## Solutions to the Exercises of Chapter 4

### 4A. Basic Analytic Geometry

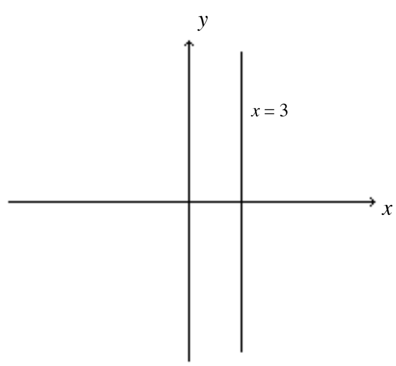
**1.** The distance between $(1,1)$ and $(4,5)$ is $\sqrt{(1-4)^2+(1-5)^2}=\sqrt{9+16}=5$ and that from $(1,-6)$ to $(-1,-3)$ is $\sqrt{(1-(-1))^2+(-6-(-3))^2}=\sqrt{(2^2+3^2)}=\sqrt{13}$.

**2.** **i.** $AB=\sqrt{(6-11)^2+(-7-(-3))^2}=\sqrt{25+16}=\sqrt{41}$.
$AC=\sqrt{(6-2)^2+(-7-(-2))^2}=\sqrt{16+25}=\sqrt{41}$.
$BC=\sqrt{(11-2)^2+(-3-(-2))^2}=\sqrt{81+1}=\sqrt{82}$.
So $AB^2+AC^2=BC^2$. So by Pythagoras, $\Delta ABC$ is a right triangle.

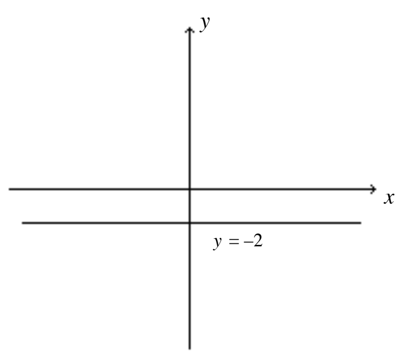
**ii.** With the side $AB$ as base, the height is $AC$. So the area is $\frac{1}{2}\left(\sqrt{41}\right)\left(\sqrt{41}\right)=\frac{41}{2}$.

**3.** $AB=\sqrt{(-1-3)^2+(3-11)^2}=\sqrt{16+64}=\sqrt{80}$.
$BC=\sqrt{(3-5)^2+(11-15)^2}=\sqrt{4+16}=\sqrt{20}$
$AC=\sqrt{(-1-5)^2+(3-15)^2}=\sqrt{36+144}=\sqrt{180}$
So $AB+BC=\sqrt{16\cdot 5}+\sqrt{4\cdot 5}=4\sqrt{5}\ +2\sqrt{5}=6\sqrt{5}=\sqrt{36\cdot 5}=AC$.

**4.** **i.**

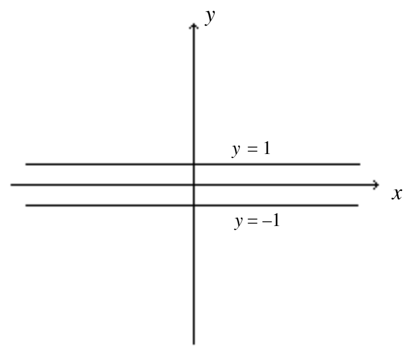



**ii.**

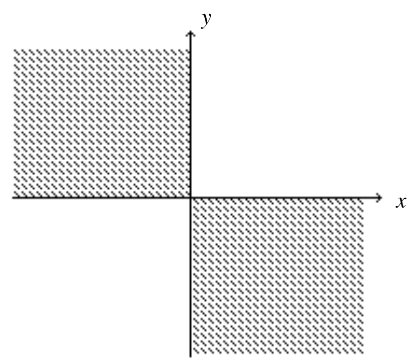



**5.** **i.** The $x$ and $y$ axes taken together.

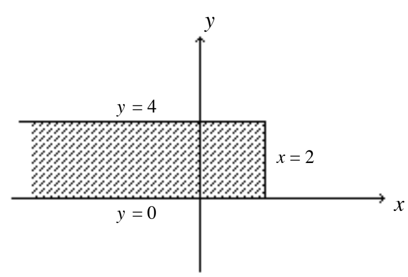
**ii.** $|y| = 1$ means that either $y = 1$ or $y = -1$. So the graph consists of the two lines



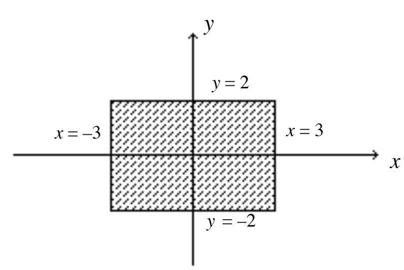

**6.** This is the set of all $(x, y)$ with either $x$ positive and $y$ negative, or $x$ negative and $y$ positive. So it is the shaded region (without the axes):



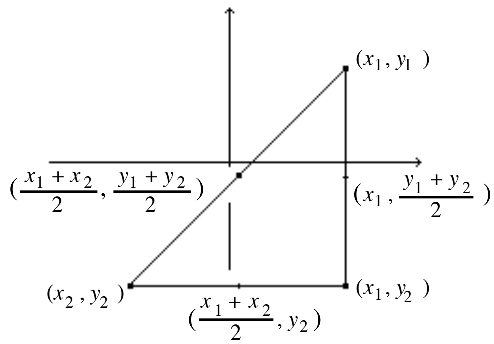

**7.** This is the strip (including the indicated boundaries):




**8.** Because $|x| < 3$ is equivalent to $-3 < x < 3$ and $|y| < 2$ is equivalent to $-2 < y < 2$, this consists of all points within, but not on, the rectangle below.



**9.** To show that the midpoint of the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ proceed as follows: Drop perpendiculars and form the point $(x_1, y_2)$. Then use the fact that the midpoint of the segment from $x_1$ to $x_2$ on the $x$ axis is $\frac{x_1+x_2}{2}$ and that from $y_1$ to $y_2$




on the $y$ axis is $\frac{y_1+y_2}{2}$. Consider the points $\left(x_1, \frac{y_1+y_2}{2}\right)$ and $\left(\frac{x_1+x_2}{2}, y_2\right)$. It remains to notice (by considering similar triangles) that

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

it is the midpoint of the segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$.

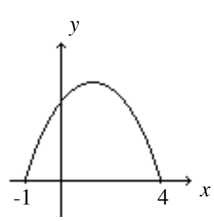
**10.** Use of the formula developed in Exercise 9 shows that the midpoint of the line segment joining the points (1, 3) and (7, 15) is (4, 9) and that of the segment joining the points $(-1, 6)$ and $(8, -12)$ is $\left(\frac{7}{2}, -3\right)$.

## 4B. Circles, Parabolas, and Ellipses

**11.** This is a parabola opening downward. To get a precise idea of the graph, complete the squares

$$\begin{aligned} y &= -(x^2 - 3x - 4) = -(x - 3x + (\frac{3}{2})^2 - (\frac{3}{2})^2 - 4) \\ &= -(x - \frac{3}{2})^2 + (\frac{9}{4} + \frac{16}{4}) = (x - \frac{3}{2})^2 + \frac{25}{4}\,. \end{aligned}$$

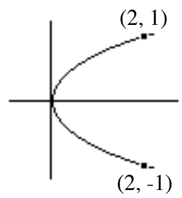
Note that the highest point on the parabola is $(\frac{3}{2}, \frac{25}{4})$. It crosses the $x$-axis when $x - \frac{3}{2} = \pm\frac{5}{2}$,




so at $x = -1$ and 4.

**12.** Dividing through by 16, we get $\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$. This is an ellipse with semimajor axis $a = 4$ and semiminor axis $b = 2$. For the graph see Figure 4.28.

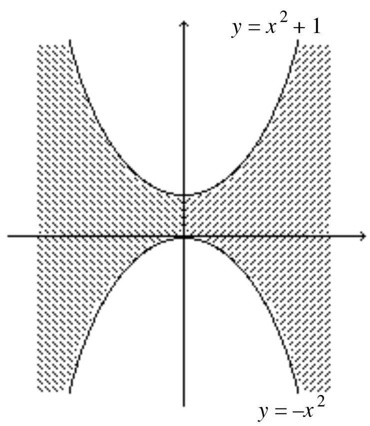
**13.** This is a parabola opening to the right starting at the origin.



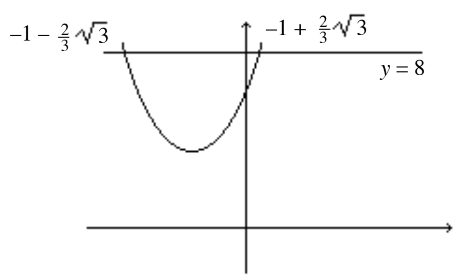

**14.** A look at the standard equation of the circle shows that this is a circle of radius $\sqrt{7}$ centered at $(3, -5)$.

**15.** Note that $\frac{x^2}{\frac{12}{9}} + \frac{y^2}{\frac{12}{2}} = 1$ and hence that $\frac{x^2}{\left(\sqrt{\frac{4}{3}}\right)^2} + \frac{y^2}{(\sqrt{6})^2} = 1$. This is an ellipse with semimajor axis $\sqrt{6}$ and semiminor axis $\sqrt{\frac{4}{3}}$. Notice that the major axis is on the $y$-axis and the minor axis is on the $x$-axis. The general shape of the graph is obtained by rotating the ellipse of Figure 4.28 by 90°.

**16.**




**17.** The points of intersection of the line and the parabola are obtained by applying the quadratic formula to the equation $3x^2 + 6x - 1 = 0$. Doing so, shows that the $x$ coordinates of the points of intersection are $-1 \pm \frac{2}{3}\sqrt{3}$. Since the parabola opens upward, the situation is as pictured.



For $P = (x, y)$ to lie in the parabolic section, both $-1 - \frac{2}{3}\sqrt{3} \leq x \leq -1 + \frac{2}{3}\sqrt{3}$ and $3x^2 + 6x + 7 \leq y \leq 8$ must hold. Why is the first of these two conditions superfluous?

**18.** Completing the square transforms $y = x^2 + 4x + 7$ to $y = (x^2 + 4x + 2^2) - 2^2 + 7$ and hence to $y = (x + 2)^2 + 3$. The smallest $y$ value is 3 and it occurs when $x = -2$. So $(-2, 3)$ is the lowest point on the graph. The points of intersection of the line $y = 7$ and the parabola are obtained by setting $x^2 + 4x + 7 = 7$ and solving for $x$. Since $x(x + 4) = 0$, this shows that $x = 0$ and $x = -4$. So the points of intersection are $S' = (-4, 7)$ and $S = (0, 7)$. The vertex of the parabolic section is $V = (-2, 3)$. The area of the triangle $\Delta S'VS$ is $\frac{1}{2}(4)(4) = 8$. Therefore by Archimedes's theorem, the area of the parabolic section $S'VS$ is $\frac{4}{3} \cdot 8 = 10\frac{2}{3}$.

**19.** Consider the equation $y = 3x^2 - 2x + 5$ together with the general equation

$$y = \left(\frac{1}{2(b-c)}\right)x^2 - \left(\frac{a}{b-c}\right)x + \left(\frac{a^2 + b^2 - c^2}{2(b-c)}\right).$$

In this case, $\frac{1}{2(b-c)} = 3, \frac{a}{b-c} = 2$, and $\frac{a^2+b^2-c^2}{2(b-c)} = 5$. So $b - c = \frac{1}{6}, a = 2(b - c) = \frac{1}{3}$, and $a^2 + b^2 - c^2 = 10(b - c) = \frac{5}{3}$. Since $(b+c)(b-c) = b^2 - c^2 = \frac{5}{3} - \frac{1}{9} = \frac{14}{9}, b + c = \frac{14}{9} \cdot 6 = \frac{28}{3} = \frac{56}{6}$. Using $b - c = \frac{1}{6}$, we get $b = \frac{57}{12}$ and $c = \frac{55}{12}$. Refer to the text and conclude that the focus is $(a, b) = \left(\frac{1}{3}, \frac{57}{12}\right)$ and that the directrix is the line $y = \frac{55}{12}$.

**20.** This is $(x - 3)^2 + (y + 1)^2 = 25$.

**21.** Completing the square with both variables, we get

$$\begin{aligned} 0 &= x^2 + y^2 - 4x + 10y + 13 = x^2 - 4x + y^2 + 10y + 13 \\ &= x^2 - 4x + (2^2 - 2^2) + y^2 + 10y + (5^2 - 5^2) + 13 \\ &= (x - 2)^2 - 2^2 + (y + 5)^2 - 5^2 + 13. \end{aligned}$$

So, $(x - 2)^2 + (y + 5)^2 = -13 + 4 + 25 = 16 = 4^2$. So the graph is a circle. Its center is $(2, -5)$ and its radius is 4.

**22.** Proceed as above and complete the square with $x^2 + y^2 + ax + by + c = 0$:

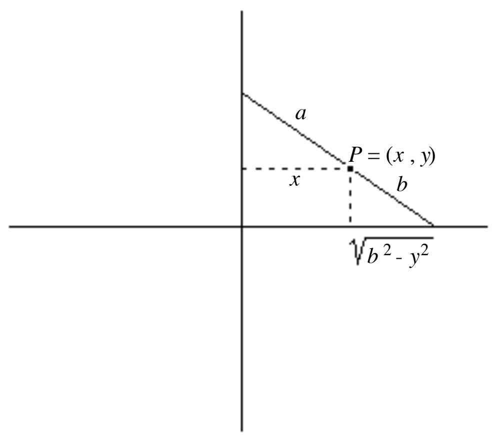
$$\begin{aligned} 0 &= x^2 + y^2 + ax + by + c = x^2 + ax + y^2 + by + c \\ &= x^2 + ax + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 + y^2 + by + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 - \left(\frac{a}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c. \end{aligned}$$

Therefore, $\left(x + \frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 - c$. Since the left side cannot be negative, $\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 - c$ must be greater than or equal to 0 if there are to be any points on the graph of this equation. So the condition is $\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 \geq c$. Let $r = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 - c}$. Since $\left(x + \frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 = r^2$, we are dealing with a circle with center $\left(-\frac{a}{2}, -\frac{b}{2}\right)$ and radius $r$.

**23.** Since the equation is $\frac{x^2}{5^2} + \frac{y^2}{2^2} = 1$, the semimajor axis is $a = 5$ and the semiminor is $b = 2$. The linear eccentricity is $e = \sqrt{a^2 - b^2} = \sqrt{5^2 - 2^2} = \sqrt{21}$, and the astronomical eccentricity is $\varepsilon = \frac{e}{a} = \frac{\sqrt{21}}{5}$.

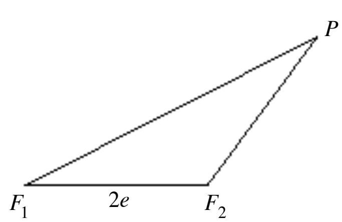
**24.** Since the string is stretched it will always form a triangle with base the segment $F_1F_2$. So the base has length $2e = 2\sqrt{a^2 - b^2}$. This means that the sum of the lengths of the remaining two sides of the triangle is equal to $2a$. Hence the sum of the distances from the tip of the pencil to the points $F_1$ and $F_2$ is equal to $2a$. Therefore what is being traced out is an ellipse with focal points $F_1$ and $F_2$, constant $k = 2a$, and linear eccentricity $e$. The rest follows from the discussion in Section 4.5, especially Figure 4.28.

**25.** Take a line segment of fixed length and let $P$ be a fixed point on it. Let $a$ and $b$ be the lengths of the segments on the two sides of $P$ as shown. Let the segment be in typical position in the first quadrant and put $P = (x, y)$. By similar triangles, $\frac{x}{a} = \frac{\sqrt{b^2 - y^2}}{b}$. Square both sides to get




$\frac{x^2}{a^2} = \frac{b^2 - y^2}{b^2}$. Therefore, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Check that this holds regardless of the quadrant in which the segment is placed. So the points $P$ produced in this way coincide with the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

**26.** Review the basics about hyperbolas from Section 3.1. Let $k$ be a positive constant and let $e$ be one half the distance between $F_1$ and $F_2$. So the focal points are $(-e, 0)$ and $(e, 0)$. Let $F_1 = (e, 0)$ and $F_2 = (-e, 0)$. The hyperbola determined by $F_1, F_2$ and $k$, consists of all points $P = (x, y)$ such that $|PF_1 - PF_2| = k$. A look at the diagram below shows that $2e + PF_2 > PF_1$. So $2e > PF_1 - PF_2$ and $2e > |PF_1 - PF_2| = k$. Note that $P = (x, y)$ is on



the hyperbola precisely if $PF_1 - PF_2 = \pm k$. This translates to

$$\sqrt{(x-e)^2+y^2} - \sqrt{(x+e)^2+y^2} = \pm k.$$

So $\sqrt{(x-e)^2+y^2} = \pm k + \sqrt{(x+e)^2+y^2}$. After squaring both sides, etc., this equation can be transformed in successive steps to

$$(x-e)^2+y^2 = k^2 \pm 2k\sqrt{(x+e)^2+y^2} + (x+e)^2+y^2$$

$$(x-e)^2 = k^2 \pm 2k\sqrt{(x+e)^2+y^2} + (x+e)^2$$

$$x^2-2ex+e^2 = k^2 \pm 2k\sqrt{(x+e)^2+y^2} + x^2+2ex+e^2$$

$$\pm 2k\sqrt{(x+e)^2+y^2} = k^2+4ex$$

$$4k^2\left((x+e)^2+y^2\right) = k^4+8k^2ex+16e^2x^2$$

$$4k^2\left(x^2+2ex+e^2+y^2\right) = k^4+8k^2ex+16e^2x^2$$

$$4k^2x^2+8k^2ex+4k^2e^2+4k^2y^2 = k^4+8k^2ex+16e^2x^2$$

$$4k^2x^2-16e^2x^2+4k^2y^2 = k^4-4k^2e^2$$

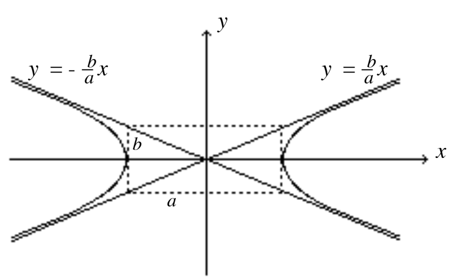
$$4\left(k^2-4e^2\right)x^2+4k^2y^2 = k^2\left(k^2-4e^2\right).$$

Dividing through by $k^2\left(k^2-4e^2\right)$, gives $\frac{4}{k^2}x^2 + \frac{4}{k^2-4e^2}y^2 = 1$. Hence $\frac{x^2}{\frac{k^2}{4}} - \frac{y^2}{\frac{4e^2-k^2}{4}} = 1\,.$

Recall that $2e > k$. So $4e^2 > k^2$, and $4e^2-k^2 > 0$. With a $= \sqrt{\frac{k^2}{4}} = \frac{k}{2}$ and $b = \sqrt{\frac{4e^2-k^2}{4}} = \frac{1}{2}\sqrt{4e^2-k^2}$, we now have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,.$$

The graph is given by



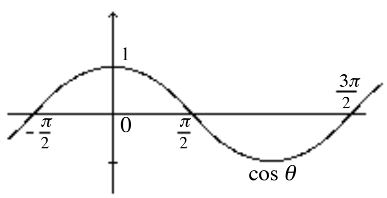

## 4C. Some Geometry and Trigonometry

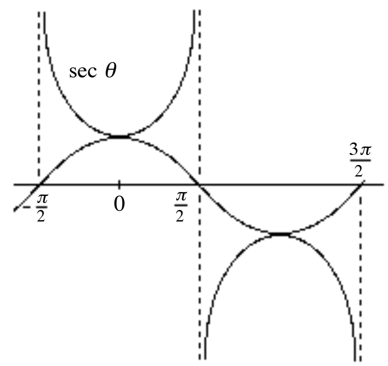
**27.** The radian measure of $\theta$ is $\frac{s}{1} = 0.693$ and $P = (\cos\theta, \sin\theta) \approx (0.769, 0.639)$.

**28.** Since $\theta$ is equal to both 5 and $\frac{s}{2}$, we get that $s = 10$.

**29.** Consider $\theta = 17.52$. Because $\frac{17.72}{2\pi} = 2.82$, we get that $17.72 = (2.82)(2\pi)$. Since $\theta$ is positive, it follows that $P_\theta$ is obtained by going around the unit circle 2.82 revolutions in the clockwise direction starting from the point (1, 0). Two complete revolutions return us to the starting point (1, 0). Since $(0.82)(2\pi) = (3.28)\left(\frac{\pi}{2}\right)$, it remains to proceed another three quarters of a revolutions in the clockwise direction to the point $(0, -1)$, and then another $0.28 \approx \frac{1}{4}$ of a quarter revolution to locate $P_\theta$. It follows that $P_\theta$ is in the fourth quadrant. More precision is obtained by recalling that $P_\theta = (\cos\theta,\ \sin\theta) = (\cos 17.52,\ \sin 17.52) \approx (0.24, -0.97)$. For $\theta = -21.83$, do a similar thing in the clockwise direction. Because $P_\theta = (\cos\theta,\ \sin\theta) = (\cos -21.83, \sin -21.83) \approx (-0.99, -0.16)$. This time $P_\theta$ is in the third quadrant, close to the point $(-1, 0)$.

**30.** Start with the portion of the graph of the cosine shown below on the left. First consider $0 \le \theta \le \frac{\pi}{2}$. As $\cos\theta$ goes from $\cos 0 = 1$ to $\cos\frac{\pi}{2} = 0$, notice that $\sec\theta = \frac{1}{\cos\theta}$ will move




from 1 to a larger and larger positive number. At $\theta = \frac{\pi}{2}$, note that $\sec\frac{\pi}{2} = \frac{1}{\cos\frac{\pi}{2}} = \frac{1}{0}$ is not defined. The graph on the right captures this information. Then do the same thing for $-\frac{\pi}{2} \le \theta \le 0$ and $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ The rest of the graph follows the pattern already established.

**31.** The first three identities follow from similar identities for the cosine. To verify the fourth, start with $\sin^2\theta + \cos^2 = 1$ and divide through by $\cos^2\theta$.

## 4D. Computing Orbital Information

**32.** From Table 4.2, $a = 5.2028$ AU and $\varepsilon = 0.0484$. So the linear eccentricity is $e = \varepsilon a = 0.2518$ AU and the semiminor axis is $b = \sqrt{a^2 - e^2} = \sqrt{27.0691 - 0.0634} = \sqrt{27.0057} = 5.1967$ AU. After consulting Figure 4.28 observe that the greatest distance from Jupiter to the Sun is $a + e = 5.4546$ AU and that the least distance is $a - e = 4.951$ AU. To convert to miles use the estimate 1 AU $= 93 \times 10^6$ miles. For example, the greatest distance from Jupiter to the Sun is $507 \times 10^6$ miles.

**33.** For Mars, $\frac{a^3}{T^2} = \frac{1.5237^3}{1.8809^2} = \frac{3.5375}{3.5378} = 0.9999$. For Jupiter, $\frac{a^3}{T^2} = \frac{5.2028^3}{11.8622^2} = \frac{140.8353}{140.7118} = 1.0009$. For Saturn, $\frac{a^3}{T^2} = \frac{9.5388^3}{29.4577^2} = \frac{867.9231}{867.7561} = 1.0002$. The fact that all these ratios are equal to one is directly related to the definition of the units used. By definition, 1 AU is the semimajor axis of the Earth and 1 year is equal to the period of the Earth's orbit. So in these units, the ratio $\frac{a^3}{T^2}$ is equal to 1 for the Earth. By Kepler's law all the ratios $\frac{a^3}{T^2}$ must be equal to 1 (or more accurately, close to 1) in these units.

**34.** **i.** The formula for the area of a circular sector of radius $r$ and angle $\theta$ is $\frac{1}{2}r^2\theta$. The radian measure of the angle $\angle PSP'$ is $\frac{\text{arc } PP'}{a-e}$. It follows that the area of the circular sector $PSP'$ is $\frac{1}{2}(a-e)^2\frac{\text{arc } PP'}{a-e} = \frac{1}{2}(a-e)(\text{arc } PP')$. The same computation verifies the other formula.

**ii.** Let $t$ be this common time. By Kepler's second law and part (i), $\frac{1}{2}(a-e)(\text{arc } PP') = \frac{1}{2}(a+e)(\text{arc } QQ')$. Since $v_P t = \text{arc } PP'$ and $v_A t = \text{arc } QQ'$, it follows that $\frac{1}{2}(a-e)v_P t = \frac{1}{2}(a+e)v_A t$. The formula $\frac{v_P}{v_A} = \frac{a+e}{a-e}$ follows. Dividing the numerator and denominator by $a$, shows that $\frac{v_P}{v_A} = \frac{1+\varepsilon}{1-\varepsilon}$. A look at Table 4.2 tells us that its value for the Earth is $\frac{1+0.0167}{1-0.0167} = \frac{1.0167}{0.9833} = 1.0340$. For Saturn the value is $\frac{1+0.0557}{1-0.0557} = \frac{1.0557}{0.9443} = 1.1180$.

**35.** **i.** Counting days, hours, and minutes, shows that $t_{ve} = 75.6313$ days for 1995. Adding the length of spring, i.e., 92.7639 days to $t_{ve} = 75.6313$, gives us $t_{ss} = 168.3952$ days for 1995.

**ii.** For $t_{ve}$, the strategy of Section 4.8 gives $\beta_1 = 1.3011$ and then $\beta = \beta_3 = \beta_4 = 1.3173$.

**iii.** Making the indicated substitutions gives $r = 0.9958$ AU and $\alpha = 1.3335$ radians.

**iv.** For $t_{ss}$, the strategy of Section 4.8 gives $\beta_1 = 2.8969$ and then $\beta = \beta_3 = \beta_4 = 2.9009$. It follows that $r = 1.0162$ AU and $\alpha = 2.9048$ radians.

**36.** Since $e = 0$, we find that $a = b$. So the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = a^2$ coincide. In reference to Figure 4.34, this means that the focus $S$ of the ellipse is the center $O$ of the circle and that the points $P$ and $P_0$ coincide. So the segment $SP$ coincides with the segment $OP_0$. It follows that $\alpha = \beta$, $r = a$, and by Kepler's formula (or a direct argument), that $\alpha = \beta = \frac{2\pi t}{T}$. The method of successive approximations of Section 4.8 is not needed because $r = a$ and $\alpha = \beta$ has an explicit expression in terms of $t$.

## 4E. The Orbit of Halley's Comet

**37.** By Kepler's third law $\frac{a^3}{T^2} = 1$ in the units AU and years because this ratio is 1 for the Earth. Since $T = 76$ years for Halley, we get $a^3 = T^2 = 76^2$. It follows that $a = 76^{2/3} = 17.94$ AU. Since the minimum distance between Halley and the Sun is $a - e = d = 0.59$ AU, where $e$ is the linear eccentricity, we see that $e = a - d = 17.94 - 0.59 = 17.35$ AU. Halley's semiminor axis is $b = \sqrt{a^2 - e^2} = 4.56$ AU and its astronomical eccentricity is $\varepsilon = \frac{e}{a} = \frac{17.35}{17.94} = 0.967$. Halley's greatest distance from the Sun is $a + e = 17.94 + 17.35 = 35.29$ AU. The ratio $\frac{v_P}{v_A}$ for Halley is $\frac{v_P}{v_A} = \frac{17.94+17.35}{17.94-17.35} = \frac{35.29}{0.59} \approx 60$.

**38.** By assumption and Figure 4.39, the Earth's orbit is a circle with center $(e, 0)$ and radius 1. So $(x-e)^2 + y^2 = 1$ is an equation of the orbit. To find the $x$-coordinates of the points $H_1$ and $H_2$ of Figure 4.40, we need to solve the equations $(x-e)^2 + y^2 = 1$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for

$x$. By substituting and taking common denominators, we get

$$1 = \frac{x^2}{a^2} + \frac{1-(x-e)^2}{b^2} = \frac{b^2x^2 + a^2 - a^2x^2 + 2a^2ex - a^2e^2}{a^2b^2} = \frac{a^2 - e^2x^2 + 2a^2ex - a^2e^2}{a^2b^2}$$

So $a^2b^2 = a^2 - e^2x^2 + 2a^2ex - a^2e^2$ and $e^2x^2 - 2a^2ex + a^2e^2 - a^2 + a^2b^2 = 0$. Because $a^2e^2 - a^2 + a^2b^2$ simplifies to $a^2(e^2 - 1 + b^2) = a^2(a^2 - 1)$, we get by using the quadratic formula that

$$x = \frac{2a^2e \pm \sqrt{4a^4e^2 - 4e^2a^2(a^2-1)}}{2e^2} = \frac{2a^2e \pm 2ae\sqrt{a^2-(a^2-1)}}{2e^2} = \frac{a^2 \pm a}{e}\,.$$

Because $\frac{a^2+a}{e} = \frac{a^2}{e} + \frac{a}{e} > \frac{a^2}{a} = a$, it is not possible for $x = \frac{a^2+a}{e}$. (Since $x = a$ is the $x$-intercept of the ellipse in Figure 4.40.) So we can conclude that $x = \frac{a^2-a}{e} = \frac{a(a-1)}{e}$. So $x = \frac{17.94(16.94)}{17.35} = 17.52$ AU. Inserting this value of $x$ into $y^2 = 1-(x-e)^2$, gives us $y^2 = 1-(17.52-17.35)^2 = 1-(0.17)^2 = 1-0.03 = 0.97$. It follows that the $y$-coordinates of the points $H_1$ and $H_2$ are 0.98 and $-0.98$ respectively. To sketch a more accurate version of Figure 4.40, place $H_1$ and $H_2$ in such a way that the vertical segment $H_1H_2$ is 0.17 units to the *right* of the Sun $S$. Notice that the trajectory of Halley (within the Earth's orbit) is much "steeper" than suggested in the figure.

**39.** The value of $r$ is equal to the length of the segment $SP = SH_1$. Since $H_1$ lies on the circle with center $S$ and radius 1 AU (in other words on the Earth's orbit), it follows that $r = 1$. Because 0.98 is the y-coordinate of $H_1$, notice that $\sin\alpha = \frac{0.98}{1} = 0.98$. Using the inverse sine button of your calculator, will give you $\alpha = 1.37$ (in radians). Now use Gauss's formula $\tan\frac{\alpha}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\beta}{2}$ to compute $\beta$. By Exercise 37, $\varepsilon = \frac{e}{a} = \frac{17.35}{17.94} = 0.967$. So

$$\tan\frac{\beta}{2} = \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{\alpha}{2} = (0.13)(0.82) = 0.11.$$

By taking an inverse tan, $\frac{\beta}{2} = 0.11$. So $\beta = 0.22$. Inserting what we already know into Kepler's formula $\beta - \varepsilon\sin\beta = \frac{2\pi t}{T}$, tells us that $0.22 - (0.967)(0.22) = \frac{2\pi t}{76}$. Solving for $t$, we get $t \approx \frac{(0.01)(76)}{2\pi} = 0.12$ years. We have shown that Halley requires about 0.12 years or 44 days to move from perihelion to $H_1$. A repetition of this discussion (or an appeal to symmetry) will show that Halley requires the same number of days to move from $H_2$ to perihelion. It follows that Halley remains inside the Earth's orbit for about 88 days.

**40.** Recall that in general $|\beta - \beta_i| \leq \varepsilon^i$. To insure that $|\beta - \beta_i| \leq 0.0002$, we need to achieve $\varepsilon^i \leq 0.0002$. For Halley, $\varepsilon = 0.967$. To be on the safe side we will take $\varepsilon = 0.968$. (There is no information about the fourth decimal place.) By squaring again and again, we see that $\varepsilon^{256} < 0.000243$. So this gets us close. Multiplying by 0.968 six more times shows that $\varepsilon^{262} < 0.0002$.

**41.** This task is left to the student.

## 4F. More Trigonometry and Gauss's Formula

**42.** Enough hints have been supplied.

**43.** The first part is obvious. For the second, use $\sin^2\varphi + \cos^2\varphi = 1$.

**44.** The equality $\tan^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{1+\cos\alpha}$ is verified by using the equalities $2\cos^2\varphi = 1+\cos 2\varphi$ and $2\sin^2\varphi = 1-\cos 2\varphi$ with $\varphi = \frac{\alpha}{2}$. That $\frac{1-\cos\alpha}{1+\cos\alpha} = \frac{1+\varepsilon}{1-\varepsilon}\frac{1-\cos\beta}{1+\cos\beta}$ follows by use of $\cos\alpha = \frac{\cos\beta-\varepsilon}{1-\varepsilon\cos\beta}$. That $\tan^2\frac{\beta}{2} = \frac{1-\cos\beta}{1+\cos\beta}$, uses $2\cos^2\varphi = 1+\cos^2\varphi$ and $2\sin^2\varphi = 1-\cos 2\varphi$ again, this time with $\varphi = \frac{\beta}{2}$. So

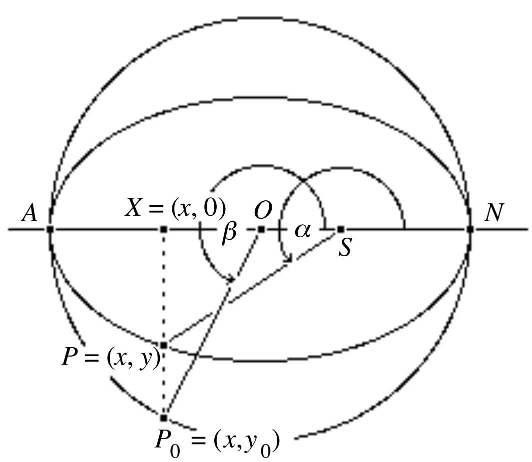
$$\tan^2\frac{\alpha}{2} = \frac{1+\varepsilon}{1-\varepsilon}\tan^2\frac{\beta}{2}.$$

Now suppose that $0 \le \beta < \pi$. Refer to the basic diagram from Kepler's discussion and notice that $0 \le \alpha < \pi$. So $0 \le \frac{\beta}{2} < \frac{\pi}{2}$ and $0 \le \frac{\alpha}{2} < \frac{\pi}{2}$. Refer to the graph of the tangent and notice that $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are both positive. So by taking square roots, $\tan\frac{\alpha}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\beta}{2}$ in this case. If $\pi < \beta < 2\pi$, then $\pi < \alpha < 2\pi$. Now both $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are negative. So again, $\tan\frac{\alpha}{2} = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}\tan\frac{\beta}{2}$. If $\beta = \pi$, then by the basic diagram from Kepler's discussion $\alpha = \pi$. So neither $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are defined in this case. (Have a look at the graph of the tangent.) This presents no problem since the basic point is to determine $\alpha$ in terms of $\beta$.

## 4G. A study of Kepler's Formulas

**Correction:** In Exercise 45 the formula $\tan\alpha = \frac{b\sin\beta}{a(\cos\beta-\varepsilon)}$ is incorrectly written as $\tan\alpha = \frac{b\sin\beta}{a(\cos\beta-e)}$.

**45.** In the new figure, $P$ and $P_0$ both lie below the $x$-axis, $X$ is on the left of $O$, and $x$ is negative. Check that $SX$ is equal to $e-x$ in this situation also. Let $\alpha' = \angle XSP$ and $\beta' = \angle XOP_0$ and notice that $\alpha = \alpha'+\pi$ and $\beta = \beta'+\pi$. By use of the Examples 4.11 and 4.12, the verifications



of the equations $r = a(1 - \varepsilon\cos\beta)$ and $\tan\alpha = \frac{b\sin\beta}{a(\cos\beta - \varepsilon)}$ go through virtually unchanged. Only Kepler's equation remains. The sectors referred to in the argument below, as well as the related sections, will be those determined by the angles $2\pi - \alpha$ and $2\pi - \beta$ rather than $\alpha$ and $\beta$. In reference to the circle,

$$\text{Area section } P_0XN = \text{Area sector } P_0ON + \text{Area } \Delta P_0XO\,.$$

Because the sector $P_0ON$ is determined by the angle $2\pi - \beta$, its area is equal to $\frac{1}{2}a^2(2\pi - \beta)$. The area of the triangle $\Delta P_0XO$ is $\frac{1}{2}(-x)(-y_0) = \frac{1}{2}(-x)(-a\sin\beta)$. So

$$\text{Area section } P_0XN = \frac{1}{2}a^2(2\pi - \beta) + \frac{1}{2}xa\sin\beta\,.$$

Now turn to the ellipse. By Cavalieri's principle,

$$\text{Area section } PXN = \frac{b}{a}\left(\frac{1}{2}a^2(2\pi - \beta) + \frac{1}{2}xa\sin\beta\right) = \frac{1}{2}ba(2\pi - \beta) + \frac{1}{2}xb\sin\beta\,.$$

Note next that $A_t$ is equal to the area of the full ellipse, minus the area of the section just computed, plus the area of $\Delta PXS$. So

$$\begin{aligned}
A_t &= ab\pi - \frac{1}{2}ba(2\pi - \beta) - \frac{1}{2}xb\sin\beta + \frac{1}{2}(e - x)(-y) \\
&= \frac{1}{2}ab\beta - \frac{1}{2}xb\sin\beta - \frac{1}{2}(e - x)b\sin\beta = \frac{1}{2}ab\beta - \frac{1}{2}eb\sin\beta \\
&= \frac{1}{2}ab(\beta - \varepsilon\sin\beta)\,.
\end{aligned}$$

The rest of the argument is identical to the one in the text. Alternatively, the argument in the text can be retained with the following understanding: Let the sector $P_0ON$ be that determined by $\beta$, the section $P_0XN$ to be that with perimeter the circular arc from $N$ to $A$ to $P_0$ and the segments $P_0X$ and $XN$, and let the elliptical section $PXN$ be that with perimeter the elliptical arc from $N$ to $A$ to $P$, and the segments $PX$ and $XN$.

**46.** Let $\alpha, \beta$, and $t$ be the parameters for the same position in the first orbit. In going from the first orbit to the second, $2\pi$ is added to both $\alpha$ and $\beta$ and $T$ is added to $t$. Observe that $a, b, \varepsilon$, and $r$ are the same for both orbits. So the question is as to whether the formulas are valid with $\alpha' = \alpha + 2\pi$ in place of $\alpha, \beta' = \beta + 2\pi$ in place of $\beta$, and $t' = t + T$ in place of $t$. This follows quickly from the identities in Examples 4.11 and 4.12. For example,

$$\beta' - \varepsilon\sin\beta' = \beta + 2\pi - \varepsilon\sin\beta = \frac{2\pi t}{T} + 2\pi = \frac{2\pi t + 2\pi T}{T} = \frac{2\pi t'}{T}.$$