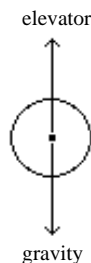


Solutions to the Exercises of Chapter 14

14A. Force and Acceleration

1. If the person is represented by a circle, then the force diagram in to this situation is



and the net force on the person is $F = \text{push by elevator} - \text{gravity}$. Note that the force of gravity on the person is $(70)(-9.8) = -686$ newtons. (The $-$ indicates the downward direction of this force.)

- i. Here, as well as in (ii) and (iii), the acceleration of the person is 0, so $F = m \cdot 0 = 0$, and the net force on the person is 0.
- iv. Here $F = (70)(2) = 140$ newtons. So the force produced by the elevator exceeds that of gravity by 140 newtons.
- v. Now $F = (70)(-2) = -140$ newtons. This time, gravity exceeds the force of the elevator by 140 newtons.
2. Let $y(t)$ be the height of the object above the ground at any time $t \geq 0$. So $y(0) = y_0$. Also, the velocity of the object is $v(t) = y'(t)$ and its acceleration is $a(t) = y''(t) = -9.81 \text{ m/sec}^2$. Taking antiderivatives of both sides of $y''(t) = -9.81$ gives us

$$y'(t) = -9.81t + C.$$

Because $y'(0) = v_0$, we get $C = v_0$ and hence

$$v(t) = -9.81t + v_0.$$

Doing the same with $y'(t) = -9.81t + v_0$, we get

$$y(t) = -\frac{1}{2}(9.81)t^2 + v_0t + y_0.$$

3. Lacking more explicit information, we will assume that the object begins its fall with a velocity of zero exactly 6 meters above the pile. Our strategy will be to:
- A. Use the formulas derived in Exercise 2 to compute the length of time t_1 of the fall and the velocity v_1 with which the object hits the top of the pile.

B. The assumption that the force F on the object exerted by the pile is constant, implies that the acceleration produced by this force is a constant, say c . The principles used in the solution of Exercise 2 can be used to determine c and hence F .

A. If $y(t)$ is the height (in meters) of the object above the pile at time $t \geq 0$, then

$$y(t) = -\frac{1}{2}(9.81)t^2 + 6,$$

and $0 = y(t_1) = -\frac{1}{2}(9.81)t_1^2 + 6$. So $t_1 = \sqrt{\frac{12}{9.81}} \approx 1.106$ seconds. The velocity of the object at that time is $v_1 = v(t_1) = -9.81t_1 \approx -10.85$ meters per second. This is the velocity with which the object hits the pile of sand.

B. Now place a vertical axis so that the origin is the point of impact and $z(t)$ is the coordinate of the bottom of the object at any time $t \geq 0$ after it first hits the pile. Observe that $z(0) = 0$, $z'(0) = -10.85$, and $z''(t) = c$. It follows that $z'(t) = ct - 10.85$ and $z(t) = \frac{c}{2}t^2 - 10.85t$. Let t_2 be the instant that the object comes to rest. Then $ct_2 - 10.85 = 0$ and (see the given of the problem) $\frac{c}{2}t_2^2 - 10.85t_2 = z(t_2) = -0.03$. So $ct_2 = 10.85$ and

$$\frac{1}{2}(ct_2)t_2 - 10.85t_2 = \frac{1}{2}(10.85)t_2 - 10.85t_2 = -0.03.$$

Therefore, $\frac{1}{2}(10.85)t_2 = 0.03$ and $t_2 = \frac{0.06}{10.85}$. Hence $c = \frac{10.85}{t_2} = \frac{(10.85)^2}{0.06} = 1962$ m/sec² and $F = mc = 2(1962) = 3924$ newtons. Another way to do (B) is as follows: Let F be the constant force exerted by the sandpile. By facts from Section 14.1B,

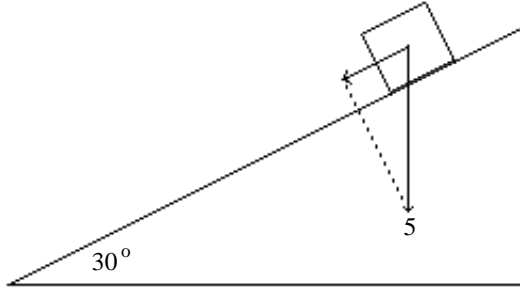
$$\int_{-0.03}^0 F dx = W = \frac{1}{2} \cdot 2(-10.85)^2 - 0$$

where W is the work done by the force F . So $0.03F = (10.85)^2$ and hence $F = 3924$ newtons. The time $t_2 = \frac{0.06}{10.85} \approx 0.0055$ can also be determined from the impulse equation of Section 14.3A, as follows:

$$\int_0^{t_2} 3924 dt = \int_0^{t_2} F dt = 0 - 2 \cdot (-10.85) = 21.7.$$

So $3924t_2 = 21.7$ and $t_2 \approx 0.0055$ seconds.

4. Let's use this problem to review some basic facts. The fact that the block weighs 5 pounds means that the force of gravity on the block is 5 pounds. If the block were in free fall, gravity would accelerate it by 32.17 feet per second². It follows that the mass m of the block satisfies $5 = m(32.17)$. So $m = \frac{5}{32.17} \approx 0.155$ slugs. The diagram



shows that the force F that pushes the block down the ramp is $F \cos 60^\circ = \frac{1}{2}F = 2.5$ lbs. If a is the acceleration of the block down the plane, then $F = ma$, or $2.5 = 0.155a$. So $a = 16.1$ ft./sec². The diagram above and the fact that acceleration is a vector quantity shows more directly that $a = \frac{1}{2}g \approx 16.1$ ft./sec².

5. i. A look at the solution of Problem 4 tells us that the car has a mass $m = \frac{3000}{32.17} \approx 93.25$ slugs. Let a be the constant deceleration of the car in feet per second². Suppose that the breaks are first applied at time $t = 0$. At any time $t \geq 0$ (in seconds) later, let $s(t)$ be the distance in feet from the car to the point (on the street) where the breaks were first applied. So $s'(t)$ is the velocity of the decelerating car, and $s''(t) = a$ is its acceleration. Because $\frac{60 \text{ miles}}{\text{hour}} = 60 \cdot \frac{5280 \text{ feet}}{3600 \text{ seconds}} = 88$ ft. sec, we find that

$$s'(t) = at + 88.$$

When $t = 15$ seconds, $s'(t) = 0$, and therefore $15a = -88$. So $a = -5.87$ ft./sec². So the force on the car is

$$F = ma = (93.25)(5.87) \approx 550 \text{ lbs.}$$

- ii. Again let a be the constant acceleration of the car and suppose that the accelerator is pushed at time $t = 0$. For any time $t \geq 0$ let $s(t)$ be the distance from the car to the point at which the accelerator pedal was first pushed. So $s''(t) = a$ and (since 10 miles/mph = 14.67 feet/sec.)

$$s'(t) = at + 14.67.$$

Because $s'(10) = 117.33$ feet/sec, we get $10a = 117.33 - 14.67$, or $a = 10.27$ feet/sec². So the force that is generated by the engine is

$$F = ma = (93.25)(10.27) \approx 960 \text{ lbs.}$$

Here too the results of Section 14.3 provide a different approach:

- i. $\int_0^{15} F dt = 0 - mv$, So $15F = -(93.25)(88)$, and $F = -550$ pounds.
- ii. $\int_0^{10} F dt = 117.33m - 14.67m = (102.66)(93.25)$. So $10F \approx 9573$, and $F \approx 960$.

14B. Work and Energy

6. The work done is $35 \times 200 = 7000$ foot-lbs.
7. In the case of the bullet, $v = 1000$ meters per second and $m = 12$ grams = 0.012 kilograms. So the kinetic energy is

$$\frac{1}{2}mv^2 = \frac{1}{2}(0.012)(1000^2) = 6000 \text{ newton-meters.}$$

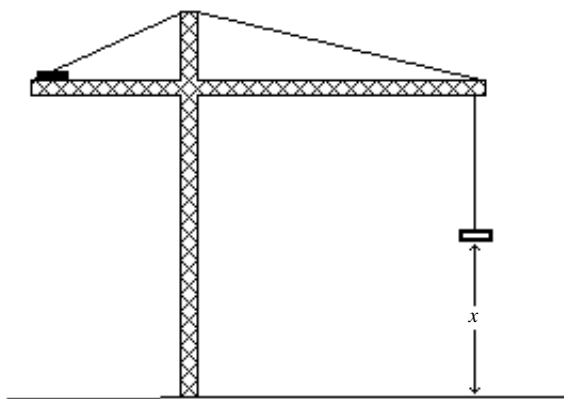
For the truck we get

$$\frac{1}{2}(10)(1000) \cdot 7^2 = 245,000 \text{ newton-meters.}$$

8. When the slab is x feet off the ground, the crane has to pull with a force of

$$f(x) = 6(90 - x) + 500$$

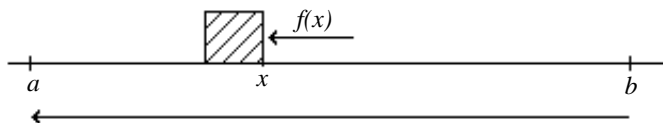
pounds in order to counteract the weight of the cable and the slab.



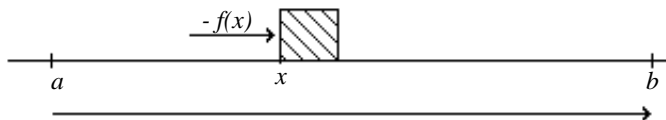
So the work done on the slab is

$$\begin{aligned} W &= \int_0^{70} [6(90 - x) + 500]dx = \int_0^{70} (1040 - 6x)dx \\ &= (1040x - 3x^2) \Big|_0^{70} = 72,800 - 14,700 \\ &\approx 58,000 \text{ foot-pounds.} \end{aligned}$$

9. Consider the situation of an object moving on an axis from b to a with $a < b$ acted on by a force $f(x)$ for $a \leq x \leq b$. How is the work W done by this force expressed as a definite integral?



Because the motion of the object is in the negative direction, the discussion of Section 14.1A does not apply. However, this discussion does apply to the situation which is physically identical but “runs” in the opposite direction. So the work done by $-f(x)$ in this



situation is equal to W . Quantitatively, everything is the same, but the direction of the forces and velocities are reversed. By Section 14.1A, the work done by $-f(x)$ in this second situation is $\int_a^b -f(x)dx$. It follows from Problem 14 in Exercise 13C that

$$W = - \int_a^b f(x)dx = \int_b^a f(x)dx.$$

Therefore, as in the situation of Section 14.1A, the work done by the force $f(x)$ is equal to the definite integral of $f(x)$ from the beginning of the objects motion at b to the end of that motion at a .

Note: If a force $f(x)$ and the motion of the object on which it acts are both in the positive direction, then the work done $\int_a^b f(x)dx$ is positive. Why? If both $f(x)$ and the motion of the object are in the negative direction, then $\int_a^b f(x)dx$ is also positive. Why? Check that if the force and the motion are in opposite directions, then the work $\int_a^b f(x)dx$ is negative. In reference to the work done by the force $g(y)$ on the pedal in Figure 14.43, our discussion shows that this work equals

$$\int_6^0 g(y)dy$$

and that this is negative. However, this pedal does support a part of the woman’s weight so that this motion requires a pushing effort on her part. So the work done by her on the two pedals is

$$\int_0^6 f(x)dx + \int_0^6 g(y)dy.$$

- i. Notice that $f(x) + g(y) = 120$ and by “pedal symmetry ” that the distances $6 - x$ and y are equal. So by changing variables,

$$\int_0^6 g(y)dy = \int_6^0 (120 - f(x))(-dx) = \int_6^0 (f(x) - 120)dx.$$

So the work done is

$$\begin{aligned} \int_0^6 f(x)dx + \int_0^6 (120 - f(x))dx &= \int_0^6 120dx = 120x \Big|_0^6 \\ &= 720 \text{ inch-pounds} = 60 \text{ foot-pounds.} \end{aligned}$$

ii. The work done in one minute is

$$50 \times 60 = 3000 \text{ foot-pounds.}$$

In 30 minutes, the work done is

$$30 \times 3000 = 90,000 \text{ foot-pounds.}$$

Because 1 foot-pound corresponds to 3.1 calories, this is equivalent to

$$\frac{90,000}{3.1} \approx 29,000 \text{ calories.}$$

iii. The same computations in the case of the 180 pound man, tell us that the work done for one “up and down” is

$$180x \Big|_0^6 = 1080 \text{ inch-pounds} = 90 \text{ foot-pounds.}$$

For 30 minutes the total is $30(50 \times 90) = 135,000$ foot-pounds, or $\frac{135,000}{3.1} \approx 43,500$ calories.

Remarks: The calorie is defined to be the amount of heat energy required to raise the temperature of one gram of water from 15° to 16° centigrade. In the context of foods and diets, the standard unit is the Calorie (“big C”) defined to be equal to 1000 “little c” calories. So 1 Calorie corresponds to about $(1000)(3.1) = 3100$ foot-pounds. We have seen that in her 30 minute work out the woman produces about

$$\frac{29,000}{1000} = 29 \text{ Calories.}$$

It is a fact, however, that a human body will burn up (i.e. convert energy stored in the body) a few hundred Calories during a steady stair climbing effort of 30 minutes. Why the big discrepancy between this fact and our computation? Two primary reasons: One is the heat and sweat that the workout produces. The second is the fact that the body burns up about 100 Calories (per half-hour) even while standing still. (This is the energy required for metabolic processes, muscle activity, etc.) To calculate the total “burn up” Calorie outputs generated by the heat, sweat and metabolic processes must be added to the 29 Calories that we computed.

14C. The Internal Combustion Engine

10. Recall that force = pressure \times area. The cross-sectional area of the piston is $A = \pi 2^2 = 4\pi$ square inches. So the force is

$$f(x) = 4\pi \cdot \frac{120}{x} \approx \frac{1508}{x} \text{ pounds.}$$

11. The work in inch-pounds is

$$\begin{aligned} \int_{0.40}^4 \frac{1508}{x} dx &= 1508 \ln x \Big|_{0.40}^4 = 1508(\ln 4 - \ln 0.40) = 1508 \ln \left(\frac{4}{0.4} \right) \\ &= 1508 \ln 10 \approx (1508)(2.30) \\ &\approx 3470 \text{ inch-pounds} \approx 290 \text{ foot-pounds.} \end{aligned}$$

12. Per cylinder, $(45)(289.35) \approx 13,020$ foot-pounds are generated per second. All six cylinders generate 78,120 foot-pounds or about 142 horsepower. Because there are 45 power strokes per second, there are a total of 90 strokes per second and hence 5400 strokes per minute.

14D. Impulse and Momentum

13. The bullet has a momentum of $mv = (0.012)(1000) = 12$ kg-m/sec and the truck has a momentum of $mv = (10,000)(7) = 70,000$ kg-m/sec.

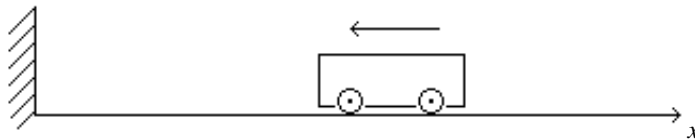
14. The momentum goes from some positive quantity mv to zero. So momentum is not conserved; but this is not a contradiction because friction is an outside force on the “system” of the sliding block.

15. The momentum is $5 \cdot 2 = 10$ kg-m/sec initially, and $5 \cdot 8 = 40$ kg-m/sec, 3 seconds later. The change is $40 - 10 = 30$ kg-m/sec. If F is the constant force on the object, then

$$\int_0^3 F dt = 40 - 10 = 30,$$

so that $3F = 30$ and $F = 10$ newtons.

16. The momentum of the cart is $(1.5)(-0.20) = -0.30$ kg-m/sec before it hits against the bumper. Let $t = 0$ be the instant of impact and F the average force on the cart during



the time of contact with the bumper.

i. The change in momentum is $0 - (-0.30) = 0.30$ kg-m/sec. So

$$\int_0^{0.1} F dt = 0.30.$$

Therefore, $(0.1)F = 0.30$, and $F = 3$ newtons.

ii. The momentum at $t = 0.1$ is $(1.5)(0.1) = 0.15$ kg-m/sec. So the change in momentum is $0.15 - (-0.30) = 0.45$ kg-m/sec. Because

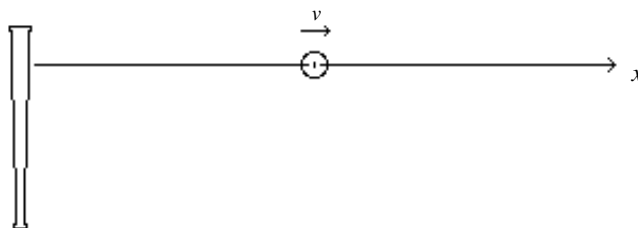
$$\int_0^{0.1} F dt = 0.45,$$

we get $(0.1)F = 0.45$, and hence $F = 4.5$ newtons.

17. The impulse is

$$\int_0^{0.01} 200dt = 2 \text{ pound-seconds.}$$

Let v be the velocity in feet/sec with which the ball leaves the bat. Because the ball



weighs 0.34 pounds, its mass m satisfies the equation $0.34 = m \cdot (32.17)$, and hence $m = 0.01$ slugs. The velocity of the ball as it comes towards the bat is $-90 \cdot \frac{5280}{3600} = -132$ feet/sec. It follows that

$$\int_0^{0.01} 200dt = (0.01)v - (0.01)(-132).$$

So $2 = (0.01)(v - 132)$. Hence $v - 132 = 200$ and $v = 332$ feet/sec.

18. The impulse that the barbell receives is $\int_0^{t_1} 40dt$, where t_1 is the duration of the first part of the lift and the change in momentum is $mv - 0$, where $m = \frac{535}{g}$ is the mass of the barbell and v is the velocity of the barbell at the end of the first part of the lift. By Section 14.1B, $v = \sqrt{\frac{40}{m}}$ feet/sec. So the change in momentum is

$$mv = m\sqrt{\frac{40}{m}} = \sqrt{40m} = \sqrt{\frac{40 \cdot 535}{g}}.$$

The acceleration of the barbell is $a = \frac{F}{m} = \frac{40}{m}$. So its velocity is $v(t) = \frac{40}{m}t$ and its height above the ground $x(t) = \frac{20}{m}t^2 + \frac{3}{4}$, both for any $t \geq 0$ during the first part of the lift. So $\frac{20}{m}t_1^2 + \frac{3}{4} = \frac{5}{4}$, and hence $t_1 = \sqrt{\frac{m}{40}}$. So the impulse is $40t_1 = 40\sqrt{\frac{m}{40}} = \sqrt{40m} = \sqrt{\frac{40 \cdot 535}{g}}$ as required.

19. The gas molecules in the piston travel in all directions and at various speeds during the explosion. (Indeed, there are lots of “random” things going on as the molecules bump against each other and ricochet off the piston and the walls of the cylinder.) We will make the simplifying assumption that the 0.6 grams of gas molecules push straight down on the piston throughout the explosion, that they have a velocity of v meters per second at the beginning of the explosion, and that they have zero velocity at the end. Under this assumption, it follows that (in M.K.S.)

$$\int_0^{0.001} 1400 dt = 0.0006v - 0.$$

So $(1400)(0.001) = 0.0006v$, and hence $v = \frac{1400}{0.6} \approx 2333$ meters/sec.

14E. More About Springs and Guns

20. With the setup as in Figure 14.10, $f(x) = kx$, where k is the spring constant. Because a force of 9 pounds displaces the spring by 1 inch, $9 = f(1) = k$. So $k = 9$ pounds/inch, and $f(x) = 9x$ pounds.

i. This force is $f(2) = 9 \cdot 2 = 18$ pounds.

ii. This is $\int_0^1 9x dx = \frac{9}{2}x^2 \Big|_0^1 = \frac{9}{2}$ inch-pounds.

iii. $\int_1^2 9x dx = \frac{9}{2}x^2 \Big|_1^2 = \frac{9}{2} \cdot 3 = \frac{27}{2}$ inch-pounds.

21. The force on the ice cube is $-kx(t)$. The $-$ is needed since $x(t)$ is negative and the force is in the positive direction. So the acceleration $a(t)$ of the ice cube is $a(t) = \frac{-k}{m}x(t)$. Therefore, $x''(t) = a(t) = \frac{-k}{m}x(t)$ as required.

22. By Example 13.33, $x(t)$ has the form

$$x(t) = A \left(\sin \sqrt{\frac{k}{m}} t \right) + B \left(\cos \sqrt{\frac{k}{m}} t \right)$$

for some constants A and B . Therefore,

$$x'(t) = A \sqrt{\frac{k}{m}} \left(\cos \sqrt{\frac{k}{m}} t \right) - B \sqrt{\frac{k}{m}} \left(\sin \sqrt{\frac{k}{m}} t \right).$$

Plugging in the conditions $x(0) = -0.03$ and $x'(0) = 0$, tells us that $B = -0.03$ and $A = 0$. So $x(t)$ and $x'(t)$ are both as indicated.

23. Note that $\sqrt{\frac{k}{m}} = \sqrt{1250}$. As t increases from 0, we find that $x(t) = 0$ when $\sqrt{\frac{k}{m}} t = \frac{\pi}{2}$. So the left edge of the ice cube arrives at the origin when $t = \frac{\pi}{2} \sqrt{\frac{m}{k}} = \frac{\pi}{2\sqrt{1250}}$. At this instant

$$\begin{aligned} x'(t) &= (0.03)\sqrt{1250} \cdot \sin \left(\sqrt{\frac{k}{m}} \cdot \frac{\pi}{2} \sqrt{\frac{m}{k}} \right) \\ &= (0.03)\sqrt{1250} \cdot \sin \frac{\pi}{2} = (0.03)\sqrt{1250} \\ &\approx 1.06 \text{ meters/sec.} \end{aligned}$$

24. If v is the recoil velocity of the gun, then $4v = (0.05)(800)$. Hence $v = 1$ meter/sec.

25. The horizontal component of the velocity of the shell is $1000(\cos 10^\circ)$ meters/sec. If v is the initial recoil velocity of the tank, then

$$50,000v = (25)(1000 \cos 10^\circ).$$

So $v = \frac{1}{2} \cos 10^\circ \approx 0.49$ meters/sec. If F is the constant horizontal component of the frictional force, then by Section 14.1B,

$$\int_0^{0.5} F dx = 0 - \frac{1}{2}(50,000)v^2.$$

So $(0.5)F \approx -25,000(0.24) = -6000$. Hence $F \approx -12,000$ newtons. The minus means that the force slows the motion.

26. For the Springfield rifle, the barrel length is $b = 2$ feet and the muzzle velocity is $v = 2800$ feet/second. So

$$t_1 \approx \frac{3b}{2v} \approx \frac{6}{5600} \approx 0.0011 \text{ seconds.}$$

Note that the estimate $t_1 \approx \frac{3b}{2v}$ translates to an average velocity of the bullet of

$$\frac{b}{t_1} = b \cdot \frac{2v}{3b} = \frac{2}{3}v.$$

27. Starting from LeDuc's equation $\frac{dx}{dt} = \frac{rx}{s+x}$, we get by separating variables, that

$$dt = \left(\frac{s+x}{rx} \right) dx = \left(\frac{s}{r} \frac{1}{x} + \frac{1}{r} \right) dx.$$

By anti-differentiating both sides,

$$t = \frac{s}{r} \ln x + \frac{1}{r}x + C.$$

28. **Correction:** Replace "So the formula of LeDuc requires the assumption that $x(0)$ is some small positive number a " by: "So the mathematical model does not reflect the reality of the situation when $x = 0$. In order for it to do so, we need to assume that $x(0)$ is equal to some small positive number a ."

Since $x = a$ when $t = 0$, we get from Exercise 27 that

$$0 = \frac{s}{r} \ln a + \frac{a}{r} + C$$

and hence that $C = -\frac{s}{r} \ln a - \frac{a}{r}$. It follows that

$$\begin{aligned} t &= \frac{s}{r} \ln x + \frac{x}{r} - \frac{s}{r} \ln a - \frac{a}{r} \\ &= \frac{s}{r} \ln \frac{x}{a} + \frac{x-a}{r}. \end{aligned}$$

This results in the equation

$$t_1 = \frac{s}{r} \ln \frac{b}{a} + \frac{b-a}{r}$$

for the barrel time t_1 . In the case of the Springfield rifle this is

$$t_1 = \frac{0.65}{3716} \ln \frac{2}{a} + \frac{2-a}{3716}$$

seconds. Taking $a = 0.1, 0.01, 0.001$, and 0.0001 feet (this corresponds successively to 1.2, 0.12, 0.012, and 0.0012 inches), we get respectively,

$$\begin{aligned} t_1 &= \frac{0.65}{3716} \ln \frac{2}{0.1} + \frac{2-0.1}{3716} \\ &\approx 0.00052 + 0.00051 \\ &\approx 0.0013 \text{ seconds;} \\ t_1 &= \frac{0.65}{3716} \ln \frac{2}{0.01} + \frac{2-0.01}{3716} \\ &\approx 0.00093 + 0.00054 \\ &\approx 0.0015 \text{ seconds;} \\ t_1 &= \frac{0.65}{3716} \ln \frac{2}{0.001} + \frac{2-0.001}{3716} \\ &\approx 0.00133 + 0.00054 \\ &\approx 0.0019 \text{ seconds;} \text{ and} \\ t_1 &= \frac{0.65}{3716} \ln \frac{2}{0.0001} + \frac{2-0.0001}{3716} \\ &\approx 0.00173 + 0.00054 \\ &\approx 0.0023 \text{ seconds.} \end{aligned}$$

29. Taking $r = 3720$ and $s = 0.64$ we get the force function

$$f(x) = \frac{6492x}{(0.64+x)^3}$$

and a maximum force of $f(0.32) \approx 2348$ pounds. This leads to a maximum pressure of 4,790,000 pounds/ft². The muzzle velocity is

$$v(2) = \frac{(3720)(2)}{2.64} \approx 2818 \text{ ft./sec.}$$

and this gives a kinetic energy of 2910 foot-pounds.

Comparing these figures with those of Section 14.2C tells us that the relatively small errors for r and s result in fairly substantial differences in the maximal force and pressure, as well as the muzzle velocity and kinetic energy.

Correction: Change the narrative of Exercise 30 as follows. Replace “Is energy conserved? In other words, is the kinetic energy of the bullet in state (a) equal to the potential energy of the block at the top of the swing?” by “Is mechanical energy conserved? In other words, is the kinetic energy of the bullet before impact equal to the kinetic energy of the block-bullet system after impact?” Then add: “Does the principle of conservation of energy apply to this situation?”

- 30.** Refer to the discussion about the ballistic pendulum in Section 14.3B and the notation there. In the typical situation the mass m of the pendulum is many times greater than the mass m_b of the bullet. So $m = km_b$ where k is a large constant. In the case of Example 14.10,

$$k = \frac{m}{m_b} \approx \frac{0.62}{0.00073} \approx 849.$$

In the general situation,

$$m_b v_b = (m_b + m)v$$

by conservation of momentum; so $m_b v_b = (m_b + km_b)v$, and hence $v_b = (1+k)v$. So the initial kinetic energy of the pendulum/bullet system right after impact is

$$\begin{aligned} \frac{1}{2}(m_b + m)v^2 &= \frac{1}{2}(m_b + km_b) \left(\frac{v_b}{1+k} \right)^2 \\ &= (1+k) \frac{m_b v_b^2}{(1+k)^2} = \frac{\frac{1}{2}m_b v_b^2}{1+k} \\ &= \left(\frac{1}{1+k} \right) \frac{1}{2}m_b v_b^2. \end{aligned}$$

Hence the initial kinetic energy (or, equivalently, the potential energy of the pendulum/bullet system at the top of the swing) is but a small fraction $\frac{1}{k+1}$ of the kinetic energy $\frac{1}{2}m_b v_b^2$ of the bullet. In case of Example 14.10 this small fraction is approximately $\frac{1}{850} \approx 0.0012$ or 0.12%. So not only is kinetic energy (or mechanical energy) not conserved, 99.88% of it is in fact lost! Where does it go? As the bullet penetrates and burries itself in the block molecules of the block and the bullet are “rearranged” and heat is produced. So it goes into heat energy. This heat energy is approximately equal to

$$0.9988 \frac{1}{2}m_b v_b^2 \approx 0.9988 \left(\frac{1}{2} \right) (0.000733)(2800)^2 \approx 1.025 \text{ ft-lbs} \approx \frac{1.025}{3.1} \approx 0.331 \text{ calories.}$$

14F. The Sliding Chain

- 31.** A look at Figure 14.46 shows that the chain is subject to a (gravitational) force of

$$F(t) = -(L - x(t))g.$$

Because the rate of change of the momentum of the chain is $\frac{d}{dt}mv(t) = \frac{d}{dt}(mx'(t)) = mx''(t)$, it follows that

$$-(L - x(t))g = mx''(t).$$

where $m = 100 \text{ grams} = 0.1\text{kg}$ is the mass of the chain. Observe that this implies that $x''(t)$ is negative. This is confirmed by thinking about the sliding chain: $x'(t)$ decreases slowly at first and then more rapidly as the gravitational force increases.

We need to solve the differential equation just derived for $x(t)$. Towards this end, we let $z(t) = L - x(t)$ to get $mz''(t) - gz(t) = 0$, or

$$(*) \quad z''(t) - \frac{g}{m}z(t) = 0.$$

Let A and B be constants and check that

$$Ae^{\sqrt{\frac{g}{m}}t} \text{ and } Be^{-\sqrt{\frac{g}{m}}t}$$

are both solutions of (*). It can be shown that $Ae^{\sqrt{\frac{g}{m}}t} + Be^{-\sqrt{\frac{g}{m}}t}$ is the general solution and hence that

$$z(t) = Ae^{\sqrt{\frac{g}{m}}t} + Be^{-\sqrt{\frac{g}{m}}t}.$$

So $z'(t) = A\sqrt{\frac{g}{m}}e^{\sqrt{\frac{g}{m}}t} - B\sqrt{\frac{g}{m}}e^{-\sqrt{\frac{g}{m}}t}$. Because $z(0) = L - x(0) = 10\text{cm} = 0.10$ meters and $z'(0) = -x'(0) = 0$, we get $0.10 = A + B$ and $0 = A - B$. So $A = B = 0.05$, and

$$z(t) = 0.05e^{\sqrt{\frac{g}{m}}t} + 0.05e^{-\sqrt{\frac{g}{m}}t}.$$

Let $a = \sqrt{\frac{g}{m}}$. Observe that we need to solve $z(t) = 1$, or

$$e^{at} + e^{-at} = 20$$

for t . To do this, let $y = e^{at}$. So $y + y^{-1} = 20$, or $y^2 - 20y + 1 = 0$. Therefore, $y = \frac{20 \pm \sqrt{400-4}}{2} = 10 \pm \sqrt{99}$. Let's examine the two possibilities separately. With $e^{at} = y = 10 + \sqrt{99}$, we get $at = \ln(10 + \sqrt{99})$ and hence

$$\begin{aligned} t &= \frac{1}{a} \ln(10 + \sqrt{99}) = \sqrt{\frac{m}{g}} \ln(10 + \sqrt{99}) \\ &= \sqrt{\frac{0.1}{9.81}} \ln(10 + \sqrt{99}) \approx (0.101)(2.993) \approx 0.30 \text{ seconds.} \end{aligned}$$

With $e^{at} = y = 10 - \sqrt{99}$, we get

$$t = \sqrt{\frac{m}{g}} \ln(10 - \sqrt{99}) = \sqrt{\frac{0.1}{9.81}} \ln(10 - \sqrt{99}) \approx (0.101)(-2.993) \approx -0.30.$$

Since this is negative, this last case cannot arise.

14G. More About Rockets

32. The acceleration of the rocket is

$$a = \frac{F}{m} = \frac{2 \cdot 10^5}{10^4}$$

meters/sec². So $v(t) = 20t$. So $v(20) = 400$ meters/sec.

33. Start with the rocket equation of Section 14.4B. Given that the air resistance is assumed to be zero and the rocket is fired straight up, we can take $R(t) = 0$ and $\varphi(t) = 0$. So $m(t)\frac{d}{dt}v(t) = T(t) - G(t)$, and hence

$$\frac{d}{dt}v(t) = \frac{T(t)}{m(t)} - \frac{GM_E}{r(t)^2},$$

where M_E is the mass of the Earth. As in the discussion in Section 14.4B, we can assume that

$$T(t) = A(\rho_e v_e^2 + p_e),$$

and that $m(t) = m_0 - rt$, where m_0 is the initial mass of the rocket and r the constant rate at which the fuel is consumed. Therefore,

$$\frac{d}{dt}v(t) = \frac{A(\rho_e v_e^2 + p_e)}{m_0 - rt} - \frac{GM_E}{r(t)^2}.$$

We know from Section 1.5 that the radius of the Earth is 3950 miles and from Section 14.4A that the V-2 reached an altitude of about 116 miles when it is fired vertically upward. So $r(t)$ satisfies $3950 \leq r(t) \leq 4066$ in miles and hence

$$2.08 \times 10^7 \leq r(t) \leq 2.15 \times 10^7$$

in feet. So we will assume that $r(t)$ is the constant $R = 2.10 \times 10^7$ feet. Hence

$$\frac{d}{dt}v(t) = \frac{A(\rho_e v_e^2 + p_e)}{m_0 - rt} - \frac{GM_E}{R^2}.$$

By another discussion of Section 14.4B, it follows that

$$v(t) = -\frac{1}{r}A(\rho_e v_e^2 + p_e) \ln(m_0 - rt) - \frac{GM_E}{R^2}t + C.$$

Because $v(0) = 0$, we get

$$C = \frac{1}{r}A(\rho_e v_e^2 + p_e) \ln m_0.$$

So

$$v(t) = \frac{1}{r}A(\rho_e v_e^2 + p_e) \ln \left(\frac{m_0}{m_0 - rt} \right) - \frac{GM_E}{R^2}t.$$

Next refer to Section 14.4A for the values of the constants:

$$\begin{aligned} A &= 4.60 \text{ ft}^2 \\ r &= 8.55 \text{ slugs/sec} \\ \rho_e &= 2.83 \times 10^{-4} \text{ slugs/ft}^3 \\ v_e &= 6560 \text{ feet/sec} \\ p_e &= 2,840 \text{ pounds/ft}^3 \\ m_0 &= \frac{28,229}{32.17} = 877.49 \text{ slugs.} \end{aligned}$$

Converting the constants G and M_E as given in Section 7.6 to the American System (see Exercises 7E), we get

$$\begin{aligned} G &= 3.46 \times 10^{-8} \text{ ft}^3/\text{slug}\cdot\text{sec}^2 \\ M_E &= 4.10 \times 10^{23} \text{ slugs.} \end{aligned}$$

Finally, recall that the mass of the fuel is equal to $\frac{19,392}{32.17} = 603$ slugs and hence that the engine of the V-2 will fire for $\frac{603}{8.55} = 70.5$ seconds. We can now compute the answer $v(70.5)$ to the problem:

$$\begin{aligned} v(70.5) &= \frac{4.60}{8.55}((2.83 \times 10^{-4})(6560)^2 + 2840) \ln \left(\frac{877.49}{877.49 - 602.78} \right) \\ &\quad - \frac{(3.46 \times 10^{-8})(4.10 \times 10^{23})(70.5)}{(2.10 \times 10^7)^2} \\ &= 9383.86 - 2267.83 = 7116 \text{ feet/sec} \\ &= 1.35 \text{ miles/sec.} \end{aligned}$$

As we would expect (from the discussion in Section 6.6), air resistance has considerable impact on a rocket in flight in the Earth's atmosphere. This is evident here as well: Compare the estimated velocity of 7116 feet/second with the fact (see Section 14.4A) that the V-2 has a maximum velocity of 5575 feet/second. A model that takes air resistance into account is a very subtle mathematical undertaking. See Exercises 14J.

34. Consider

$$\frac{d}{dt}\mu(t) = \lim_{\Delta t \rightarrow 0} \frac{\mu(t + \Delta t) - \mu(t)}{\Delta t}.$$

The change $\mu(t + \Delta t) - \mu(t)$ in the momentum of the exhaust gases is equal to the mass $m_e(t + \Delta t) - m_e(t)$ of the exhaust gas ejected during the time interval $[t, t + \Delta t]$ times the velocity of this gas. The latter varies from $v_e(t)$ to $v_e(t + \Delta t)$ and is thus, for small Δt , approximated by $v_e(t)$. It follows that

$$\frac{\mu(t + \Delta t) - \mu(t)}{\Delta t} \approx \frac{[m_e(t + \Delta t) - m_e(t)]v_e(t)}{\Delta t}$$

for small Δt , and that

$$\begin{aligned} \frac{d}{dt}\mu(t) &= \lim_{\Delta t \rightarrow 0} \Delta t v_e(t) \frac{m_e(t + \Delta t) - m_e(t)}{\Delta t} \\ &= v_e(t) \lim_{\Delta t \rightarrow 0} \Delta t \frac{m_e(t + \Delta t) - m_e(t)}{\Delta t} \\ &= v_e(t) \frac{d}{dt}m_e(t). \end{aligned}$$

14H. More About Gravity

35. Instead of engaging in the detailed analysis of Section 14.5B, we will use the “short hand” version described in Section 5.6. So we consider a typical circular band (or circle with mass) on the sphere as shown in Figure 14.47 with center at x on the axis, and ask about the force with which it acts on the particle P . An application of the formula Section 14.5A tells us that this force is equal to

$$\frac{GmM_0(c-x)}{((c-x)^2+r^2)^{\frac{3}{2}}}$$

where M_0 is the mass of the circular band and r is its radius. Notice that the point (x, r) is on the circle $x^2 + y^2 = R^2$, so that $x^2 + r^2 = R^2$ and $r^2 = R^2 - x^2$. Next, let dx be the width of the circular band. It follows from Example 13.6 that the area of the circular band is $2\pi R dx$. Because the area of a sphere of radius R is $4\pi R^2$ (see Section 5.6) the mass per unit area of the sphere is $\frac{M}{4\pi R^2}$. It follows that

$$M_0 = \frac{M}{4\pi R^2} \cdot 2\pi R dx = \frac{M}{2R} dx.$$

By substituting, we see that the force of this typical band on the particle P is

$$\begin{aligned} \frac{GmM_0(c-x)}{((c-x)^2+r^2)^{\frac{3}{2}}} &= \frac{GmM}{2R} \frac{(c-x)dx}{(c^2-2cx+x^2+R^2-x^2)^{\frac{3}{2}}} \\ &= \frac{GmM}{2R} \frac{(c-x)dx}{(c^2-2cx+R^2)^{\frac{3}{2}}}. \end{aligned}$$

By adding up all the forces on P exerted by all the bands as x varies from $-R$ to c , we get that the portion of the spherical shell to the left of c exerts a force of

$$\frac{GmM}{2R} \int_{-R}^c \frac{(c-x)dx}{(c^2-2cx+R^2)^{\frac{3}{2}}}.$$

on P . We know from Section 14.5B that

$$\int \frac{(c-x) dx}{(c^2-2cx+R^2)^{\frac{3}{2}}} = \frac{1}{c}(c-x)(c^2-2cx+R^2)^{-\frac{1}{2}} - \frac{1}{c^2}(c^2-2cx+R^2)^{\frac{1}{2}} + C.$$

So

$$\begin{aligned} \int_{-R}^c \frac{(c-x) dx}{(c^2-2cx+R^2)^{\frac{3}{2}}} &= 0 - \frac{1}{c^2}(R^2-c^2)^{\frac{1}{2}} - \frac{1}{c}(c+R)(c^2+2cR+R^2)^{-\frac{1}{2}} \\ &\quad + \frac{1}{c^2}(c^2+2cR+R^2)^{\frac{1}{2}}. \end{aligned}$$

Because $c^2 + 2cR + R^2 = (c+R)^2 = (R+c)^2$ we see that $(c^2 + 2cR + R^2)^{\frac{1}{2}} = R+c$. So the value of the integral is

$$-\frac{1}{c^2}(R^2-c^2)^{\frac{1}{2}} - \frac{1}{c} + \frac{1}{c^2}(R+c) = \frac{1}{c^2}[R - (R^2-c^2)^{\frac{1}{2}}].$$

Note: The “summing up” process just illustrated is more subtle than might appear. For example, observe that the area of the thin circular band that was discussed is approximately equal to

$$2\pi r dx = 2\pi\sqrt{R^2 - x^2} dx,$$

so that $M_0 \approx \frac{M}{4\pi R^2} \cdot 2\pi\sqrt{R^2 - x^2} dx = \frac{M}{2R^2}\sqrt{R^2 - x^2} dx$. The use of this approximation would have led to a different and hence incorrect result.

36. Correction: Exercise 36 should read “to the right of the plane through c ” and not “to the left of the plane through c .”

Consider a typical circular band to the right of c , i.e. suppose that $c < x < R$. The force exerted by the band on the particle P is (again by the formula in Section 14.5A)

$$\frac{GmM_0(x - c)dx}{((x - c)^2 + r^2)^{\frac{3}{2}}}$$

where M_0 is the mass of the circular band and r is its radius. Letting dx be the width of the circular band, we get (as in the solution of Exercise 35) that the force is

$$\frac{GmM}{2R} \frac{(x - c)dx}{(c^2 - 2cx + R^2)^{\frac{3}{2}}}.$$

Letting x vary from c to R we get that the force exerted on P by the portion of the shell lying to the right of c is

$$\frac{GmM}{2R} \int_c^R \frac{(x - c)dx}{(c^2 - 2cx + R^2)^{\frac{3}{2}}}.$$

37. Because

$$\int \frac{(x - c)dx}{(c^2 - 2cx + R^2)^{\frac{3}{2}}} = \frac{1}{c^2}(c^2 - 2cx + R^2)^{\frac{1}{2}} - \frac{1}{c}(c - x)(c^2 - 2cx + R^2)^{-\frac{1}{2}} + c,$$

we get that

$$\int_c^R \frac{(x - c)dx}{(c^2 - 2cx + R^2)^{\frac{3}{2}}} = \frac{1}{c^2}(c^2 - 2cR + R^2)^{\frac{1}{2}} - \frac{1}{c}(c - R)(c^2 - 2cR + R^2)^{-\frac{1}{2}} - \frac{1}{c^2}(R^2 - c^2)^{\frac{1}{2}}.$$

The fact that $R > c$ implies that $(c^2 - 2cR + R^2)^{\frac{1}{2}} = [(c - R)^2]^{\frac{1}{2}} = R - c$. Therefore,

$$\int_c^R \frac{(x - c)dx}{(c^2 - 2cx + R^2)^{\frac{3}{2}}} = \frac{1}{c^2}(R - c) + \frac{1}{c} - \frac{1}{c^2}(R^2 - c^2)^{\frac{1}{2}} = \frac{1}{c^2}[R - (R^2 - c^2)^{\frac{1}{2}}].$$

38. It follows from the diagram that the force on P exerted by the entire spherical shell is zero.

14I. About Kepler's Laws

39. Recall from Section 14.6A that

$$\frac{dy}{dt} = \frac{dr}{dt} \sin \theta + r \cos \theta \cdot \frac{d\theta}{dt}.$$

By differentiating both sides with respect to t we get

$$\frac{d^2y}{dt^2} = \left[\frac{d^2r}{dt^2} \sin \theta + \frac{dr}{dt} \cos \theta \cdot \frac{d\theta}{dt} \right] + \left[\left(\frac{dr}{dt} \cos \theta - r \sin \theta \cdot \frac{d\theta}{dt} \right) \frac{d\theta}{dt} + r \cos \theta \cdot \frac{d^2\theta}{dt^2} \right].$$

Rearranging the right side of this equality gives us,

$$\frac{d^2y}{dt^2} = \frac{d^2r}{dt^2} \sin \theta - r \sin \theta \cdot \left(\frac{d\theta}{dt} \right)^2 + \cos \theta \left(2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} \right).$$

Because $2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} = 0$, we get $\frac{d^2y}{dt^2} = \sin \theta \left[\frac{d^2r}{dt^2} - r \cdot \left(\frac{d\theta}{dt} \right)^2 \right]$.

40. We know that

$$F = mr \left(\frac{d\theta}{dt} \right)^2 - m \frac{d^2r}{dt^2} = mr \left(\frac{2\kappa}{r^2} \right)^2 - m \frac{d^2r}{dt^2}.$$

So, $\frac{d^2r}{dt^2} - \frac{4\kappa^2}{r^3} = -\frac{F}{m} = -\frac{1}{m} C \frac{m}{r^2}$, and therefore, $\frac{d^2r}{dt^2} - \frac{4\kappa^2}{r^3} = -\frac{C}{r^2}$.

41. By the chain rule,

$$\frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} \quad \text{and} \quad \frac{du}{d\theta} = -[r(\theta)]^{-2} \frac{dr}{d\theta}.$$

So $\frac{dr}{d\theta} = -r^2 \frac{du}{d\theta}$, and hence

$$\frac{dr}{dt} = -r^2 \frac{du}{d\theta} \cdot \frac{d\theta}{dt} = -2\kappa \frac{du}{d\theta}.$$

By differentiating $\frac{dr}{dt} = -2\kappa \frac{du}{d\theta}$, we get

$$\begin{aligned} \frac{d^2r}{dt^2} &= \frac{d}{dt} \left(\frac{dr}{dt} \right) = -2\kappa \frac{d}{dt} \left(\frac{du}{d\theta} \right) \\ &= -2\kappa \frac{d}{d\theta} \left(\frac{du}{d\theta} \right) \cdot \frac{d\theta}{dt} = -2\kappa \frac{d^2u}{d\theta^2} \cdot \frac{d\theta}{dt} \\ &= -2\kappa \frac{d^2u}{d\theta^2} \left(\frac{2\kappa}{r^2} \right) = -4\kappa^2 u^2 \frac{d^2u}{d\theta^2}. \end{aligned}$$

42. By plugging the last equation of Exercise 41 into the equation of Exercise 40, we get

$$-4\kappa^2 u^2 \frac{d^2u}{d\theta^2} - 4\kappa^2 u^3 = -Cu^2.$$

So $\frac{d^2u}{d\theta^2} + u = \frac{Cu^2}{4\kappa^2 u^2} = \frac{C}{4\kappa^2}$, and therefore, $\frac{d^2u}{d\theta^2} + u - \frac{C}{4\kappa^2} = 0$. Hence $\frac{d^2z}{d\theta^2} + z = 0$.

43. The appropriate result is that the general solution of the second-order equation

$$\frac{d^2 y}{dx^2} + \frac{k}{m} y = 0$$

is $A \left(\sin \sqrt{\frac{k}{m}} x \right) + B \left(\cos \sqrt{\frac{k}{m}} x \right)$, where A and B are constants. Applying this fact with $k = 1, m = 1, y = z$, and $x = \theta$, we get that the general solution of

$$\frac{d^2 z}{d\theta^2} + z = 0$$

is $z = A \sin \theta + B \cos \theta$. Therefore, $u - \frac{C}{4\kappa^2} = z = A \sin \theta + B \cos \theta$ for some constants A and B .

Addition to the introduction to Exercises 44 and 45. Recall that the distance $r(\theta)$ from P to S is positive for any θ . Because $\sin \theta$ and $\cos \theta$ are continuous functions for $0 \leq \theta \leq 2\pi$,

$$u(\theta) = \frac{1}{r(\theta)} = A \sin \theta + B \cos \theta + \frac{C}{4\kappa^2}$$

is also continuous on $[0, 2\pi]$. See Section 8.2. It follows that $u(\theta)$ has a maximum value, say, M on $0 \leq \theta \leq 2\pi$. Because $\sin \theta$ and $\cos \theta$ are periodic of period 2π (see Section 10.8), it follows that M is the maximum value of $u(\theta)$ (this time not restricted to $[0, 2\pi]$). It follows that $d = \frac{1}{M}$ is the minimum value of the function $r(\theta)$.

44. By Exercise 43,

$$u(\theta) = A \sin \theta + B \cos \theta + \frac{C}{4\kappa^2}.$$

Because

$$u'(\theta) = A \cos \theta - B \sin \theta$$

and $u'(0) = 0$, we find that $0 = A \cdot 1 - B \cdot 0 = A$. So $u'(\theta) = -B \sin \theta$, and $u''(\theta) = -B \cos \theta$. If $B < 0$, then $u''(0) = -B > 0$. But this is impossible, since in this case, $u(\theta)$ has a local minimum at $\theta = 0$. (Use the second derivative test.) It follows that $B \geq 0$. Because $r(0) = d$, we get

$$\frac{1}{d} = u(0) = B \cos 0 + \frac{C}{4\kappa^2} = B + \frac{C}{4\kappa^2}.$$

So $B = \frac{1}{d} - \frac{C}{4\kappa^2}$. Therefore,

$$u(\theta) = \left(\frac{1}{d} - \frac{C}{4\kappa^2} \right) \cos \theta + \frac{C}{4\kappa^2}.$$

It follows that

$$\begin{aligned} r(\theta) &= \frac{1}{\left(\frac{1}{d} - \frac{C}{4\kappa^2} \right) \cos \theta + \frac{C}{4\kappa^2}} = \frac{1}{\left(\frac{4\kappa^2 - Cd}{4\kappa^2 d} \right) \cos \theta + \frac{Cd}{4\kappa^2 d}} \\ &= \frac{4\kappa^2 d}{(4\kappa^2 - Cd) \cos \theta + Cd} = \frac{4\kappa^2 d}{Cd \left(1 - \left(\frac{4\kappa^2}{Cd} - 1 \right) \cos \theta \right)} \\ &= \frac{\frac{4\kappa^2}{C}}{1 - \left(\frac{4\kappa^2}{Cd} - 1 \right) \cos \theta} = \frac{\frac{8\kappa^2}{C}}{2 \left(1 - \left(\frac{4\kappa^2}{Cd} - 1 \right) \cos \theta \right)}. \end{aligned}$$

Recall that $B = \frac{1}{d} - \frac{C}{4\kappa^2} \geq 0$. So $\frac{1}{d} \geq \frac{C}{4\kappa^2}$, and hence $\frac{4\kappa^2}{C} \geq d$. So $\frac{4\kappa^2}{dC} \geq 1$, and $\frac{4\kappa^2}{dC} = 1 \geq 0$ as required.

45. Taking $L = \frac{8\kappa^2}{C}$ and $\varepsilon = \frac{4\kappa^2}{dC} - 1$, we see that

$$r = r(\theta) = \frac{L}{2(1 + \varepsilon \cos \theta)}$$

is in the standard form of Section 13.4. Conclusions (1) - (5) of Section 13.4 complete Exercise 45.

14J. Free Fall with Air Resistance

Addition to the introduction to Exercises 14J. Insert “Since the object is falling near the Earth’s surface, we will assume that ρ is constant.” after “and v is the velocity of the object.”

46. Because $F = ma = m\frac{dv}{dt}$, we get

$$m\frac{dv}{dt} = F = -mg + \frac{1}{4}\rho Av^2.$$

So $\frac{dv}{dt} = -g + \frac{\rho A}{4m}v^2 = \frac{\rho A}{4m}v^2 - g$. Hence $\frac{4m}{\rho A}\frac{dv}{dt} = v^2 - \frac{4mg}{\rho A}$, and therefore,

$$\frac{k^2}{g}\frac{dv}{dt} = v^2 - k^2.$$

The force of gravity exceeds the drag. So the net force is negative; hence the acceleration $\frac{dv}{dt}$ is negative. Therefore, $v^2 - k^2 = (v - k)(v + k) < 0$. It follows that either

$$(v - k > 0 \text{ and } v + k < 0) \text{ or } (v - k < 0 \text{ and } v + k > 0).$$

The first option implies that $v > k$ and $v < -k$. But this forces $k < -k$; impossible, because $k > 0$. So the second option,

$$v < k \text{ and } v > -k$$

is the only possibility. So $-k < v < k$, as asserted.

Note: The inequality just verified holds for any time $t \geq 0$. But what if, you ask, the object is given an initial velocity $v(0)$ such that either $v(0) \geq k$ or $v(0) \leq -k$? This means that at time $t = 0$, $v(0)^2 \geq k^2$ and hence that

$$\frac{1}{4}\rho Av(0)^2 \geq \frac{1}{4}\rho Ak^2 = \frac{1}{4}\rho A\frac{4mg}{\rho A} = mg.$$

So the drag (upward) exceeds the gravitational force (downward). This does not correspond to the facts and means that the drag as given by $\frac{1}{4}\rho Av^2$ is an invalid starting point under this assumption on the initial velocity. The mathematical model for the drag must be different in this case.

47. By separating variables, we get

$$\frac{dv}{v^2 - k^2} = \frac{g}{k^2} dt$$

and hence $\int \frac{dv}{v^2 - k^2} = \int \frac{g}{k^2} dt$. The integral $\int \frac{dv}{v^2 - k^2}$ is solved as follows:

$$\begin{aligned} \frac{1}{v^2 - k^2} &= \frac{1}{(v - k)(v + k)} = \frac{A}{v - k} + \frac{B}{v + k} \\ &= \frac{A(v + k) + B(v - k)}{(v - k)(v + k)} = \frac{(A + B)v + (A - B)k}{v^2 - k^2}. \end{aligned}$$

Because v is the variable and k the constant, we get $A + B = 0$ and $(A - B)k = 1$. So, $2Ak = 1$. Hence $A = \frac{1}{2k}$ and $B = -A = -\frac{1}{2k}$. Therefore

$$\frac{1}{v^2 - k^2} = \frac{1}{2k} \frac{1}{v - k} - \frac{1}{2k} \frac{1}{v + k}.$$

Therefore,

$$\begin{aligned} \int \frac{dv}{v^2 - k^2} &= \frac{1}{2k} \int \frac{1}{v - k} dv - \frac{1}{2k} \int \frac{1}{v + k} dv \\ &= \frac{1}{2k} \ln(v - k) - \frac{1}{2k} \ln(v + k) + C_1. \end{aligned}$$

Let's stop for a moment. Because $k > v$, we see that $v - k < 0$. So $\ln(v - k)$ does not make sense! (See Section 10.3.) We need to retrace our steps, beginning with

$$\frac{1}{v^2 - k^2} = \frac{1}{2k} \frac{1}{v - k} - \frac{1}{2k} \frac{1}{v + k}.$$

Rewriting this equation

$$\frac{1}{v^2 - k^2} = -\frac{1}{2k} \frac{1}{k - v} - \frac{1}{2k} \frac{1}{k + v},$$

we get

$$\begin{aligned} \int \frac{1}{v^2 - k^2} dt &= -\frac{1}{2k} \int \frac{1}{k - v} dv - \frac{1}{2k} \int \frac{1}{k + v} dv \\ &= -\frac{1}{2k} (-\ln(k - v)) - \frac{1}{2k} \ln(k + v) + C_1. \end{aligned}$$

To see that this is true, check that

$$\begin{aligned} \frac{d}{dv} \ln(k - v) &= \frac{1}{k - v} \frac{d}{dv} (k - v) \\ &= \frac{-1}{k - v}. \end{aligned}$$

Also observe that $k + v > 0$. We have shown that

$$\int \frac{1}{v^2 - k^2} dt = \frac{1}{2k} \ln(k - v) - \frac{1}{2k} \ln(k + v) + C_1.$$

Because $\int \frac{g}{k^2} dt = \frac{g}{k^2}t + C_2$, we get the required

$$\frac{1}{2k} \ln(k-v) - \frac{1}{2k} \ln(k+v) = \frac{g}{k^2}t + C.$$

Under the assumption that $v = 0$ when $t = 0$, we get $\frac{1}{2k} \ln k - \frac{1}{2k} \ln k = 0 + C$. So $C = 0$, and

$$\frac{1}{2k} \ln(k-v) - \frac{1}{2k} \ln(k+v) = \frac{g}{k^2}t.$$

By properties of the logarithm (See Section 10.3), $\frac{1}{2k} \ln\left(\frac{k-v}{k+v}\right) = \frac{g}{k^2}t$. Hence

$$\ln\left(\frac{k-v}{k+v}\right) = \frac{2kg}{k^2}t = \frac{2g}{k}t,$$

and $\frac{k-v}{k+v} = e^{\frac{2g}{k}t}$. Solving for v , we get $(k+v)e^{\frac{2g}{k}t} = k-v$, hence $v\left(e^{\frac{2g}{k}t} + 1\right) = k - ke^{\frac{2g}{k}t} = -k\left(e^{\frac{2g}{k}t} - 1\right)$. Therefore,

$$v = \frac{-k(e^{\frac{2g}{k}t} - 1)}{e^{\frac{2g}{k}t} + 1}$$

as asserted.

- 48. Clarification** of the statement of Exercise 48. Insert “the same” between “at” and “time” and insert “from possibly different altitudes” after “initial velocity.”

Since $\frac{m}{A}$ is the same for both objects, $k = \sqrt{\frac{4mg}{\rho A}}$ is the same for both objects. Therefore, see the formula above, $s(t)$ is the same for both objects.

- 49. Correction:** Replace Exercise 49 by “Verify that the speed $s(t)$ is an increasing function of time t ”. In the Hint, replace “decreasing” by “increasing.”

Let’s start with the hint. The function $f(x) = \frac{x-1}{x+1}$ with $x \geq 0$ is indeed one-to-one, but the fact is that it is increasing and not decreasing as asserted. To see this, check that

$$f'(x) = \frac{1 \cdot (x+1) - (x-1) \cdot 1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

and observe that $f'(x) > 0$. So by Section 8.7, $f(x)$ is an increasing function.

Next, consider the increasing function $x = e^{\frac{2g}{k}t}$ and conclude that the composite function $f(e^{\frac{2g}{k}t})$ is an increasing function of t . Because

$$s(t) = kf(e^{\frac{2g}{k}t}),$$

it follows that the speed of a falling object is an increasing function of t . More explicitly, applying the chain rule to the equation

$$s(t) = kf(e^{\frac{2g}{k}t})$$

we get

$$s'(t) = kf'(e^{\frac{2g}{k}t}) \cdot e^{\frac{2g}{k}t} \cdot \frac{2g}{k} = \frac{2}{(e^{\frac{2g}{k}t} + 1)^2} \cdot 2ge^{\frac{2g}{k}t} = \frac{4ge^{\frac{2g}{k}t}}{(e^{\frac{2g}{k}t} + 1)^2}.$$

So $s'(t) > 0$ for all t , and $s(t)$ is increasing.

50.
$$\lim_{t \rightarrow \infty} \frac{k(e^{\frac{2g}{k}t} - 1)}{e^{\frac{2g}{k}t} + 1} = \lim_{t \rightarrow \infty} \frac{k \cdot e^{\frac{2g}{k}t} \cdot \frac{2g}{k}}{e^{\frac{2g}{k}t} \cdot \frac{2g}{k}} = \lim_{t \rightarrow \infty} k = k.$$

51. The domain of

$$s(t) = \frac{k(e^{\frac{2g}{k}t} - 1)}{e^{\frac{2g}{k}t} + 1}$$

is all $t \geq 0$. Because

$$\lim_{t \rightarrow \infty} s(t) = k,$$

we know that $k = s_\infty$ is a horizontal asymptote. We saw in Exercise 48 that $s'(t) > 0$ for all t and hence that $s(t)$ is an increasing function. From the solution of Exercise 48,

$$s'(t) = \frac{4ge^{\frac{2g}{k}t}}{(e^{\frac{2g}{k}t} + 1)^2}.$$

It remains to check that the curve is concave down by computing $s''(t)$.

$$\begin{aligned} s''(t) &= \frac{4ge^{\frac{2g}{k}t} \cdot \frac{2g}{k}(e^{\frac{2g}{k}t} + 1)^2 - 4ge^{\frac{2g}{k}t} 2(e^{\frac{2g}{k}t} + 1) \cdot e^{\frac{2g}{k}t} \cdot \frac{2g}{k}}{(e^{\frac{2g}{k}t} + 1)^4} \\ &= \frac{4ge^{\frac{2g}{k}t} \cdot \frac{2g}{k}[e^{\frac{2g}{k}t} + 1 - 2e^{\frac{2g}{k}t}]}{(e^{\frac{2g}{k}t} + 1)^3} \\ &= \frac{\frac{8g^2}{k} \cdot e^{\frac{2g}{k}t}[1 - e^{\frac{2g}{k}t}]}{(e^{\frac{2g}{k}t} + 1)^3} = \frac{-\frac{8g^2}{k}e^{\frac{2g}{k}t}[e^{\frac{2g}{k}t} - 1]}{(e^{\frac{2g}{k}t} + 1)^3}. \end{aligned}$$

Because $e^{\frac{2g}{k}t} > 1$ for $t > 0$, it follows that $s''(t) < 0$ and hence that the curve is concave down.

52. This equation follows from the earlier equation

$$v(t) = \frac{-k(e^{\frac{2g}{k}t} - 1)}{e^{\frac{2g}{k}t} + 1}$$

and the fact that $\frac{dy}{dt} = v(t)$.

53. Since $\frac{du}{dt} = ae^{at} = au$, we see that $\frac{1}{a} \frac{du}{u} = dt$. Therefore

$$\int \frac{e^{at} - 1}{e^{at} + 1} dt = \int \frac{u - 1}{u + 1} \frac{1}{au} du = \frac{1}{a} \int \frac{u - 1}{u(u + 1)} du.$$

54. Put $\frac{u-1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$. Now,

$$\frac{A}{u} + \frac{B}{u+1} = \frac{A(u+1) + Bu}{u(u+1)} = \frac{(A+B)u + A}{u(u+1)}.$$

It follows that $A = -1$ and $A + B = 1$. So $B = 2$ and hence $\frac{u-1}{u(u+1)} = \frac{-1}{u} + \frac{2}{u+1}$.
By Section 10.3,

$$\frac{d}{du} \ln u = \frac{1}{u} \quad \text{and} \quad \frac{d}{du} \ln(u+1) = \frac{1}{u+1}.$$

So,

$$\int \frac{u-1}{u(u+1)} du = - \int \frac{1}{u} du + 2 \int \frac{1}{u+1} du = -\ln u + 2 \ln(u+1) + C.$$

55. Start with the conclusion $\frac{dy}{dt} = -s_\infty \frac{e^{at}-1}{e^{at}+1}$ of Exercise 52. So $y = -s_\infty \int \frac{e^{at}-1}{e^{at}+1} dt$ and using Exercise 53, $y = -\frac{s_\infty}{a} \int \frac{u-1}{u(u+1)} du$. By Exercise 54,

$$y = -\frac{s_\infty}{a} [-\ln u + 2 \ln(u+1) + C].$$

Because $u = e^{at}$, $y = -\frac{s_\infty}{a} [-\ln e^{at} + 2 \ln(e^{at} + 1) + C]$, and hence

$$y(t) = \frac{s_\infty}{a} \cdot at - 2 \frac{s_\infty}{a} \ln(e^{at} + 1) + C_1.$$

Because $y_0 = y(0) = -2 \frac{s_\infty}{a} \ln 2 + C_1$, we get $C_1 = 2 \frac{s_\infty}{a} \ln 2 + y_0$. Therefore,

$$y(t) = s_\infty t - \frac{2s_\infty}{a} \ln(e^{at} + 1) + 2 \frac{s_\infty}{a} \ln 2 + y_0.$$

Because $a = \frac{2g}{s_\infty}$, we get

$$y(t) = s_\infty t - \frac{(s_\infty)^2}{g} \ln \left(e^{\frac{2g}{s_\infty} t} + 1 \right) + \frac{(s_\infty)^2}{g} \ln 2 + y_0.$$

One more simplification gives us,

$$y(t) = s_\infty t - \frac{(s_\infty)^2}{g} \ln \frac{1}{2} \left(e^{\frac{2g}{s_\infty} t} + 1 \right) + y_0.$$

14K. Returning to the Leaning Tower

56. By Exercise 50,

$$s_\infty = k = \sqrt{\frac{4mg}{\rho A}} = \sqrt{\frac{4(0.9956)(32.1700)}{(0.0023)(0.1964)}} = \sqrt{283613.318} = 532.5536.$$

So $\frac{2g}{s_\infty} = \frac{64.3400}{532.5536} = 0.1208$.

57.

$$\begin{aligned}s(2) &= 532.5536 \frac{e^{0.2416} - 1}{e^{0.2416} + 1} = (532.5536)(0.1202) = 64.0214 \\s(5) &= (532.5536)(0.2931) = 156.1137 \\s(10) &= (532.5536)(0.5399) = 287.5202.\end{aligned}$$

Finally, we need to solve

$$0.9s_\infty = s(t) = s_\infty \frac{e^{0.1208t} - 1}{e^{0.1208t} + 1}$$

for t . Because $0.9(e^{0.1208t} + 1) = e^{0.1208t} - 1$, we get $(0.1)e^{0.1208t} = 1.9$, and hence $0.1208t = \ln 19$. So $t = 24.37$ seconds.

58. Check that $\frac{(s_\infty)^2}{g} = \frac{(532.5536)^2}{32.17} = 8816.0813$ and

$$y_0 + \frac{(s_\infty)^2}{g} \ln 2 = 177 + 6110.8419 = 6287.8419.$$

59. $y(3.3172) = 1766.5868 - 8816.0813(0.9134) + 6287.8419 = 1.4231$.

60. $s(3.3307) = (532.5536)(0.1985) = 105.7138$. Note that this answer is different from the 105.7119 given in the text. The present computation rounded off only at the end.

14L. Dropping a Basketball from the Leaning Tower

61. $A = \pi(0.39)^2 = 0.4778$. If m is the mass of the ball, then $1.3 = mg$. So $m = \frac{1.3}{32.17} = 0.0404$ slugs. The terminal velocity is

$$s_\infty = \sqrt{\frac{4mg}{\rho A}} = \sqrt{\frac{4(0.0404)(32.17)}{(0.0023)(0.4778)}} = \sqrt{4730.6240} = 68.7795.$$

62. Because $\frac{2g}{s_\infty} = \frac{64.34}{68.7795} = 0.9355$ and

$$\frac{(s_\infty)^2}{g} = \frac{(68.7795)^2}{32.17} = 147.0507,$$

we get, by substituting into the formulas of Exercises 50 and 55, that

$$s(t) = 68.7795 \frac{e^{0.9355t} - 1}{e^{0.9355t} + 1} \quad \text{and} \quad y(t) = 68.7795t - 147.0507 \ln(e^{0.9355t} + 1) + 278.9278.$$

63. You are being asked to fill the missing information into the column on the right and to determine the time at which the basketball strikes the ground.

$$\begin{aligned}
y(1) &= 68.7795 - 186.2429 + 278.9278 = 161.4644 \\
y(2) &= 137.559 - 296.1907 + 278.9278 = 120.2961 \\
y(3) &= 206.3385 - 421.3240 + 278.9278 = 63.9423 \\
y(3.3172) &= 228.1554 - 462.7928 + 278.9278 = 44.2904 \\
y(3.3307) &= 229.0839 - 464.5706 + 278.9278 = 43.4411.
\end{aligned}$$

Solving $y(t) = 0$ for t by Newton's method (see Exercises 8K), we get $t = 4.0051$ seconds.

14M. More About Springs

- 64.** The object exerts a downward force of mg on the spring. This is the force that stretches the spring h units beyond its natural length. So by Section 14.2A, $mg = kh$. Note that we have ignored, i.e. considered as negligible, the stretching effect produced by gravity acting on the spring itself.
- 65.** When the object is at $y(t)$, it will be subjected to an (upward) pull of $k(-y(t) + h)$ by the spring and a (downward) pull of mg by gravity. So

$$\begin{aligned}
F(t) &= k(-y(t) + h) - mg \\
&= -ky(t) + kh - mg \\
&= -ky(t).
\end{aligned}$$

By Newton's law, $F(t) = my''(t)$, and therefore,

$$y''(t) = -\frac{k}{m}y(t)$$

as required.

- 66.** Because $y(t)$ satisfies the differential equation $y''(t) = -\frac{k}{m}y(t)$ we know from the analysis following Example 13.3 of Section 13.5 that

$$y(t) = a \sin\left(\sqrt{\frac{k}{m}}t\right) + b \cos\left(\sqrt{\frac{k}{m}}t\right)$$

and

$$y'(t) = a \left(\cos\sqrt{\frac{k}{m}}t\right) \sqrt{\frac{k}{m}} - b \left(\sin\sqrt{\frac{k}{m}}t\right) \sqrt{\frac{k}{m}},$$

for some constants a and b . (We use lower case in order to avoid confusion with the A that is already being used.) Because, $y(0) = -A$ and $y'(0) = 0$, we get $-A = y(0) = b$ and $0 = y'(0) = a$. It follows that

$$y(t) = -A \left(\cos\sqrt{\frac{k}{m}}t\right).$$

67. A look at the graph of $\cos x$ shows that $\cos x$ moves through a complete cycle when x goes from 0 to 2π . So

$$y(t) = -A \left(\cos \sqrt{\frac{k}{m}} t \right)$$

moves through a complete cycle over $0 \leq \sqrt{\frac{k}{m}} t \leq 2\pi$. Dividing by $\sqrt{\frac{k}{m}}$, we see that this corresponds to the time frame

$$0 \leq t \leq \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\sqrt{\frac{m}{k}}\pi.$$

68. As in the solution of the Problem 66,

$$y(t) = a \left(\sin \sqrt{\frac{k}{m}} t \right) + b \left(\cos \sqrt{\frac{k}{m}} t \right)$$

and

$$y'(t) = a \left(\cos \sqrt{\frac{k}{m}} t \right) \sqrt{\frac{k}{m}} - b \left(\sin \sqrt{\frac{k}{m}} t \right) \sqrt{\frac{k}{m}}.$$

Notice that $-A = y(0) = b$ and $v_0 = y'(0) = a\sqrt{\frac{k}{m}}$. Therefore,

$$y(t) = v_0 \sqrt{\frac{m}{k}} \left(\sin \sqrt{\frac{k}{m}} t \right) - A \left(\cos \sqrt{\frac{k}{m}} t \right).$$

69. The relevant system of units is M.K.S. By the definition of the spring constant k , we get $40 = k(0.05)$, and hence $k = 800$. Notice that $m = \frac{1}{2}$, $A = 0.1$, and $v_0 = -2$. Plugging into the equation derived in Problem 68, we get

$$\begin{aligned} y(t) &= -2\sqrt{\frac{\frac{1}{2}}{800}} \left(\sin \sqrt{\frac{800}{\frac{1}{2}}} t \right) - 0.1 \left(\cos \sqrt{\frac{800}{\frac{1}{2}}} t \right) \\ &= -2\sqrt{\frac{1}{1600}} \left(\sin \sqrt{1600} t \right) - 0.1 \left(\cos \sqrt{1600} t \right) \\ &= -\frac{1}{20}(\sin 40t) - \frac{1}{10}(\cos 40t). \end{aligned}$$

Using the solution of Problem 67, we see that the object will move through a cycle in

$$2\sqrt{\frac{m}{k}}\pi = 2\sqrt{\frac{\frac{1}{2}}{400}}\pi = \frac{2}{40}\pi \approx 0.16 \text{ seconds.}$$

14N. About the Universe

70. Recall that $H_0 = H(t_0) = \frac{R'(t_0)}{R(t_0)}$ and $14 \leq H_0 \leq 23$, where $R'(t_0)$ is given in kilometers per second and $R(t_0)$ in mega light years. Since $R'(t_0)$ is the velocity of a galaxy at the outer fringe of the universe,

$$0.9c \leq R'(t_0) \leq c.$$

The fact that $R'(t_0) = H_0 R(t_0)$ gives $0.9c \leq H_0 R(t_0) \leq c$ and hence

$$\frac{0.9c}{H_0} \leq R(t_0) \leq \frac{c}{H_0}.$$

Taking the upper bound for H_0 on the left and the lower bound on the right gives us

$$\frac{0.9c}{23} \leq R(t_0) \leq \frac{c}{14}.$$

So

$$\frac{(0.9)(3.0 \times 10^5)}{23} \leq R(t_0) \leq \frac{3.0 \times 10^5}{14}$$

in mega light years. It follows that

$$11 \times 10^9 \leq R(t_0) \leq 22 \times 10^9$$

in light years.

71. Recall that $\Omega = 1$ is equivalent to $E = 0$. So equation (*) of Section 14.7 reduces to $\frac{1}{2}(R'(t))^2 = \frac{GM_u}{R(t)}$. Therefore, $M_u = \frac{1}{2G}(R'(t))^2 R(t)$. By the assumption that M_u is constant,

$$M_u = \frac{1}{2G}(R'(t_0))^2 R(t_0).$$

It remains to plug appropriate estimates into this formula. We know that $G \approx 6.67 \times 10^{11}$ in M.K.S. Using Problem 70, we will take

$$R'(t_0) \approx c \quad \text{and} \quad R(t_0) \approx 17 \times 10^9 \text{ light years.}$$

We need to convert these estimates into M.K.S. Since 1 kilometer = 10^3 meters, $c \approx 3 \times 10^8$ meters per second. Because 1 year has approximately 32×10^6 seconds, light travels

$$(3 \times 10^8)(32 \times 10^6) \approx 10^{16}$$

meters in one year. So $R'(t_0) \approx 3 \times 10^8$ and $R(t_0) \approx 17 \times 10^{25}$ in M.K.S. By substituting, we now get

$$\begin{aligned} M_u &\approx \left(\frac{1}{2(6.67)} \times 10^{11} \right) (9 \times 10^{16})(17 \times 10^{25}) \\ &\approx \frac{153}{14} \times 10^{52} \approx 10^{53} \text{ kilograms.} \end{aligned}$$