Mathematics from The DaVinci Code

We turn to assess the mathematics in Dan Brown's novel *The DaVinci Code*. The disussion that follows provides an analysis of Professor Langdon's assertions about the Fibonacci sequence and the number Phi. The mathematics involved is related to several of the themes of previous chapters, especially Chapter 3. These include connections with the mathematical world of the Greeks, the transfusion of sophisticated mathematics from the Islamic world into medieval Europe, the convergence of sequences, trigonometric matters, geometrical constructions, factorizing polynomials, and the applications of mathematics.

The central character of this story is Leonardo of Pisa (about 1175-1250). The name Fibonacci, a contraction of *filius Bonacci*, "son of Bonacci" in Latin, was conferred on him by a mathematician in the 18th century. In the eleventh and twelfth centuries Greek and Arabic mathematics was streaming through Spain and Sicily into Western Europe. Along with the goods of merchants it came to Venice, Genoa, and Pisa (illustrating the words of a famous historian that "transmission is to civilization what reproduction is to life"). Leonardo's father (presumably the Bonacci just referred to) was managing a Pisa trade agency in what is Algeria today. Young Leonardo joined him there and was taught by an Islamic scholar. He learned to calculate "by a marvelous method through the nine figures" of the Hindus and became aquainted with the mathematics of Euclid, Archimedes, and Diophantus. In 1202 he published "a book about the abacus" the *Liber Abacci* that signalled the rebirth of mathematics in Latin Christendom. This historic work introduced Arabic algebra to Western Europe and began a practice of using letters instead of numbers to generalize and abbreviate equations. It provided the first thorough exposition of the Hindu-Arabic numerals - including zero - and the method of calculating with them. This is the system we use today: the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, the positional notation for larger numbers, and the arithmetic with which we compute. This system was resisted by the merchants of Europe who calculated with the abacus and recorded the results with Roman numerals. Indeed, as late as 1299 a law was passed in Florence that outlawed the use of the "newfangled figures." Only few realized at the time that the new symbols and the organization of a number into units, tens, hundreds, ... would open the way to mathematical developments completely beyond the reach of any scheme based on the numerals of the Greeks and Romans. It was not until the 16th century that the Hindu-Arabic scheme began to prevail in Western Europe.

The sequence of numbers

$$1, 1, 2, 3, 5, 8, 13, 21$$

that cryptologist Sophie Neveu mentions goes on beyond these eight numbers. The pattern that is already apparent persists: Just add the previous two numbers to get the next. So the ninth number in the sequence is $13 + 21 = 34$, the tenth is $21 + 34 = 55$, the eleventh is $34 + 55 = 89$, and so on. The 24-th number turns out to be 46,368 and the 25-th is 75,025. Why is the name Fibonacci attached to this progression of numbers? On pages 123 and 124 of a surviving 1228 edition of the

Liber Abacci, Fibonacci raises the question "How many pairs of rabbits are born of one pair in a year?" He then goes on to formulate the problem this way:

47. "Someone placed a pair of rabbits in a certain place, enclosed on all sides by a wall, to find out how many pairs of rabbits will be born there in the course of one year, it being assumed that every month a pair of rabbits produces another pair, and that rabbits begin to bear young two months after their own birth." Under these assumptions, Fibonacci determines how many pairs rabbits are born in each month. By adding them, he gets the total number of pairs born during the year. What answers did he get? Draw a flow chart that organizes what is happening and determine the twelve monthly numbers as well as the total.

Fibonacci's approach to this problem illustrates the important fact that assumptions need to be made in any application of mathematics. If the application is to be useful, then such assumptions need to reflect - at least in an idealized or approximate way - basic properties of the particular situation. In this regard, comment on the assumptions that Fibonacci makes.

Fibonacci's Sequence and the Number ϕ

The *divine proportion* Phi, or ϕ in Greek rendition is also known as the *golden ratio*. This is the number that Professor Langdon rapsodizes about. We have already encountered it in the section "More about Rational and Real Numbers" of the Additional Exercises for Chapter 1. Recall from there that

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618033989\ldots .$$

so that the value 1.618 that Professor Langdon provides is only an approximation.

48. Show that this number is the positive root of the polynomial $x^2 - x - 1$. Deduce that $\phi^{-1} = \phi - 1 = \frac{\sqrt{5}-1}{2}$.

Professor Langdon's assertion that the sequence of ratios of successive numbers of the Fibonacci sequence closes in on $\phi = \frac{1+\sqrt{5}}{2}$ is correct. The first few ratios are $\frac{1}{1} = 1$, $\frac{2}{1} = 2$, $\frac{3}{2} = 1.5$, $\frac{5}{3} \approx 1.666$, $\frac{8}{5} = 1.6$, $\frac{13}{8} = 1.625$, $\frac{21}{13} \approx 1.615$, $\frac{34}{21} \approx 1.619$, $\frac{55}{34} \approx 1.617$, and $\frac{89}{55} \approx 1.618$. So the ratio of the eleventh Fibonacci number over the tenth agrees with ϕ up to three decimal places. The ratio obtained by dividing the 25-th number by the 24-th is the first ratio that agrees with ϕ up to the nine decimal places of the approximation provided above. So it appears that the sequence of ratios does close in on ϕ. But can we be sure?

Let's take any sequence of numbers $u_1, u_2, u_3, \ldots, u_n, u_{n+1}, u_{n+2}, \ldots$. We will say that it is of *Fibonacci type* if u_1 and u_2 are any two positive numbers and every subsequent number is obtained by adding the two previous ones. So, $u_3 = u_1 + u_2$, $u_4 = u_2 + u_3$, and so on. The equation $u_{n+2} = u_n + u_{n+1}$, valid for any $n \geq 1$, provides the pattern.

49. Let $u_1, u_2, u_3, \ldots, u_n, u_{n+1}, u_{n+2}, \ldots$ be any sequence of Fibonacci type. Show that

$$\frac{u_{n+2}}{u_{n+1}} = \frac{1}{\frac{u_{n+1}}{u_n}} + 1.$$

Now *assume* that a the sequence of ratios $\frac{u_2}{u_1}, \frac{u_3}{u_2}, \ldots, \frac{u_{n+1}}{u_n}, \ldots$ closes in on *some positive number* and call it α. Show that $\alpha = \frac{1}{\alpha} + 1$. Deduce that α must be equal to ϕ.

50. Write the first eight numbers of the sequence of Fibonacci type that the choice $u_1 = 1, u_2 = 100$ provides. Compute the seven ratios of consecutive numbers with the larger one in the numerator. Are your computations in line with the conclusion of Exercise 49? Do the same thing starting with $u_1 = 7, u_2 = 19$.

To reach the answer desired conclusion in Exercise 49 it was assumed that the ratios $\frac{u_2}{u_1}, \frac{u_3}{u_2}, \ldots,$ $\frac{u_{n+1}}{u_n}$ close in on some positive number. This is in fact correct, but it is a missing piece of our puzzle that is not easy to verify. In the case of the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots,$ this is done in

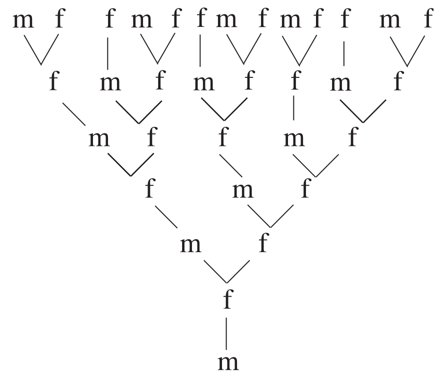
N. N. Vorob'ev, *FIBONACCI NUMBERS*, Blaisdell Publishing Company, New York · London, 1961.

In any case, when our Professor Langdon says that "the *quotients* of adjacent terms possessed the astonishing property of approaching the number 1.618−PHI!" he is correct (provided that PHI is understood to be $\frac{1+\sqrt{5}}{2}$). Even more astonishing is the fact that this is so for any sequence of Fibonacci type. It does not matter whether the process starts with the pair of numbers 1, 1, the pair 57, 100,359, or the pair π, $\sqrt{2}$, the resulting sequence of Fibonacci type has the property that the "flow" of ratios of consecutive terms closes in on ϕ.

Occurrences of ϕ in Nature and Art

Recall Professor Langdon's question "... did you know that if you divide the number of female bees by the number of male bees in any beehive in the world, you always get the same number?" The bi major's reply "You do?" reflects astonishment. The Professor pushes on "Yup. PHI" and the bio major's "NO WAY!" confirms disbelief. Who has in right? The professor or the bio major?

Let's begin with some information about the reproduction of bees. A queen bee mates with a male bee (called drone). Some of her eggs - they number in the thousands - are fertilized and some are not. The fertilized eggs grow into female worker bees (and an occasional queen). The unfertilized eggs grow into drones. So a female bee has two "parents," but a male bee has only one. With this information, you can sketch the ancestral tree of a drone - with f for female and m for male - as follows:



Now count the females in each generation. Starting at the bottom of the tree, there are 1 in the first, 1 in the second, 2, 3 in the third, 5 in the fourth, and 8 females in the fifth and last. A similar count of the males gives 1, 1, 2, 3, and 5. Both sets of numbers - and this is so no matter how far up into the ancestral tree you crawl - belong to the Fibonacci sequence. Note that in each generation, the ratio of the number of females to males is a ratio of successive Fibonacci numbers with the larger in the numerator. For example, at the top of the tree, the fifth generation, the ratio is $\frac{8}{5}$. Recall from Exercise 49 that the ratios of successive Fibonacci numbers close in on ϕ. So the higher up you go in the ancestral tree, the closer the ratio of female to male bees in the generation (at that "height") will be to ϕ. Refer to earlier computations and observe that this convergence is already evident from first few ratios. We can conclude that the ratio of female bees to male bees in the entire ancestral tree of the drone is approximately equal to ϕ. Case closed! Professor Langdon has it right! Or does he? Does Langdon's claim about bee populations follow from the analysis just undertaken?

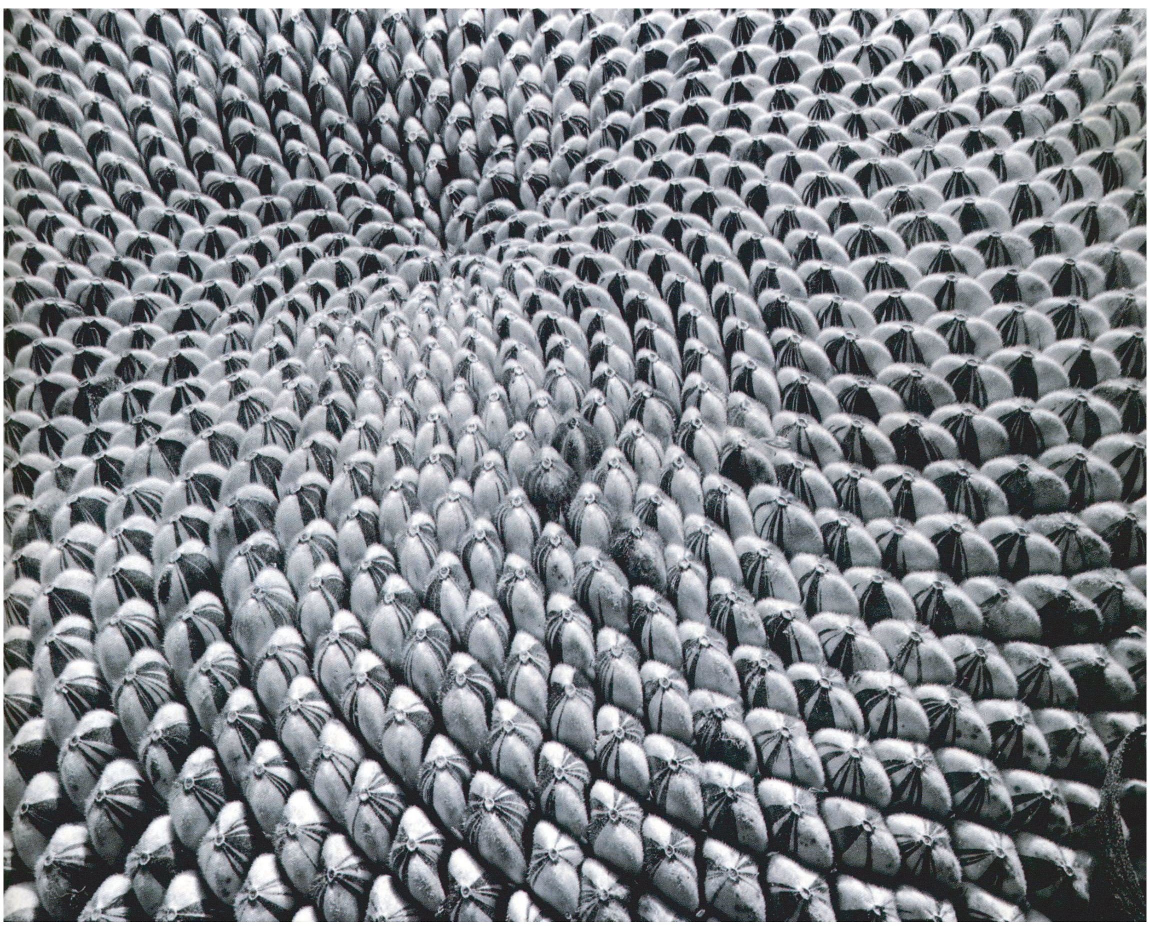
NO WAY! To see why not, we need to know a little more about bees. In the bee scheme of things (with very rare exception) only queen bees produce offspring and female worker bees are sterile. This means that the females appearing in an ancestral tree of a bee are queens (a worker bee appears only at the very bottom of its own tree). In fact, typically, a honey bee colony is one gigantic family where all the workers and virtually all the drones are the daughters and sons of a single queen. Over her life span a queen will mate with a dozen or so drones. In particular, only a handful of bees are involved in reproduction. So in the initial part of the ancestral tree of a drone sketched above, all the fs will likely represent the same queen and some of the ms will be the same drone. So there is lot's of duplication. So the count of bees that the Fibonacci sequence provides is abstractly correct but provides no information about the number of individual bees. It turns out that in a typical hive there are some 20,000 worker bees and a few hundred drones. So the ratio of females to males is much larger than ϕ. Professor Langdon is wrong! For an informed and wide-ranging discussion about bees and their hives, Langdon is invited to turn to

T. S. Seeley, *The Wisdom of the Hive,* Harvard University Press, Cambridge, MA, 1995.

Recall next that Langdon's slide of a close-up of a sunflower's seed head revealed that "sunflower seeds grow in opposing spirals." This is true. Have a look below at the striking close up of the sunflower. This photograph by Yasuhiro Ishimoto was scanned from

D. Mancoff, *Sunflowers,* The Art Institute of Chicago, Thames and Hudson, New York, 2001.

Observe that the seeds are distributed in two interlaced arrangements of spirals. One winds clockwise, the other counterclockwise, and the two arrangements seem to fit through each other. Now follow the spiral that ends at the lower right corner of the photograph to its beginning at the center of the vortex, and assess Professor Langdon's question: Can you guess the ratio of each rotation's diameter to the next?" What could he mean by that? What ratio exactly is it that is equal to ϕ? Not clear?! But there is a surprising connection between this eddy of seeds and the Fibonacci sequence. The number of clockwise spirals and the number of counterclockwise spirals are consecutive Fibonacci numbers. For example, there might be 34 clockwise spirals and 55 counterclockwise spirals, or 55 and 89, or even 89 and 144. Is this merely a mathematical curiosity? Or is there a deeper underlying mathematical reality? Why is it that nature arranges sunflower



seeds in this way? And it's not only sun flowers. In the plant kingdom the arrangement of leaves along stems, the number of petals in blossoms, the geometry of pinecones, and so on, display an "astonishing obedience" to the Fibonacci sequence.

As another example, consider the blossom of the daisy on the next page. It was taken by Rutherford Platt and can be found in

D. Bergamini, *Mathematics*, Life Science Library, Time Inc., New York, 1963.

Notice that the "florets" are arranged in spirals in similar pattern as the seeds of the sun flower. Count the number of clockwise spirals and then the number of counterclockwise spirals. What do you get?

Any understanding of the forces that underlie the shape and growth patterns of plants and animals must distinguish those of mathematics and physics from those that have been molded by evolution. It must "disentangle universal mathematical constraints from flexible genetic instructions." The search for an explanation of the persistent occurence of Fibonacci's numbers in nature has gone on for over three centuries. Significant progress was made in 1992 by two French mathematicians, who demonstrate conclusively that these mathematical patterns are not mere curiuosity but arise from mathematical laws of the physical world. Consider for example, the arrangement of the seeds on the "face" of the sunflower. The tip of the emerging shoot of a sunflower reveals under magnification the microscopic bits and elements from which all the parts of the sunflower develop. At the center of the tip is a circular region of tissue around which tiny lumps form. Over time, these migrate away from the organizing circle to become the leaves, petals, seeds, stem elements, and so on. The interlaced spiral pattern of seeds is determined by the physics of efficient arrangements and the biological dynamics that produce them. The numbers of Fibonacci arise as a mathematical consequence of these. We can now - in computer simulation - grow "realistic" grasses, flowers, bushes, and trees from mathematical rules. By a strange twist of history, Fibonacci's rabbit puzzle has a mathematical message for the biology of plants. This and much more information can be found in

Ian Stewart, *Life's Other Secret The New Nathematics of the Living World,* John Wiley, New York, 1998.

Mathematics arises in nature in the form of another example mentioned by Professor Langdon, namely the shell of the chambered pearly nautilus. The pearly nautilus is an ocean animal with a smooth, spiraling shell about 25 cm (10 inches) in diameter consisting of about 30 chambers. The animal lives in the outermost of these. The chambers are connected by a tube that lets the nautilus adjust the gases in the chambers. This ability allows it to change the depth at which it is swimming. A bottom feeder, the nautilus uses its suckerless tentacles (up to about 100 in number) for capturing prey. It lives at 1000 foot depths and is observable in its natural habitat only through the televison cameras of unmanned submersibles. The last surviving genus of an old (in the sense of evolution) order of shells it is used for dating geologic strata and provides information about the fossil record. The attractive photograph of the cross-section of the shell on the next page was taken by Rutherford Platt and scanned from

D. Bergamini, *Mathematics*, Life Science Library, Time Inc., New York, 1963.

Let's turn to Professor Langdon's reply to the bio major: "Correct. And can you guess what the ratio is of each spiral's diameter to the next?" What exactly is Landon after? The ratio of one diameter to the next? Let's take a diameter of the shell at any time in its life to be any segment from one outer wall through the center to the other outer wall. For the sake of our discussion, focus on the sequence of horizontal diameters from the last or outermost, to the preceeding one, to the one before that, and so on. OK? Print out the image (it is to scale) and measure the lengths of these diameters. Are the ratio's equal to ϕ? Sorry Langdon, no "eerie exactitude" here!

However, the spiral of the shell does satisfy a mathematical property. At any time in the life of the nautilus, take a line segment from the center of the spiral outward and let P be the point at which the segment crosses the shell's outer wall. Consider the tangent line at P to the curved outer wall and focus on the larger of the two angles at P between the segment and the tangent. If you were to print out the image of the shell and get yourself a "protractor" you would be able to check that the angle just descibed is close to $100°$ (give or take $2°$) *no matter what* segment you start with. So the spiral is a so-called *equiangular* or *logarithmic* spiral. It can be described mathematically (see the Additional Exercises for Chapter 13). Surely a mathematical mechanism that is similar in spirit to the one described for the seed arrangement of the sunflower must underlie the growth

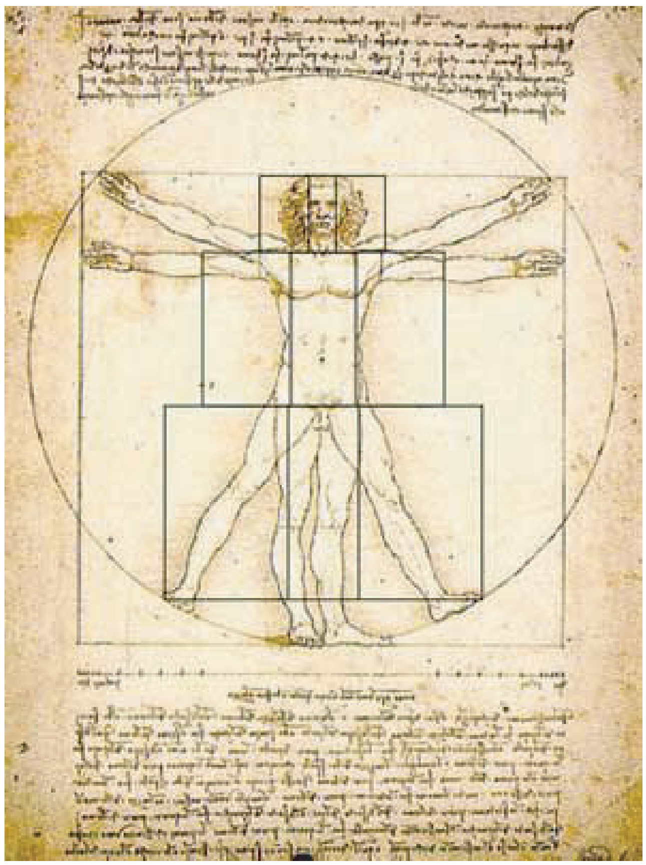
dynamic of the shell of the nautilus. Somehow it ought to be related to the shell's programmed sense to preserve its equiangular shape.

We next turn to Da Vinci's sketch the *Vitruvian Man* and Langdon's contention that Da Vinci understood "that the human body is literally made of building blocks whose proportional ratios *always* equal PHI." You will need to know that a rectangle is said to be *golden*, if the ratio of the longer side to the shorter is equal to ... you guessed it ... ϕ. The rendition of the Vitruvian man on the next page has the golden rectangles that DaVinci apparently had in mind superimposed. Convinced? Or not? The article

G. Markowsky, Misconceptions about the Golden Ratio, *The College Mathematics Journal*, Vol 23(1), 1992.

argues that many assertions commonly made about the golden rectangle lack precision and are not convincing.

We conclude this section with a discussion of a purely geometric game of growth related to the concerns above. The game begins with any rectangle. Pick one of the longer sides (if the rectangle is a square, then any of the four sides will do) and complete it to a square. Form a new larger rectangle by attaching this square to the rectangle at the side you have picked. Now repeat the process just described with this new rectangle. At each step the previous rectangle is enlarged by the addition of a square. We'll call this growth process "add a square."



51. **i.** Play "add a square" by starting with a square of side length 1. Show that the rectangles of the process get closer and closer to a golden rectangle.

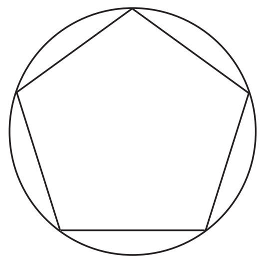
ii. Play "add a square" by starting with a golden rectangle. Show that every rectangle at every step is a golden rectangle.

52. Use one of the properties of a sequence of Fibonacci type to show that "add a square" provides a sequence of rectangles that get closer and closer to a golden rectangle *no matter what* rectangle the game is started with.

About the Pentagram

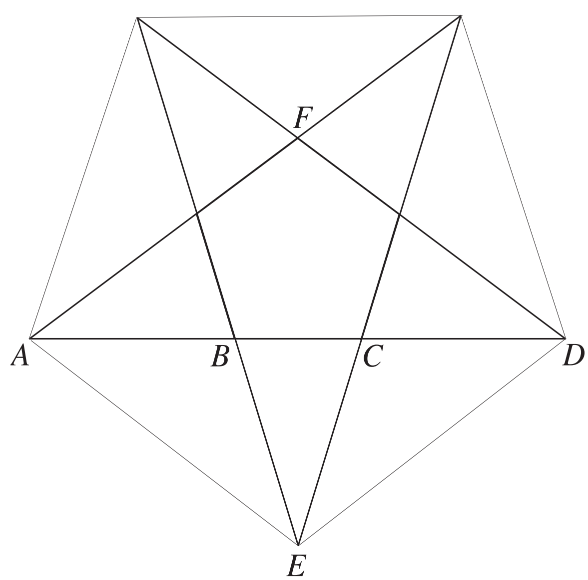
In conclusion we turn to examine Professor Langdon's comment: "Nice job. Yes, the ratios of line segments in a pentacle *all* equal PHI, making this symbol the *ultimate* expression of the Divine Proportion." Our presentation follows the little volume by Vorob'ev already mentioned.

The figure below shows a regular pentagon. All of its five vertices lie on a circle and the lengths of its five sides are the same.



53. Draw line segments from the center of the pentagon to two successive vertices. What is the angle between the segments? Show that the angle between successive sides of the pentagon is $108°$.

Extend all five sides of the pentagon to form a five pointed star. This star is the *pentacle* or *pentagram*. Connecting the points of the star gives us another pentagon.



We next recall a few facts from the matter discussed in the Additional Exercises for Chapter 2. First, for an obtuse angle θ, $\sin\theta = \sin(\pi - \theta)$. Next (see Exercise 37) there are the trigonometric identities

$$\sin 2\alpha = 2(\sin\alpha)(\cos\alpha) \quad \text{and} \quad \cos 2\alpha = 1 - 2\sin^2\alpha.$$

Finally (see Exercise 28), recall the Law of Sines.

54. Apply the law of sines to the triangle ΔADF to show that

$$\frac{AD}{AF} = \frac{\sin 108°}{\sin 36°} = \frac{\sin 72°}{\sin 36°} = 2\cos 36°.$$

55. Show that $\sin 72° = 4(\sin 18°)(\cos 18°)(1 - 2\sin^2 18°)$. Deduce that $\sin 18°$ is a root of the equation $8x^3 - 4x + 1 = 0$.

56. Use the strategy of Exercises 3F and the quadratic formula to factor the polynomial $8x^3 - 4x + 1$ into linear factors. To check your result deduce that $\sin 18° = \frac{\sqrt{5}-1}{4}$.

Recalling that $\cos 36° = 1 - 2\sin^2 18°$, we get by using Exercise 54 that

$$\tfrac{AD}{AF} = 2(1 - 2(\tfrac{\sqrt{5}-1}{4})^2) = 2(1 - \tfrac{5-2\sqrt{5}+1}{8}) = 2 - \tfrac{6-2\sqrt{5}}{4} = \tfrac{1+\sqrt{5}}{2} = \phi\,.$$

By the symmetry of the pentagram, $AF = AC$ and thus

$$\frac{AD}{AC} = \phi\,.$$

So the segment AD is divided at C according in the golden section. Therefore,

$$\frac{AC}{CD} = \phi\,.$$

Observing that $AB = CD$, we obtain

$$\frac{AC}{AB} = \phi\,.$$

So AC is divided by B in the golden section. Therefore, $\frac{AB}{BC} = \phi$. Thus of the segments

$$BC,\ AB,\ AC,\ AD$$

each is ϕ times greater than the preceding one. So Langdon is absolutely right when he tells us that "the ratios of line segments in a pentacle *all* equal PHI, making this symbol the *ultimate* expression of the Divine Proportion."