Solutions to the Exercises of Chapter 2

2A. Some Geometry

1. Consider any rectangle and let A, B, C, and D be a successive listing of its vertices. Notice that $AC = BD, AB = CD$, and $AD = BC$. Since the points A, B, C, and D all lie on the circle, we get by Ptolemy's theorem that $(AC)(BD) = (AB)(CD) + (AD)(BC)$. Therefore, $AC^2 = AB^2 + BC^2$.

2. Since arc $AB = 1\frac{1}{2} = \frac{3}{2}$, we get $\theta = \frac{\text{arc AB}}{2} = \frac{3}{4}$ radians. This is equal to $\frac{3}{4} \cdot \frac{180}{\pi} = 42.97°$.

3. Let $\theta = \angle ACB$. Observe that $\theta = \frac{\text{arc AB}}{3} = \frac{4}{3}$ radians. By Proposition 2.1 of the text, $\angle ADB = \frac{1}{2}\theta = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}$ radians. In degrees this is $\frac{2}{3} \cdot \frac{180}{\pi} = 38.20°$.

2B. Some "Inverse" Trigonometry

4. **i.** $\sin 32.28° = 0.534$

ii. $\sin 12.65° = 0.219$

iii. $\sin 56.51° = 0.834$

iv. $\sin 0.002 = 0.002$

v. $\sin 0.726 = 0.664$

vi. $\tan 37.74° = 0.774$

vii. $\tan 55.92° = 1.478$

viii. $\tan 78.69° = 5.000$

ix. $\tan 1.476 = 10.473$

x. $\tan 1.535 = 27.664$

5. $\sin 30° = \frac{1}{2} = \sin \frac{\pi}{6};\ \tan 45° = 1 = \tan \frac{\pi}{4};\ \sin 60° = \frac{\sqrt{3}}{2} = \sin \frac{\pi}{3}$.

2C. More Trigonometry

6. We will be working with Figure 2.30.

i. The third angle of ΔACB is $\pi - (\alpha + \beta)$. So $(\pi - (\alpha + \beta)) + \gamma = \pi$. Hence, $\gamma = \alpha + \beta$.

ii. That $\frac{BD}{AB} = \frac{EC}{AC}$ follows from the fact that ΔAEC and ΔADB are similar.

iii. Since $BD = AB \cdot \frac{EC}{AC} = (AE + EB) \cdot \frac{EC}{AC} = \frac{EC}{AC} \cdot (EB + AE)$, we get

$$\begin{aligned}
\sin(\alpha+\beta) = \sin\gamma &= \frac{BD}{BC} = \frac{EC}{AC} \cdot \frac{(EB+AE)}{BC} = \frac{EC}{AC} \cdot \frac{EB}{BC} + \frac{EC}{AC} \cdot \frac{AE}{BC} \\
&= \frac{EC}{AC} \cdot \frac{EB}{BC} + \frac{AE}{AC} \cdot \frac{EC}{BC} \\
&= (\sin\alpha)(\cos\beta) + (\cos\alpha)(\sin\beta)
\end{aligned}$$

iv. By similar triangles, $\frac{AD}{BD} = \frac{AE}{EC}$. It follows that $CD = AE \cdot \frac{BD}{EC} - AC$. By (ii) and the Pythagorean Theorem,

$$\begin{aligned}
AC \cdot CD &= AE \cdot BD \cdot \frac{AC}{EC} - AC^2 = AE \cdot BD \cdot \frac{AB}{BD} - AE^2 - EC^2 \\
&= AE(AB - AE) - EC^2 = AE \cdot EB - EC^2.
\end{aligned}$$

So, $CD = \frac{AE}{AC} \cdot EB - \frac{EC}{AC} \cdot EC$, and hence,

$$\frac{CD}{BC} = \frac{AE}{AC} \cdot \frac{EB}{BC} - \frac{EC}{AC} \cdot \frac{EC}{BC}.$$

It now follows that $\cos(\alpha+\beta) = (\cos\alpha)(\cos\beta) - (\sin\alpha)(\sin\beta)$.

Correction: In Figure 2.31, C should be the center of the circle and the segment CB a radius.

7. **i.** That $\angle BDC = \frac{\theta}{2}$ follows by an application of Proposition 2.1. By another application of Proposition 2.1 we get that $\angle ABD = \frac{\pi}{2}$.

ii. $\tan\left(\frac{\theta}{2}\right) = \frac{BE}{DC+CE} = \frac{\sin\theta}{1+\cos\theta}$

iii. $\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1+\cos\theta} \cdot \frac{1-\cos\theta}{1-\cos\theta} = \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta} = \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta} = \frac{1-\cos\theta}{\sin\theta}$

iv. $1 - \tan^2\left(\frac{\theta}{2}\right) = 1 - \frac{\sin\theta}{1+\cos\theta} \cdot \frac{1-\cos\theta}{\sin\theta} = 1 - \frac{1-\cos\theta}{1+\cos\theta} = \frac{2\cos\theta}{1+\cos\theta}$

v. Using (ii) and (iv) together,

$$\frac{2\tan\left(\frac{\theta}{2}\right)}{1-\tan^2\left(\frac{\theta}{2}\right)} = \left(\frac{2\sin\theta}{1+\cos\theta}\right) \Big/ \left(\frac{2\cos\theta}{1+\cos\theta}\right) = \tan\theta$$

2D. Ptolemy's Mathematics

8. The task is to refer to Section 2.3 and to substitute different numbers. Taking $\omega = 0.02$ radians, gives $\angle JDC = (94.5)(0.02) = 1.89$ radians. So arc $JC = 1.89r$. In the same way, arc $CG = (92.5)(0.02)r = 1.85r$. Hence arc $JC +$ arc $CG = 3.74r$. Since $2(\text{arc } JK) + \pi r =$ arc JC + arc $CG = 3.74r$, it follows that

$$2(\text{arc } JK) = 3.74r - 3.14r = 0.60r.$$

Thus arc $JK = 0.30r$. So, $DL = 0.30r$. Continuing Ptolemy's argument (with $\omega = 0.02$), we get

$$\begin{aligned} \text{arc } BC &= \text{arc } JC - \text{arc } JB = \text{arc } JC - (\text{arc } JK + \text{arc } KB) \\ &= 1.89r - 0.30r - \frac{\pi}{2}r \\ &= 1.59r - 1.57r \\ &= 0.02r. \end{aligned}$$

So $EL = \text{arc } BC = 0.02r$. Putting this information into the right triangle DLE and using the Pythagorean Theorem gives:

$$e = \sqrt{DL^2 + EL^2} = \left(\sqrt{0.09 + 0.0004}\right) r \approx 0.30r.$$

Since $\sin \lambda_A = \frac{DL}{e} \approx 1$, we get $\lambda_A \approx 90°$.

Note: These values of e and λ_A are substantially different from those achieved in Section 2.3. The point is that in Ptolemy's calculations, a relatively small change initially (the value of ω) has a serious impact on the final results (the values of e and λ_A). As one might expect, therefore, Ptolemy's results were very sensitive to inaccuracies in the observations.

9. Refer to Section 2.3 and in particular to Figure 2.20. Recall that $\angle DSL = 76°$. So $\angle ADF = 76°$. Hence $\angle BDA = 14°$. So $\angle BDA = 0.244$ radians. Because arc $CB = 0.007r, \angle CDB = 0.007$ radians. So $\angle CDA = 0.251$ radians. It follows that it takes the Earth $\frac{0.251}{0.0172} = 14.6$ days to travel from C to the aphelion position A. So aphelion occurs 14.6 days after summer solstice. Since the latter occurs on either June 21 or June 22, the prediction of Ptolemy's model is that the Earth is in aphelion position on July 5 or 6. Refer again to Figure 2.20 and consider the Earth's perihelion position P on the circle. Notice that the points $A, D,$ and P lie on a straight line. So the prediction is that the Earth will reach P about $\frac{1}{2}\left(365\frac{1}{4}\right) \approx 183$ days after aphelion. Verify that this would put the date of perihelion on January 5 or 6.

Note: The *1998 World Almanac* - turn to the Astronomy and Calendar section, then to Celestial Events Highlights subsection - lists perihelion and aphelion for 1998 as occurring on January 4 and July 4 respectively. The website http://aa.usno.navy.mil/data/ of the *U.S. Naval Observatory* contains this information in the section entitled Earth's Seasons - Equinoxes, Solstices, Perihelion, and Aphelion, 1992-2005.

10. **i.** Consider the triangle ΔEAM. The angle at A is $\pi - \alpha$. Adding the three angles of this triangle we get $\pi - \alpha + \mu + \beta = \pi$. So $\alpha = \beta + \mu$.

ii. Using the formula $\sin(\mu + \beta) = (\sin \mu)(\cos \beta) + (\cos \mu)(\sin \beta)$, Ptolemy gets that $\sin \alpha = (\sin \mu)(\cos \beta) + (\cos \mu)(\sin \beta)$. Observe that

$$\sin \mu = \frac{AQ}{AM}, \ \cos \beta = \frac{EQ}{r_E}, \ \cos \mu = \frac{MQ}{AM}, \ \text{and } \sin \beta = \frac{AQ}{r_E}.$$

Therefore, $\sin\alpha = \frac{AQ}{AM} \cdot \frac{EQ}{r_E} + \frac{MQ}{AM} \cdot \frac{AQ}{r_E} = \frac{AQ(EQ+MQ)}{AM \cdot r_E} = \frac{AQ \cdot D_M}{AM \cdot r_E}$. So $D_M = \frac{r_E \cdot AM \cdot \sin\alpha}{AQ}$. Since $\sin\mu = \frac{AQ}{AM}$, we get $D_M = \frac{r_E \sin\alpha}{\sin\mu}$.

iii. We have already see that $\alpha = \mu + \beta$. So $\mu = \alpha - \beta = 1°7'$. Now to a hand calculator: $\sin\alpha = \sin 50°55' = \sin 50.92° = 0.776$, and $\sin\mu = \sin 1°7' = \sin 1.12° = 0.020$. Taking $r_E = 3850$, gives

$$D_M = r_E \cdot \frac{\sin\alpha}{\sin\mu} = \frac{3850(0.776)}{0.020} \approx 150{,}000 \text{ miles}.$$

iv. The modern value is about 240,000 miles.

2E. Diophantus of Alexandria

11. Let x be Diophantus's age when he died. Check that

$$x = \frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \left(\frac{x}{2} + 4\right) = \frac{x}{6} + \frac{x}{12} + \frac{x}{7} + \frac{x}{2} + 9.$$

Multiplying through by the common denominator $7 \cdot 12 = 84$, we get

$$14x + 7x + 12x + 42x + 9 \cdot 84 = 84x.$$

So $x = 84$.

12. Let x, y, z, and w be the four numbers. We know that

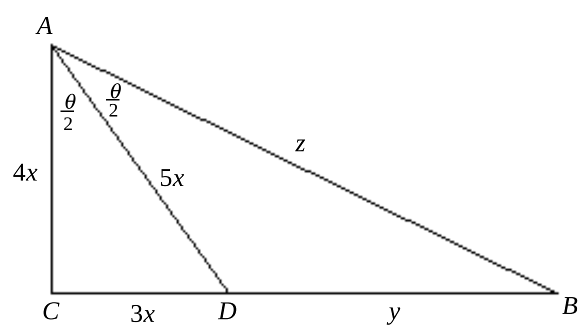
$$x + y + z = 22,\ x + y + w = 24,\ x + z + w = 27, \text{and } y + z + w = 20.$$

Substracting the last equation from each of the other three gives us:

$$(a)\ x - w = 2,\ (b)\ x - z = 4,\ \text{ and } (c)\ x - y = 7.$$

Subtracting (a) from (b) and then from (c), gives $w - z = 2$ and $w - y = 5$. So $z = w - 2$ and $y = w - 5$. Therefore, using $y + z + w = 20$, we get $(w - 5) + (w - 2) + w = 20$. So $3w = 27$. Hence $w = 9, z = 7, y = 4$, and $x = 11$.

13. Since the ratios $DC : CA : AD$ are equal to $3 : 4 : 5$, we let $DC = 3x$, and get that $CA = 4x$ and $AD = 5x$. Since DC and CA are integers, $CA - DC = x$ is an integer. Let $DB = y$ and $AB = z$. Putting all this into the triangle we get:



Observe that $\tan\frac{\theta}{2} = \frac{3x}{4x} = \frac{3}{4}$, and $\tan\ \theta = \frac{3x+y}{4x}$. Using the formula, $\tan\theta = \frac{2\tan\left(\frac{\theta}{2}\right)}{1-\tan^2\left(\frac{\theta}{2}\right)}$, we get

$$\frac{3x+y}{4x} = \frac{\frac{3}{2}}{1-\frac{9}{16}} = \frac{3}{2}\cdot\frac{16}{7} = \frac{24}{7}.$$

So $21x + 7y = 96x$, and $75x = 7y$. Since 7 is a prime that divides $75x$, it must divide 75 or x. But it does not divide 75, so 7 divides x. Therefore, 7 is the smallest possible value for x. Does $x = 7$ work? With $x = 7, DC = 21, CA = 28$, and $AD = 35$. Since $75x = 7y$, $y = 75$. It remains to compute z and check that it is an integer. By Pythagoras's theorem:

$$\begin{aligned} z^2 &= 16x^2 + (3x+y)^2 = 16\cdot 49 + (21+75)^2 = 16\cdot 49 + (6\cdot 16)^2 \\ &= 16(49 + 6^2 16) = 16(49 + 576) = 16\cdot 625 = 4^2\cdot 25^2 = 100^2. \end{aligned}$$

Therefore $z = 100$, and we are done.