2-3^2}}=\frac{2}{3}\frac{x}{\sqrt{x^2-3^2}}$.

**iii.** The area under the upper half of this hyperbola and over the interval from 3 to 7 is

$$\int_3^7 f(x)\;dx = \int_3^7 \frac{2}{3}\sqrt{x^2-3^2}\;dx\,.$$

**iv.** The volume of the solid obtained by rotating the region of one complete revolution about the $x$-axis is

$$\int_3^7 \pi f(x)^2\;dx = \int_3^7 \frac{4\pi}{9}(x^2-3^2)\;dx\,.$$

**v.** Using (ii), we get that the length of the hyperbolic arc from the point $\big(4,\frac{2}{3}\sqrt{7}\big)$ to the point $\big(7,\frac{2}{3}\sqrt{40}\big)$ is

$$\int_4^7 \sqrt{1+f'(x)^2}\;dx = \int_4^7 \sqrt{1+\frac{4x^2}{9(x^2-3^2)}}\;dx\,.$$

## 5I. Theorems of Pappus of Alexandria

**36.** We need to use Pappus's theorem A because the concern in this exercise is surface area. Consider the circle on the right in Figure 5.45 and drop a perpendicular segment from $C$ to the axis shown. If the segment with the circle attached is rotated one revolution around the axis, the circle traces out the surface of a donut. Incidentally, the surface of such a perfect donut is called a *torus* in mathematics. Now turn to the specifics of the exercise and in particular to Figure 5.46. What was just described is now viewed from the top. The axis is perpendicular to the page and goes through the point of intersection of the segments labeled $r$ and $R$. The circle being rotated is now being viewed from the top as the thicker black segment; a frontal view of this circle is also shown in black (in this exercise only the boundary counts and not the inside). Pappus's theorem A tells us that the surface area of the donut is the product of the circumference of the rotated circle and the distance traveled by the centroid $C$ of the circle. Consider Figures 5.45 (the diagram on the right) and 5.46 together. We see that the diameter of the rotated circle is $R-r$, so its radius is $\frac{1}{2}(R-r)$, and hence its circumference is $2\pi\left(\frac{1}{2}(R-r)\right) = \pi(R-r)$. The centroid of the circle is its center $C$. Observe that the distance from $C$ to the axis is $\frac{1}{2}(R-r)+r = \frac{1}{2}(R+r)$. So in one complete rotation, $C$ traces out a circle of radius $\frac{1}{2}(R+r)$. Because this circle has a circumference of $2\pi\left(\frac{1}{2}(R+r)\right) = \pi(R+r)$, we see that this is the distance traveled by the centroid $C$. We can therefore conclude that

$$\text{length of arc } \times \text{ distance traced out by centroid } = \pi(R-r)\cdot\pi(R+r) = \pi^2(R^2-r^2).$$

So the surface area of the torus in question is $\pi^2(R^2-r^2)$.

**37.** This is another application of Pappus's theorem A. Refer to diagram in Figure 5.45 (on the left). Let $r$ be the radius of the semicircular arc and let $C$ be its centroid. By the vertical symmetry of the semicircular arc, its centroid $C$ lies on the radius that is perpendicular to the axis. Its general position is shown in the figure. But where precisely is it located? When the semicircle is rotated one revolution about the axis a sphere of radius $r$ is produced. Pappus's theorem A says that

$$\text{Surface area } = \text{ length of arc } \times \text{ distance traveled by centroid.}$$

So $4\pi r^2 = \pi r \times$ distance traveled by centroid. Therefore, the distance traveled by centroid is $4r$. Let $d$ be the distance from the centroid to the axis and note that the centroid traces out a circle of radius $d$. So the distance traveled by the centroid is $2\pi d$. Therefore $2\pi d = 4r$, and hence $d = \frac{2}{\pi}r \approx 0.64r$. The location of the centroid of the semicircular arc has been determined.

**38.** Consider the semicircular region on the left in Figure 5.45. This is no longer the arc itself, but the entire semicircular area. Let $C$ be the centroid of the region and let $r$ be the radius of the semicircle. The solid that is traced out by one full rotation of the region is a sphere of radius $r$. Now apply Pappus's theorem B. Since the volume of the sphere of radius $r$ is $\frac{4}{3}\pi r^3$, and the area of the semicircle that is being rotated is $\frac{1}{2}\pi r^2$, we now get

$$\frac{4}{3}\pi r^3 = \frac{1}{2}\pi r^2 \times \text{ distance traveled by } C.$$

By the symmetry of the region, $C$ lies somewhere on the radius that is perpendicular to the axis. Let its distance from the axis be $d$ and note that the distance traveled by $C$ during one complete rotation is $2\pi d$. Therefore, $\frac{4}{3}\pi r^3 = \frac{1}{2}\pi r^2 \cdot 2\pi d = \pi^2 r^2 d$. Solving for $d$ gives us $d = \frac{4}{3\pi}r \approx 0.42r$.