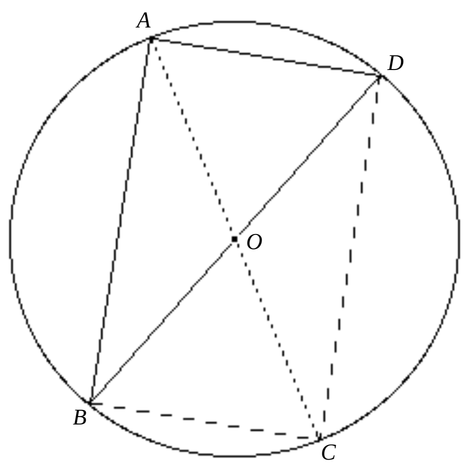
Solutions to the Exercises of Chapter 3

3A. The Circle and Related Areas

Correction: In the statement of Exercise 1, "diagonal" should be "diameter."

1. Let ΔBAD be the triangle in question with BD the diameter of the circle. Let O be the center of the circle. By Proposition 2.1, $\angle BAD$ is equal to one-half the angle given by O and the arc BCD. Since the latter is π, the former is $\frac{\pi}{2}$. So ΔBAD is a right triangle. Next, extend a line from A through O to some point C and complete this to the quadrilaterial

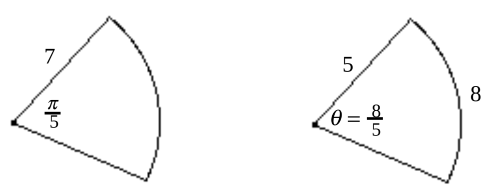


$ABCD$. Note that the triangles ΔAOD and ΔBOC are similar (side-angle-side). It follows that $AD = BC$. In the same way, $AB = CD$. By Ptolemy's Theorem,

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$

So $BD^2 = AB^2 + AD^2$. Again, ΔBAD is a right triangle by Pythagoras's Theorem.

2. The area of a circular sector is $\frac{1}{2}r^2\theta$ where r is the radius of the circle and θ the central angle of the sector. In the first case, $\theta = \frac{\pi}{5}$ and $r = 7$, so that the area is $\frac{1}{2}(7^2)\frac{\pi}{5} = \frac{49\pi}{10} \approx 15.4$. In



the second case, $\theta = \frac{8}{5}$ and the area is $\frac{1}{2}(5^2)\frac{8}{5} = 20$.

3. The area of the full circular sector ACB is $\frac{1}{2}r^2\theta$. We need to subtract from this the area of the triangle ΔABC. Let D be the midpoint of the segment AB. So ΔADC is one half of the triangle ΔABC. Note that $\sin\frac{\theta}{2} = \frac{AD}{r}$ and $\cos\frac{\theta}{2} = \frac{CD}{r}$. So the area of ΔADC is

$\frac{1}{2}CD \cdot AD = \frac{1}{2}r^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2}$. So the area of ΔABC is $r^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2}$.

Comment: By an application of Exercise 6(iii), $\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{1}{2}\sin\theta$. Therefore, the shaded area in Figure 3.30 is equal to $\frac{1}{2}r^2(\theta - \sin\theta)$. So it measures the difference between θ and $\sin\theta$.

4. Since $\theta = 50\left(\frac{\pi}{180}\right) \approx 0.873$ radians, we get that the area of the shaded section is

$$\begin{aligned}\frac{1}{2}r^2\theta - r^2\left(\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}\right) &\approx \frac{1}{2}5^2(0.873) - 5^2\left(\sin\frac{0.873}{2}\right)\left(\cos\frac{0.873}{2}\right)\\ &\approx 10.9125 - 25(0.4228)(0.9062) \approx 1.3340.\end{aligned}$$

5. All that we need to do is to show that the area of the quarter circle AOB is equal to the area of the semicircle whose diameter is AB. Why? Let $r = OB$. So the area of the quarter circle is $\frac{1}{4}(\pi r^2)$. Let $s = BC$. This is the radius of the semicircle. So the area of the semicircle is $\frac{1}{2}(\pi s^2)$. Is there a connection between r and s? By Pythagoras, $(2s)^2 = r^2 + r^2$. So $4s^2 = 2r^2$, and hence $2s^2 = r^2$. Plugging this into the expression for the area of the quarter circle gives the area of the semicircle.

6. Since the area of the triangle AOB is equal to $\frac{1}{2}r^2$, this is also the area of the lune. The area of the section of the circle between the lune and the triangle is found by applying the formula in Exercise 3 with $\theta = \frac{\pi}{2}$.

3B. Sigma Notation and Areas

7.
$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4.$$
$$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2.$$
$$\sum_{i=1}^{6} i^i = 1 + 2^2 + 3^3 + 4^4 + 5^5 + 6^6.$$

8. Consider the pattern and observe that the areas of the black regions are given by the progression of numbers

$$\frac{1}{2},\ \frac{1}{2}+\frac{1}{2^2},\ \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3},\ \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4},\ldots,$$

and after n steps, by $\sum_{i=1}^{n}\left(\frac{1}{2}\right)^i = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$. Let n get larger and larger. Since the area of the square is 1, the areas of the black regions close in on 1. Rewriting this, we have: $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{2}\right)^i = 1$.

9. By using similar triangles, the height of rectangle R_1 is $\frac{1}{2}h$. So Area $R_1 = \frac{1}{2}b\cdot\frac{1}{2}h$. Hence Area $R_1 = \frac{1}{4}bh$. In the same way, the height of R_2 is $\frac{1}{4}h$. So Area $R_2 = \frac{1}{4}b\cdot\frac{1}{4}h = \frac{1}{16}bh$.

Therefore, Area R_2 + Area $R_2^{'}$ = $\frac{1}{8}bh$. At the next step, there are four rectangles, each with base $\frac{1}{8}b$ and height $\frac{1}{8}h$, and hence area $\frac{1}{64}bh$. The total area added by this step is $4\left(\frac{1}{64}bh\right) = \frac{1}{16}bh$. So the progression of numbers,

$$\text{Area } R_1,\ [\text{Area } R_1 + (\text{Area } R_2 + \text{Area } R_2^{'})], \ldots$$

has the pattern

$$\frac{1}{4}bh\,,\ \frac{1}{4}bh + \frac{1}{8}bh\,,\ \frac{1}{4}bh + \frac{1}{8}bh + \frac{1}{16}bh\,, \ldots .$$

But this is equal to

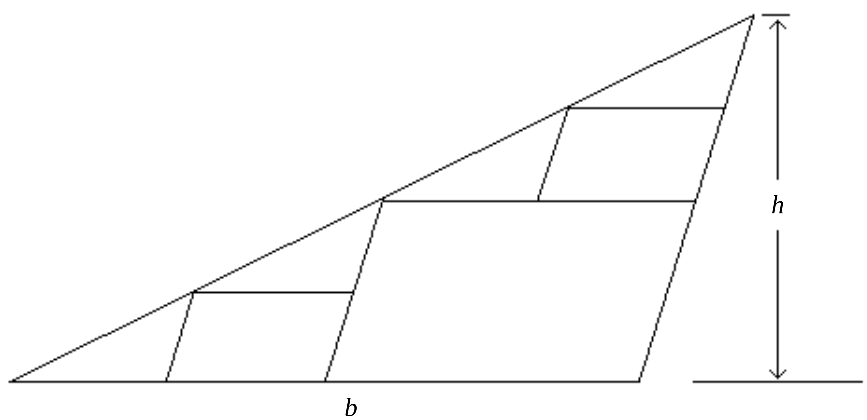
$$\frac{1}{2}bh\left(\frac{1}{2}\right),\ \frac{1}{2}bh\left(\frac{1}{2} + \frac{1}{2^2}\right),\ \frac{1}{2}bh\left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}\right), \ldots .$$

In view of Exercise 8, the sequence of numbers $\frac{1}{2}, \frac{1}{2} + \frac{1}{2^2}, \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}, \ldots$ closes in on 1. So,

$$\text{Area } R_1, [\text{Area } R_1 + (\text{Area } R_2 + \text{Area } R_2')], \ldots$$

closes in on $\frac{1}{2}bh$. But from a geometric point of view, these numbers close in on the area of the triangle.

Note: Exercise 9 can be carried out for any triangle - not just a right triangle - by replacing



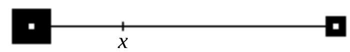
the rectangles of Figure 3.33 by parallelograms.

10. By Archimedes's theorem, this area is equal to $\frac{4}{3}\left(\frac{1}{2}(7)(4)\right) = 18\frac{2}{3}$.

3C. Archimedes's Law of the Lever

11. For equilibrium, $m_1 d_1 = m_2 d_2$. So $2000 = m_2 \cdot 10$ and $m_2 = 200$. So a force of 200 units must be applied at P_2 to attain equilibrium.

12. Put both masses on a lever and place the fulcrum a distance x from the larger mass. For



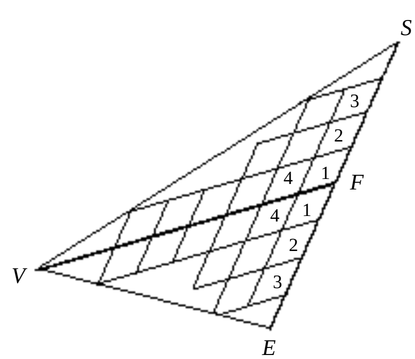
equilibrium, $80x = 15(9 - x)$. So $95x = 135$, and $x = 1.42$ meters.

3D. Facts needed in the Method

Note: The concept *center of mass* or *centroid* has already been used but not defined. Let's do so in intuitive terms. Consider any connected region in the plane and assume it to be made of a thin, homogeneous, and rigid material. If you were to try to balance the region on the tip of one of your fingers (or, to insure precision, at the tip of a pin) in such a way that the region remains in stable horizontal position, then the point at which you would have to place your finger (or the pin) is called the center of mass, or centroid, of the region. Note that this definition is consistent with the use of the concept in Exercise 12.

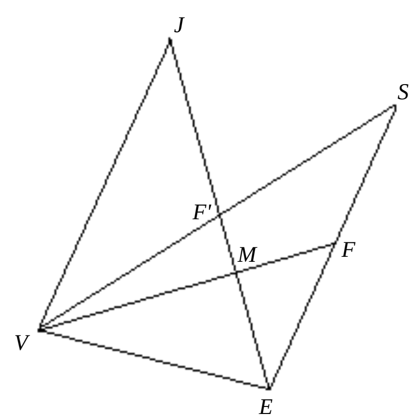
13. Refer to Figure 3.36. Because F is the midpoint of ES, it follows that $EF = FS$. So taking EF and FS as the respective bases of the triangles ΔVFE and ΔVFS, we know that these bases are equal. To see that their heights (from the perspective of these two bases) are the same simply drop a perpendicular from V to an extension of the segment ES. It follows that the areas of ΔVFE and ΔVFS are equal. Does this fact allow us to conclude that the centroid M must lie on the segment VF? The following consideration does make this conclusion plausible: Consider the triangle ΔVES to be made of thin, homogeneous, and rigid material. Suppose you have a wire that is stretched taught and horizontal. Place the triangle ΔVES horizontally in such a way that the wire lies below and along the segment VF. Since the overhanging triangles ΔVFE and ΔVFS have the same area and hence the same weight, it is plausible that ΔVES should be balanced on top of the wire in horizontal position and that, therefore, the centroid M should lie on VF.

Correction: The argument above makes it plausible that M should lie on VF. But this argument is not sufficient. Why? Simply because the requirements for balance (as we have seen) involve not



only weight, but also distance. Therefore more needs to be said. Archimedes's verification is another example of his technique "conquer by subdividing." The figure illustrates what Archimedes has in mind. In the key step he fills the two triangles ΔVFE and ΔVFS with similar arrays of small identical parallelograms so that each parallelogram on ΔVFE has a corresponding partner on ΔVFS. (Four such pairs are shown in the figure.) By inserting a huge number of parallelograms Archimedes fills out the two halves of the triangle. Since each parallelogram of ΔVFE balances its partner on ΔVFS (with fulcrum on VF), it follows that the point of balance M of the triangle ΔVES must lie on VF. So the center of mass of ΔVES lies on VF. For the details, see "On Plane Equilibriums I," in Sir Thomas Heath, *A History of Greek Mathematics, Volume II, From Aristarchus to Diophantus,* Dover Publications, Inc., New York, 1981.

14. Refer to Figure 3.36 and the figure below. The segment EF' has been extended to a segment $EF'J$ in such a way that VJ is parallel to ES. Two triangles that have equal corresponding angles are similar. So $\Delta VF'J$ is similar to $\Delta EF'S$. Since F' is the midpoint of VS, it follows that $VJ = ES$. In the same way, ΔEFM is similar to ΔVMJ and hence $\frac{EF}{VJ} = \frac{FM}{VM}$. There-



fore, $\frac{EF}{ES} = \frac{FM}{VM}$. But F is the midpoint of ES. So $\frac{FM}{VM} = \frac{1}{2}$. Therefore, $FM = \frac{1}{2}VM$. It follows that $FM = \frac{1}{3}VF$, as required.

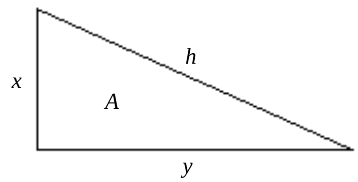
15. Go through the proof and note that the location of XZ is irrelevant.

Correction: In Figure 3.37 the segment FK needs to be shortened so that its length is equal to that of the segment VF.

3E. Some Mathematics from the Middle Ages

Correction: In the statement of Exercise 16, replace "height" by "hypothenuse".

16. Consider a right triangle with hypotenuse h and area A. Let x and y be the other two



sides. We must solve for x and y in terms of h and A: On the one hand, $x^2+y^2=h^2$, and on the other, $A=\frac{1}{2}xy$. So $xy=2A, y=\frac{2A}{x}$ and $y^2=\frac{4A^2}{x^2}$. Therefore, $x^2+\frac{4A^2}{x^2}=h^2$. Multiply through by x^2, to get

$$x^4-h^2x^2+4A^2=0.$$

Now what? Notice that you can solve for x^2 by use of the quadratic formula:

$$(*) \quad x^2 = \frac{h^2\pm\sqrt{(h^2)^2-4(4A^2)}}{2} = \frac{h^2\pm\sqrt{h^4-16A^2}}{2}.$$

Solve for x by taking square roots. Then plug x^2 into $y^2=\frac{4A^2}{x^2}$ and solve for y. This leaves the $\pm$ ambiguity. To resolve it, do the following: Notice that $x^2=\frac{4A^2}{y^2}$. Substituting this into $x^2+y^2=h^2$ shows (in the same way as in the earlier case of x^2) that

$$(**) \quad y^2=\frac{h^2\pm\sqrt{h^4-16A^2}}{2}.$$

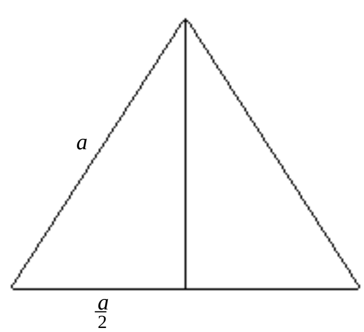
Taking x to be the longer leg of the triangle, we will assume that $x\geq y$. A look at both $(*)$ and $(**)$ shows that we must have

$$x^2=\frac{h^2+\sqrt{h^4-16A^2}}{2} \quad\text{and}\quad y^2=\frac{h^2-\sqrt{h^4-16A^2}}{2}.$$

Since x and y are positive, we now get

$$x=\sqrt{\frac{h^2+\sqrt{h^4-16A^2}}{2}} \quad\text{and}\quad y=\sqrt{\frac{h^2-\sqrt{h^4-16A^2}}{2}}.$$

17. Consider the equilateral triangle with side a. Let its height be h. By Pythagoras,



$h^2+\left(\frac{a}{2}\right)^2=a^2$. So $h^2=\frac{3}{4}a^2$, and $h=\frac{\sqrt{3}}{2}a$. Therefore, the area of the triangle is $\frac{\sqrt{3}}{4}a^2$. Setting this equal to Gerbert's expression for the area, we get $\frac{\sqrt{3}}{4}a^2=\left(\frac{a}{2}\right)\left(a-\frac{a}{7}\right)=\left(\frac{a}{2}\right)\left(\frac{6a}{7}\right)=\frac{6a^2}{14}$. It follows that $\sqrt{3}=\frac{12}{7}=1.714$. The correct value of $\sqrt{3}$ up to decimal places is 1.732.

Note: Gerbert, a brilliant student as a young monk in France, was sent to study mathematics in Spain, then under the control of the Islamic Moors. He carried back information from and about the very advanced centers of Islamic learning (such as the one in Cordoba). Most importantly in current context, he brought back from Spain the Hindu Arabic numeral system and introduced it to Christian Europe. Later, as schoolmaster of the Cathedral school at Rheims, he built up a sizable library and reformed the rigid and limited curricula. Gerbert, and in turn his students, brought an element of enlightenment to the medieval Europe of the 11^{th} and 12^{th} centuries.

3F. Mathematics in the 16th Century

18. Note that $x = 1$ is a root of $x^3 + 3x - 4$. So $x - 1$ divides $x^3 + 3x - 4$. Carry out the division

$$x - 1 | \overline{x^3 + 3x - 4}$$

to show that $x^3 + 3x - 4 = (x - 1)(x^2 + x + 4)$. Can $x^2 + x + 4$ be factored? If yes, then it must have a root. By the quadratic formula this must have the form $\frac{-1 \pm \sqrt{1-16}}{2}$. The appearance of $\sqrt{-15}$ in this expression tells us that this is impossible. So $x^3 + 3x - 4$ cannot be factored further. Now consider $x^3 - 7x - 6$ and observe that $x = -2$ is a root. So $x + 2$ divides $x^3 - 7x - 6$. Carry out the division

$$x + 2 | \overline{x^3 - 7x - 6}$$

to get that $x^3 - 7x - 6 = (x + 2)(x^2 - 2x - 3)$. Does $x^2 - 2x - 3$ factor? Note that $x^2 - 2x - 3 = (x - 3)(x + 1)$. So $x^3 - 7x - 6 = (x + 2)(x - 3)(x + 1)$.

3G. Celestial Navigation

19. Denote the distance from the ship to the equator by d and let r_E be the radius of the Earth. If β is given in radian measure, then $\beta = \frac{d}{r_E}$. So $d = \beta r_E$. Because $\alpha + \beta = \frac{\pi}{2}$, we get that $d = \left(\frac{\pi}{2} - \alpha\right) r_E$.

20. If $\alpha = 53°$, then $\alpha \approx 0.925$ radians. So by Exercise 19, $d \approx (1.571 - 0.925)(3950) = (0.646)(3950) \approx 2550$ miles.