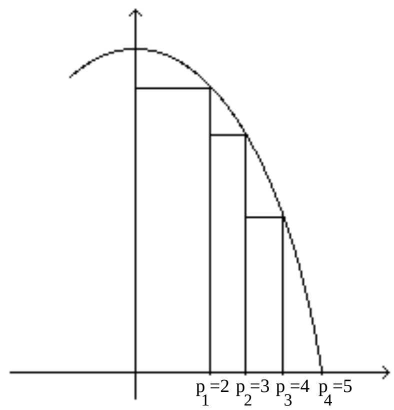
Solutions to the Exercises of Chapter 13

13A. Riemann Sums

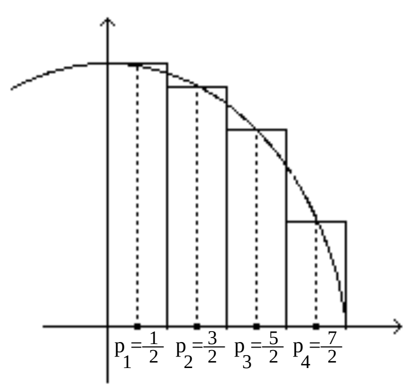
1. $a = 0,\ b = 5,\ m = 4$. The partition is $0 < 2 < 3 < 4 < 5$. So $\Delta x_1 = 2-0 = 2,\ \Delta x_2 = 3-2 = 1,\ \Delta x_2 = 4-3 = 1, \Delta x_4 = 5-4 = 1$, and $p_1 = x_1 = 2, p_2 = x_2 = 3,\ p_2 = x_3 = 4$, and



$p_4 = x_4 = 5$. So

$$\begin{aligned}\sum_{i=1}^{4} f(p_i)\Delta x_i &= f(p_1)\cdot 2 + f(p_2)\cdot 1 + f(p_3)\cdot 1 + f(p_4)\cdot 1\\ &= 2(25-2^2) + (25-3^2) + (25-4^2) + (25-5^2)\\ &= 42 + 16 + 9 + 0 = 67.\end{aligned}$$

2. $a = 0, b = 4, n = 4$. The partition is $0 < 1 < 2 < 3 < 4$. $\Delta x_1 = 1-0 = 1,\ \Delta x_2 = 2-1 = 1,\ \Delta x_3 = 3-2 = 1,\ \Delta x_4 = 4-3 = 1$, and $p_1 =$ midpoint between 0 and $1 = \frac{1}{2},\ p_2 =$ midpoint

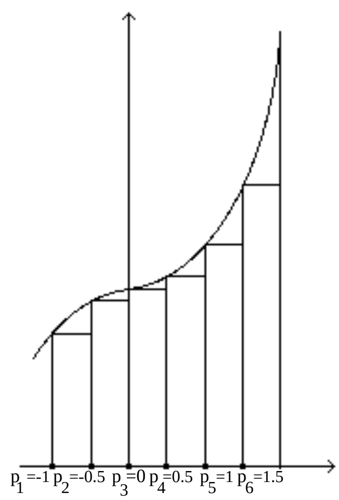


between 1 and $2 = \frac{3}{2},\ p_3 =$ midpoint between 2 and $3 = \frac{5}{2},\ p_4 =$ midpoint between 3 and 4

$= \frac{7}{2}$. So

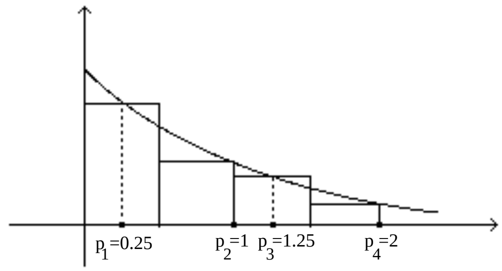
$$\begin{aligned}
\sum_{i=1}^{n} f(p_i)\Delta x_i &= \sum_{i=1}^{4} f(p_i)\cdot 1 = f(p_1)+f(p_2)+f(p_3)+f(p_4) \\
&= 16-\left(\frac{1}{2}\right)^2+16-\left(\frac{3}{2}\right)^2+16-\left(\frac{5}{2}\right)^2+16-(\frac{7}{2})^2 \\
&= 4\cdot 16-\frac{1}{4}-\frac{9}{4}-\frac{25}{4}-\frac{49}{4} = 64-\frac{1+9+25+49}{4} \\
&= 64-\frac{84}{4} = 64-21 = 43.
\end{aligned}$$

3. $a=-1,\ b=2,\ m=6,\ \Delta x_1=-0.5-(-1)=0.5,\ \Delta x_2=0-(-0.5)=0.5,\ \Delta x_3=0.5-0=0.5,\ \Delta x_4=1.0-0.5=0.5,\ \Delta x_5=1.5-1.0=0.5,\ \Delta x_6=2-1.5=0.5$, and $p_1=-1,\ p_2=-0.5,\ p_3=0,\ p_4=0.5,\ p_5=1.0, p_6=1.5$. So



$$\begin{aligned}
\sum_{i=1}^{n} f(p_i)\Delta x_i &= \sum_{i=1}^{6} f(p_i)(0.5) = (0.5)\left[f(p_1)+\cdots+f(p_6)\right] \\
&= (0.5)\left[\big((-1)^3+2\big)+\big((-0.5)^3+2\big)+(0^3+2)+\big((0.5)^3+2\big)\right. \\
&+ \left.\big((1.0)^3+2\big)+\big((1.5)^3+2\big)\right] \\
&= (0.5)\left[(-1+2)+(-0.125+2)+2+(0.125+2)+(1+2)+(3.375+2)\right] \\
&= (0.5)\left[2+2+2+2+2+2-1-0.125+0.125+1+3.375\right] \\
&= (0.5)\left[12+3.375\right] \\
&= (0.5)(15.375) = 7.6875
\end{aligned}$$

4. $a=0,\ b=2,\ m=4,\ \Delta x_1=0.5-0=0.5,\ \Delta x_2=1.0-0.5=0.5,\ \Delta x_3=1.5-1.0=0.5,\ \Delta x_4=2-1.5=0.5$. So



$$
\begin{aligned}
\sum_{i=1}^{n} f(p_i)\Delta x_i &= \sum_{i=1}^{4} f(p_i)(0.5) \\
&= (0.5)\left[f(p_1)+f(p_2)+f(p_3)+f(p_4)\right] \\
&= (0.5)\left[\frac{1}{0.25+1}+\frac{1}{1+1}+\frac{1}{1.25+1}+\frac{1}{2+1}\right] \\
&= (0.5)\left[\frac{1}{1.25}+\frac{1}{2}+\frac{1}{2.25}+\frac{1}{3}\right] \\
&= (0.5)\left[0.8+0.5+0.44+0.33\right] \\
&= 1.035
\end{aligned}
$$

13B. Pushing a Riemann Sum to the Limit

5. **i.** The partition is $0 < \frac{3}{7} < \frac{6}{7} < \frac{9}{7} < \frac{12}{7} < \frac{15}{7} < \frac{18}{7} < \frac{21}{7} = 3$. It follows that the points, $p_1, \ldots, p_7$ are as listed.

ii.

$$
\begin{aligned}
\sum_{i=1}^{7} f(p_i)\Delta x_i &= \sum_{i=1}^{7} f(p_i)\cdot\frac{3}{7} \\
&= \left[f(p_1)+f(p_2)+f(p_3)+f(p_4)+f(p_5)+f(p_6)+f(p_7)\right]\left(\frac{3}{7}\right) \\
&= \Big[16-\left(1\cdot\frac{3}{7}\right)^2+16-\left(2\cdot\frac{3}{7}\right)^2+16-\left(3\cdot\frac{3}{7}\right)^2+16-\left(4\cdot\frac{3}{7}\right)^2 \\
&+ 16-\left(5\cdot\frac{3}{7}\right)^2+16-\left(6\cdot\frac{3}{7}\right)^2+16-\left(7\cdot\frac{3}{7}\right)^2\Big]\left(\frac{3}{7}\right) \\
&= \Big[16\cdot 7-\Big(1^2\cdot\left(\frac{3}{7}\right)^2+2^2\cdot\left(\frac{3}{7}\right)^2+3^2\cdot\left(\frac{3}{7}\right)^2+4^2\cdot\left(\frac{3}{7}\right)^2 \\
&+ 5^2\cdot\left(\frac{3}{7}\right)^2+6^2\cdot\left(\frac{3}{7}\right)^2+7^2\cdot\left(\frac{3}{7}\right)^2+7^2\cdot\left(\frac{3}{7}\right)^2\Big)\Big]\left(\frac{3}{7}\right) \\
&= \left[16\cdot 7-\left(\frac{3}{7}\right)^2\left(1^2+2^2+3^2+4^2+5^2+6^2+7^2\right)\right]\left(\frac{3}{7}\right)
\end{aligned}
$$

iii. The use of the sum of squares formula shows that this equals

$$\begin{aligned}
\Big[16\cdot 7 \quad &- \quad \left(\frac{3}{7}\right)^2\left(\frac{7(7+1)(14+1)}{6}\right)\Big]\left(\frac{3}{7}\right) \\
&= \quad 16\cdot 3 - \frac{9}{2}\frac{(7+1)(14+1)}{7^2} \\
&= \quad 48 - \frac{9}{2}\left(\frac{120}{7^2}\right) \\
&= \quad 48 - \frac{540}{7^2} = 48 - 11.02 = 36.98
\end{aligned}$$

6. **i.** $p_1 = 1\cdot\frac{3}{n},\ p_2 = 2\cdot\frac{3}{n},\ p_3 = 3\cdot\frac{3}{n},\ p_4 = 4\cdot\frac{3}{n},\cdots,$
$p_i = i\cdot\frac{3}{n},\cdots,\ p_n = n\cdot\frac{3}{n} = 3.$

ii. $f(p_i) = 16 - (p_i)^2 = 16 - \left(i\frac{3}{n}\right)^2 = 16 - i^2\cdot\left(\frac{3}{n}\right)^2$

iii.

$$\begin{aligned}
\sum_{i=1}^{n} f(p_i)\Delta x_i \quad &= \quad \left[\sum_{i=1}^{n} f(p_i)\right]\left(\frac{3}{n}\right) = \left[\sum_{i=1}^{n}\left(16 - i^2\left(\frac{3}{n}\right)^2\right)\right]\left(\frac{3}{n}\right) \\
&= \quad \left[16\cdot n - \sum_{i=1}^{n} i^2\left(\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right) = 48 - \left(\sum_{i=1}^{n} i^2\right)\left(\frac{3}{n}\right)^2\left(\frac{3}{n}\right) \\
&= \quad 48 - \left[\frac{n(n+1)(2n+1)}{6}\right]\left(\frac{3}{n}\right)\left(\frac{3}{n}\right)^2 = 48 - \frac{(n+1)(2n+1)}{2}\cdot\frac{9}{n^2} \\
&= \quad 48 - \frac{9}{2}\frac{(n+1)(2n+1)}{n^2} = 48 - \frac{9}{2}\left(\frac{2n^2+3n+1}{n^2}\right) \\
&= \quad 48 - \frac{9}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)
\end{aligned}$$

7. **i.** When n is pushed to infinity, $\|P_n\| = \frac{3}{n}$ goes to zero.

ii. Review Steps (1) - (3) of the limit process

$$\lim_{\|P\|\to 0}\left(\sum_{i=1}^{n} f(p_i)\Delta x_i\right).$$

Notice that in Step (1) you were free to take *any* partition P and that in Step (2) you were free to take *any* points $p_1,\cdots,p_n$ with p_1 in the first subinterval, p_2 in the second, and so on. The assertion in Step (3) was that the numbers

$$\sum_{i=1}^{n} f(p_i)\Delta x_i$$

produced by the repetition of Steps (1) and (2) with partitions P whose norms go to zero, will close in on some number

$$\lim_{\|P\|\to 0}\sum_{i=1}^{n} f(p_i)\Delta x_i\,.$$

This number is the same regardless of how the partitions P and the points p_i were selected along the way. This fact was not proved in the text. Rather, it was only made plausible by making the connection with areas.

Now turn to the procedure outlined in Exercise 6. Notice that it is a specific instance of the process described in Steps (1) and (2). It follows that the limiting number

$$\lim_{n\to\infty}\left[48-\frac{9}{2}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\right]$$

must be equal to

$$\lim_{\|P\|\to 0}\left(\sum_{i=1}^{n} f(p_i)\Delta x_i\right).$$

iii. Because

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0}\left(\sum_{i=1}^{n} f(p_i)\Delta x_i\right)$$

by definition, we find that

$$\begin{aligned}\int_0^3 (16-x^2)dx &= \lim_{n\to\infty}\left[48-\frac{9}{2}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\right]\\ &= 48-\frac{9}{2}\cdot 2 = 48-9=39.\end{aligned}$$

Correction: Note that the answer 37 (as stated in the text) is incorrect.

13C. Applying the Fundamental Theorem of Calculus

8. From the fact that $\frac{d}{dx}\cos x = -\sin x$, we see that $F(x) = -\cos x$ is an anti-derivative of $f(x) = \sin x$. Therefore,

$$\int_0^{\frac{\pi}{2}} \sin x = (-\cos x)\Big|_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} - (-\cos 0).$$

A look at the graph of $\cos x$ (see Figure 10.29) shows that $\cos\frac{\pi}{2} = 0$ and $\cos 0 = 1$. So

$$\int_0^{\frac{\pi}{2}} \sin x\,dx = 1.$$

9. We need an anti-derivative of $\tan x = \frac{\sin x}{\cos x}$. Let $g(x) = \cos x$. So $g'(x) = -\sin x$, and $\tan x = -\frac{g'(x)}{g(x)}$. A review of Section 10.3 tells us that $\frac{d}{dx}[-\ln\ g(x)] = -\frac{g'(x)}{g(x)}$. So $\frac{d}{dx}[-\ln(\cos x)] = \tan x$. Therefore,

$$
\begin{aligned}
\int_0^{\frac{\pi}{4}} \tan x dx &= -\ln(\cos x)\Big|_0^{\frac{\pi}{4}} = -\ln\cos\frac{\pi}{4} - (-\ln\cos 0) \\
&= -\ln\frac{\sqrt{2}}{2} + \ln 1 = -\ln 2^{\frac{1}{2}} + \ln\ 2 + 0 \\
&= -\frac{1}{2}\ln 2 + \ln 2 = \frac{1}{2}\ln 2 \approx 0.347.
\end{aligned}
$$

10. Recall from Section 8.6 that $\frac{d}{dx}\tan x = \sec^2 x$. So

$$
\int_0^{\frac{\pi}{4}} \sec^2 dx = \tan x\,\Big|_0^{\frac{\pi}{4}} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1.
$$

The answers to the next two exercises depend on formulas from Sections 10.1 and 10.3.

11. $\displaystyle\int_{\ln 2}^{\ln 5} e^x\, dx = e^x\Big|_{\ln 2}^{\ln 5} = e^{\ln 5} - e^{\ln 2} = 5 - 2 = 3.$

12. $\displaystyle\int_3^7 \frac{1}{x}\, dx = \ln x\,\Big|_3^7 = \ln 7 - \ln 3 = \ln\frac{7}{3} \approx 0.847.$

13. This integral can be solved by use of the formula

$$
\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a\ln\left(\frac{a + (a^2 - x^2)^{\frac{1}{2}}}{x}\right) + C
$$

as follows:

$$
\begin{aligned}
\int_1^4 \frac{\sqrt{25 - x^2}}{x} dx &= \left[\sqrt{25 - x^2} - 5\ln\left(\frac{5 + \sqrt{25 - x^2}}{x}\right)\right]\Bigg|_1^4 \\
&= \sqrt{9} - 5\ln\left(\frac{5 + \sqrt{9}}{4}\right) - \sqrt{24} + 5\ln\left(\frac{5 + \sqrt{24}}{1}\right) \\
&\approx 3 - 5\ln 2 - \sqrt{24} + 5\ln(5 + \sqrt{24}) \\
&\approx 3 - 3.466 - 4.899 + 11.462 \ \approx\ 6.097.
\end{aligned}
$$

14. Making use of the discussion preceding this exercise, we get

$$
\begin{aligned}
\int_{20}^{6} (2x - 7) dx &= -\int_6^{20} (2x - 7) dx = -(x^2 - 7x)\Big|_6^{20} \\
&= -[(400 - 140) - (36 - 42)] = -(260 + 6) = -266.
\end{aligned}
$$

15. By the discussion preceding Exercise 14,

$$\begin{aligned}
\int_8^1 \left(x^{\frac{3}{2}} + 4x^{\frac{1}{2}} - \pi\right) dx &= -\int_1^8 \left(x^{\frac{3}{2}} + 4x^{\frac{1}{2}} - \pi\right) dx \\
&= -\left[\left(\frac{2}{5}x^{\frac{5}{2}} + 4\cdot\frac{2}{3}x^{\frac{3}{2}} - \pi x\right)\Big|_1^8\right] \\
&= -\left[\left(\frac{2}{5}\,8^{\frac{5}{2}} + \frac{8}{3}\,8^{\frac{3}{2}} - 8\pi\right) - \left(\frac{2}{5} + \frac{8}{3} - \pi\right)\right] \\
&\approx -[72.408 + 60.340 - 25.133 - (0.4 + 2.667 - 3.142)] \\
&\approx -107.69.
\end{aligned}$$

16. Once again by the discussion preceding Exercise 14,

$$\begin{aligned}
\int_\pi^0 \left(\cos x - 8x^2\right) dx &= -\int_0^\pi \left(\cos x - 8x^2\right) dx \\
&= -\left[\left(\sin x - \frac{8}{3}x^3\right)\Big|_0^\pi\right] = -\left[\left(0 - \frac{8}{3}\pi^3\right) - 0\right] \\
&= \frac{8}{3}\pi^3.
\end{aligned}$$

The fact to use in the next three exercises is:

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x).$$

17. $F'(x) = x(1+x^3)^9$.

18. $F'(x) = \sin x \left(\cos x - 4x^2\right)$.

19. Let $F(x) = \int_0^x \left(\sin t + 2t^{-3}\right) dt$. By the formula already referred to, $F'(x) = \sin x + 2x^{-3}$. Notice that $G(x) = F(3x^2)$. An application of the chain rule shows us that

$$\begin{aligned}
G'(x) &= F'(3x^2)\cdot 6x = \left[\sin 3x^2 + 2\left(3x^2\right)^{-3}\right] 6x \\
&= 6x \sin 3x^2 + 12x\frac{1}{(3x^2)^3} = 6x\sin 3x^2 + \frac{12}{27}\frac{1}{x^5} \\
&= 6x \sin 3x^2 + \frac{4}{9x^5}.
\end{aligned}$$

13D. The Substitution Method

20. With $u = 4x - 5$, we get $\frac{du}{dx} = 4$. So $du = 4dx$ and$dx = \frac{du}{4}$. So

$$\begin{aligned}\int (4x-5)^{\frac{1}{2}} dx &= \int u^{\frac{1}{2}} \cdot \frac{du}{4} = \frac{1}{4}\int u^{\frac{1}{2}}\, du = \frac{1}{4}\left[\frac{2}{3}\, u^{\frac{3}{2}} + C'\right] \\ &= \frac{2}{12}(4x-5)^{\frac{3}{2}} + \frac{C'}{4} = \frac{1}{6}(4x-5)^{\frac{3}{2}} + C.\end{aligned}$$

21. With $u = 1 - 5x^2$, we get $\frac{du}{dx} = -10x$. So $du = -10x\, dx$. Therefore,

$$\begin{aligned}\int 10x(1-5x^2)^{\frac{2}{3}}\, dx &= \int u^{\frac{2}{3}}(-du) = -\int u^{\frac{2}{3}}\, du = -\left[\frac{3}{5}u^{\frac{5}{3}} + C'\right] \\ &= -\frac{3}{5}\left(1-5x^2\right)^{\frac{5}{3}} - C' \\ &= -\frac{3}{5}\left(1-5x^2\right)^{\frac{5}{3}} + C.\end{aligned}$$

22. Because $du = 2x\, dx$, we get

$$\begin{aligned}\int x\cos x^2\, dx &= \int (\cos u)(\frac{1}{2}\, du) = \frac{1}{2}\int \cos u\, du \\ &= \frac{1}{2}\left[\sin u + C'\right] \\ &= \frac{1}{2}\sin x^2 + C.\end{aligned}$$

23. Since $\frac{du}{dt} = \cos t$, we get

$$\begin{aligned}\int \sin^3 t \cos t\, dt &= \int u^3\, du = \frac{u^4}{4} + C \\ &= \frac{1}{4}\sin^4 t + C.\end{aligned}$$

24. Note that $du = dx$ and $x = u - 1$. So

$$\begin{aligned}\int (x-1)(x+1)^{\frac{1}{2}}\, dx &= \int (u-2)\, u^{\frac{1}{2}}\, du = \int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right) du \\ &= \frac{2}{5}u^{\frac{5}{2}} - 2\cdot\frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{4}{3}(x+1)^{\frac{3}{2}} + C.\end{aligned}$$

25. Because $dx = du$ and $x = u - 3$, we get

$$\begin{aligned}
\int x^2(x+3)^{\frac{1}{2}}\,dx &= \int (u-3)^2 u^{\frac{1}{2}}\,du = \int (u^2-6u+9)u^{\frac{1}{2}}\,du \\
&= \int \left(u^{\frac{5}{2}} - 6u^{\frac{3}{2}} + 9u^{\frac{1}{2}}\right)\,du \\
&= \frac{2}{7}u^{\frac{7}{2}} - 6\cdot\frac{2}{5}u^{\frac{5}{2}} + 9\cdot\frac{2}{3}u^{\frac{3}{2}} + C \\
&= \frac{2}{7}(x+3)^{\frac{7}{2}} - \frac{12}{5}(x+3)^{\frac{5}{2}} + 6(x+3)^{\frac{3}{2}} + C.
\end{aligned}$$

26. Because $dx = du$ and $x = u + 2$, we see that

$$\begin{aligned}
\int \frac{x^2}{(x-2)^3}\,dx &= \int \frac{(u+2)^2}{u^3}\,du = \int \frac{u^2+2u+4}{u^3}\,du \\
&= \int \left(u^{-1} + 2u^{-2} + 4u^{-3}\right)\,du \\
&= \ln u - 2u^{-1} - 2u^{-2} + C \\
&= \ln(x-2) - 2(x-2)^{-1} - 2(x-2)^{-2} + C.
\end{aligned}$$

27. Note that $\frac{du}{d\varphi} = \sec^2\varphi$, so $du = \sec^2\varphi d\varphi$. Therefore,

$$\int \frac{\sec^2\varphi}{\tan^2\varphi + 1}\,d\varphi = \int \frac{du}{u^2+1}\,.$$

By a formula from Section 10.5, $\int \frac{du}{u^2+1} = \tan^{-1} u + C$. So

$$\begin{aligned}
\int \frac{\sec^2\varphi}{\tan^2\varphi+1}\,d\varphi &= \tan^{-1}(\tan\varphi) + C \\
&= \varphi + C.
\end{aligned}$$

There is a simpler approach to this problem. Dividing the identity

$$\sin^2\varphi + \cos^2\varphi = 1$$

by $\cos^2\varphi$, gives us the identity $\tan^2\varphi + 1 = \sec^2\varphi$. So

$$\int \frac{\sec^2\varphi}{\tan^2\varphi+1}d\varphi = \int d\varphi = \varphi + C.$$

28. Try $u = 12x^7 + 19$. So $\frac{du}{dx} = 84x^6$. Hence $du = 84x^6\,dx$ and (looking ahead), $x^6dx = \frac{1}{84}du$. So

$$\begin{aligned}
\int \frac{5x^6}{12x^7+19}dx &= \int \frac{5(\frac{1}{84}du)}{u} = \int \frac{5}{84}u^{-1}du = \frac{5}{84}\ln u + C \\
&= \frac{5}{84}\ln(12x^7+19) + C.
\end{aligned}$$

29. Try $u = 1 + 2x + 4x^2$. So $\frac{du}{dx} = 2 + 8x = 2(1+4x)$. Hence $du = 2(1+4x)dx$. Therefore,

$$\begin{aligned}\int (1+4x)(1+2x+4x^2)^{\frac{1}{2}}dx &= \int u^{\frac{1}{2}}\frac{du}{2} = \int \frac{1}{2}u^{\frac{1}{2}}du \\ &= \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{1}{3}(1+2x+4x^2)^{\frac{3}{2}} + C.\end{aligned}$$

30. With $u = \sin t$, we get $du = \cos t\ dt$, and hence

$$\begin{aligned}\int \sin^6 t \cos t\, dt &= \int u^6 du = \frac{u^7}{7} + C \\ &= \frac{1}{7}\sin^7 t + C.\end{aligned}$$

31. Try $u = e^z + 1$. So $du = e^z dz$ and $e^z = u - 1$. We now get

$$\begin{aligned}\int (e^z+1)^{\frac{1}{2}}e^{2z}dz &= \int (e^z+1)^{\frac{1}{2}}e^z\cdot e^z dz \\ &= \int u^{\frac{1}{2}}(u-1)du \ = \int (u^{\frac{3}{2}} - u^{\frac{1}{2}})du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{5}(e^z+1)^{\frac{5}{2}} - \frac{2}{3}(e^z+1)^{\frac{3}{2}} + C.\end{aligned}$$

32. Let $u = 1 + 4x^{\frac{1}{3}}$. So $\frac{du}{dx} = \frac{4}{3}x^{-\frac{2}{3}}$ and hence $x^{-\frac{2}{3}}dx = \frac{3}{4}du$. Also, when $x = 1$ and $x = 8$, $u = 5$ and $u = 9$, respectively. Therefore,

$$\begin{aligned}\int_1^8 x^{-\frac{2}{3}}\sqrt{1+4x^{\frac{1}{3}}}dx &= \int_5^9 u^{\frac{1}{2}}\cdot\frac{3}{4}du = \int_5^9 \frac{3}{4}u^{\frac{1}{2}}du \\ &= \left[\frac{3}{4}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_5^9\right] = \frac{1}{2}9^{\frac{3}{2}} - \frac{1}{2}5^{\frac{3}{2}} \\ &= \frac{1}{2}\cdot 3^3 - \frac{1}{2}\left(\sqrt{5}\right)^3 = \frac{1}{2}\left(27 - 5\sqrt{5}\right).\end{aligned}$$

33. Let $u = 1 + 2x$. So $\frac{du}{dx} = 2$ and $dx = \frac{1}{2}du$. Also, when $x = 0$ and 3 respectively, $u = 1$ and 7. Therefore,

$$\begin{aligned}\int_0^3 \frac{dx}{\sqrt[3]{(1+2x)^2}} &= \int_1^7 \frac{1}{2}\frac{du}{(u^2)^{\frac{1}{3}}} \ = \int_1^7 \frac{1}{2}u^{-\frac{2}{3}}\,du \\ &= \left[\frac{1}{2}(3)u^{\frac{1}{3}}\Big|_1^7\right] \ = \frac{3}{2}u^{\frac{1}{3}}\Big|_1^7 \ = \frac{3}{2}\sqrt[3]{7} - \frac{3}{2}.\end{aligned}$$

34. Let $u = t^{10}$. So $\frac{du}{dt} = 10t^9$ and (looking ahead) $t^9 dt = \frac{du}{10}$. For $t = 0$ and 1, $u = 0$ and 1 respectively. Therefore,

$$\begin{aligned}\int_0^1 t^9 \tan(t^{10}) dt &= \int_0^1 \tan u \cdot \frac{du}{10} \\ &= \frac{1}{10}\int_0^1 \tan u \, du.\end{aligned}$$

It remains to find an antiderivative of $\tan u$. But this was already done in the solution of Exercise 9 above, $\frac{d}{du}[-\ln(\cos u)] = \tan u$. So

$$\begin{aligned}\int_0^1 t^9 \tan(t^{10}) dt &= \frac{1}{10}\left[-\ln(\cos u) \Big|_0^1 \right] = \frac{1}{10}[-\ln(\cos 1) + \ln 1] \\ &\approx \frac{1}{10}(-\ln(0.540) + 0) \approx \frac{0.616}{10} \approx 0.062.\end{aligned}$$

35. Let $u = \ln x$. So $\frac{du}{dx} = \frac{1}{x}$ and $du = \frac{dx}{x}$. For $x = 1$ and 3, respectively $u = 0$ and $\ln 3$. Therefore,

$$\begin{aligned}\int_3^1 \frac{(\ln x)^2}{x} dx &= -\int_1^3 \frac{(\ln x)^2}{x} dx \\ &= -\int_0^{\ln 3} u^2 \, du = -\left[\frac{1}{3} u^3 \Big|_0^{\ln 3} \right] \\ &= -(\frac{1}{3}(\ln 3)^3 - 0) = -\ln 3.\end{aligned}$$

13E. Computations of Areas

36. Because $f(x) \geq 0$ over the interval in question, this area is equal to

$$\int_{-1}^1 x^4 \, dx = \frac{1}{5} x^5 \Big|_{-1}^1 = \frac{1}{5} - \frac{1}{5}(-1) = \frac{2}{5}.$$

37. Because $f(x) \geq 0$ over $[-2, -1]$, the area is

$$\begin{aligned}\int_{-2}^{-1} \frac{1}{x^2} dx &= \int_{-2}^{-1} x^{-2} \, dx = -x^{-1}\Big|_{-2}^{-1} \\ &= -(-1)^{-1} - \left(-(-2)^{-1}\right) = 1 - \frac{1}{2} = \frac{1}{2}.\end{aligned}$$

38. The area is equal to

$$\begin{aligned}\int_1^4 x^{\frac{1}{2}} dx &= \frac{2}{3} x^{\frac{3}{2}}\Big|_1^4 = \frac{2}{3} 4^{\frac{3}{2}} - \frac{2}{3} \\ &= \frac{2}{3}(2^3 - 1) = \frac{14}{3}.\end{aligned}$$

39. Refer back to Figure 5.25 and note that the graph of $f(x) = x^{\frac{1}{3}}$ lies below the x-axis for $-3 \le x \le 0$ and above the x-axis for $0 \le x \le 8$. Hence

$$\int_{-3}^{0} x^{\frac{1}{3}}\, dx = -(\text{area below the } x\text{-axis and above the graph from } x = -3 \text{ to } x = 0)$$

and the area that is to be computed is equal to

$$\begin{aligned} -\int_{-3}^{0} x^{\frac{1}{3}}\, dx + \int_{0}^{8} x^{\frac{1}{3}}\, dx &= -\left[\frac{3}{4}\, x^{\frac{4}{3}}\Big|_{-3}^{0}\right] + \left[\frac{3}{4}\, x^{\frac{4}{3}}\Big|_{0}^{8}\right] \\ &= -\left(0 - \frac{3}{4}(-3)^{\frac{4}{3}}\right) + \left(\frac{3}{4}\, 8^{\frac{4}{3}} - 0\right) \\ &= \frac{3}{4}(-\sqrt[3]{3})^4 + \frac{3}{4}2^4 = \frac{3}{4}(3\sqrt[3]{3}) + 12 \\ &= \frac{9}{4}\sqrt[3]{3} + 12. \end{aligned}$$

40. Because $\cos x \ge 0$ for $0 \le x \le \frac{\pi}{4}$, the same is true for $f(x) = \sec x$. So by the first part of Section 13.3, the area is

$$\begin{aligned} \int_{0}^{\frac{\pi}{4}} \sec x\, dx &= \ln(\sec x + \tan x)\Big|_{0}^{\frac{\pi}{4}} \\ &= \ln\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \ln\,(\sec 0 + \tan 0) \\ &= \ln\left(\frac{2}{\sqrt{2}} + 1\right) - \ln\,(1+0) \;=\; \ln\left(1 + \sqrt{2}\right) - 0 \\ &\approx 0.881. \end{aligned}$$

41. The area is $\displaystyle\int_{0}^{3} \sqrt{x+1}\, dx$. This integral is easily solved by the method of substitution: $u = x + 1$; $du = dx$; $u = 1$ and 4, respectively, when $x = 0$ and 3. So

$$\int_{0}^{3} \sqrt{x+1}\, dx \;=\; \int_{1}^{4} u^{\frac{1}{2}} du \;=\; \frac{2}{3}u^{\frac{3}{2}}\Big|_{1}^{4} \;=\; \frac{2}{3}(8-1) = \frac{14}{3}\,.$$

Can you think of a reason why this area is the same as that of Exercise 38? [Hint: Refer to Section 10.8.]

42. Because the graph of $f(x) = \frac{x}{(x^2+1)^2}$ lies below the x-axis for $-1 \le x \le 0$ and above the x-axis for $0 \le x \le 2$, the area in question is

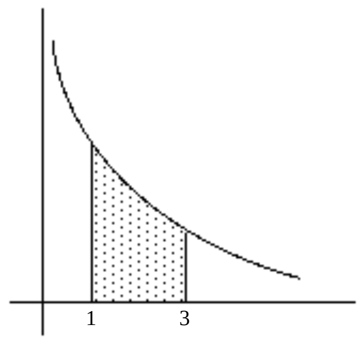
$$-\int_{-1}^{0} \frac{x}{(x^2+1)^2}\, dx + \int_{0}^{2} \frac{x}{(x^2+1)^2}\, dx.$$

These integrals are solved by the substitution $u = x^2 + 1$ as follows: $\frac{du}{dx} = 2x$, so $xdx = \frac{1}{2}du$.

Therefore, the sum of the two integrals is equal to

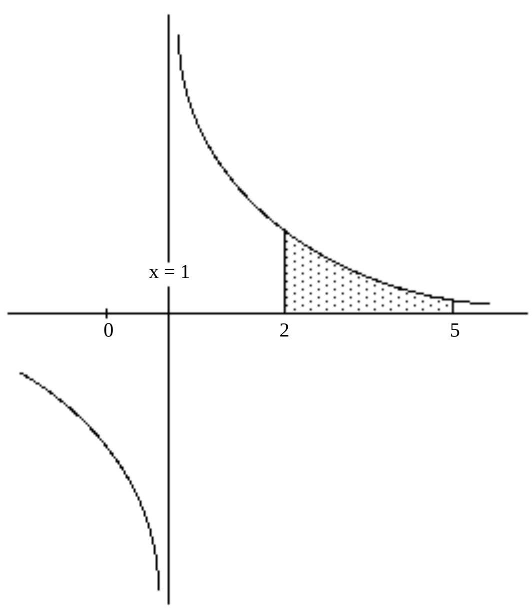
$$
\begin{aligned}
-\int_2^1 \frac{1}{2}\frac{du}{u^2} + \int_1^5 \frac{1}{2}\frac{du}{u^2} &= \frac{1}{2}\int_1^2 u^{-2}du + \frac{1}{2}\int_1^5 u^{-2}du \\
&= \frac{1}{2}\left(-u^{-1}\Big|_1^2\right) + \frac{1}{2}\left(-u^{-1}\Big|_1^5\right) \\
&= \frac{1}{2}\left(-\frac{1}{2}+1\right) + \frac{1}{2}\left(-\frac{1}{5}+1\right) \\
&= \frac{1}{4} + \frac{2}{5} = \frac{13}{20}.
\end{aligned}
$$

43. $\displaystyle\int_1^3 \frac{1}{x}dx = \ln x\Big|_1^3 = \ln 3 - \ln 1 = \ln 3 \approx 1.099.$



44. The area is given by $\displaystyle\int_2^5 \frac{1}{x-1}dx$. This integral is solved by the substitution $u = x - 1$ as follows:

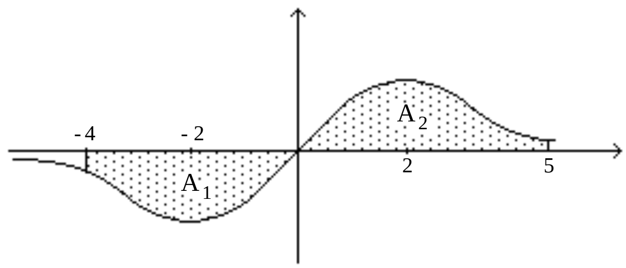
$$\int_2^5 \frac{1}{x-1}dx = \int_1^4 \frac{1}{u}\,du = \ln u\Big|_1^4 = \ln 4 = 2\ln 2 \approx 2(0.693) \approx 1.386.$$



45. Let $f(x) = \frac{x}{x^2+4}$. Because

$$f'(x) = \frac{(x^2+4) - x(2x)}{(x^2+4)^2} = -\frac{x^2-4}{(x^2+4)^2},$$

observe that $f'(x) \geq 0$ for $-2 \leq x \leq 2$ and $f'(x) \leq 0$ for all other x. So the graph of $f(x)$ is decreasing over $x = -4 \leq x \leq -2$, increasing over $-2 \leq x \leq 2$, and then decreasing over $2 \leq x \leq 5$. A rough sketch of the situation is



So the area to be computed is equal to

$$-\int_{-4}^{0} \frac{x}{x^2+4}\,dx + \int_{0}^{5} \frac{x}{x^2+4}\,dx.$$

These integrals can be solved by the substitution $u = x^2 + 4$. Because $\frac{du}{dx} = 2x$ and hence $xdx = \frac{1}{2}du$, we get that the area is

$$\begin{aligned}
-\int_{20}^{4} \frac{1}{2}\frac{du}{u} + \int_{4}^{29} \frac{1}{2}\frac{du}{u} &= -\frac{1}{2}\left[\ln u\Big|_{20}^{4}\right] + \frac{1}{2}\left[\ln u\Big|_{4}^{29}\right] \\
&= -\frac{1}{2}(\ln 4 - \ln 20) + \frac{1}{2}(\ln 29 - \ln 4) = \frac{1}{2}\left[\ln 29 + \ln 20 - 2\ln 4\right] \\
&= \frac{1}{2}\ln\frac{(29)(20)}{16} = \frac{1}{2}\ln\frac{145}{4} \approx 1.795.
\end{aligned}$$

13F. Integration by Parts

46. Starting with $u = x$ and $dv = \sin x\,dx$, we get $du = dx$ and $v = -\cos x$. So

$$\begin{aligned}
\int x\sin x\,dx &= \int u\,dv = uv - \int v\,du \\
&= -x\cos x - \int -\cos x\,dx \\
&= -x\cos x + \int \cos x\,dx \\
&= -x\cos x + \sin x + C.
\end{aligned}$$

47.

$$\begin{aligned}\int_0^{\frac{\pi}{2}} (x + x\cos x)\,dx &= \int_0^{\frac{\pi}{2}} x\,dx + \int_0^{\frac{\pi}{2}} x\cos x\,dx \\ &= \frac{1}{2}x^2\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} x\cos x\,dx \\ &= \frac{1}{8}\pi^2 + \int_0^{\frac{\pi}{2}} x\cos x\,dx.\end{aligned}$$

To compute $\int x\cos x\,dx$, we let $u = x$ and $dv = \cos x\,dx$ and proceed by integration by parts: $du = dx$, $v = \sin x$, and

$$\begin{aligned}\int x\cos x\,dx &= \int u\,dv = uv - \int v\,du \\ &= x\sin x - \int \sin x\,dx \\ &= x\sin x + \cos x + C.\end{aligned}$$

So

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x\cos x\,dx &= (x\sin x + \cos x)\Big|_0^{\frac{\pi}{2}} \\ &= \left(\frac{\pi}{2}\cdot 1 + 0\right) - (0+1) \;=\; \frac{\pi}{2} - 1.\end{aligned}$$

Therefore,

$$\int_0^{\frac{\pi}{2}} (x + x\cos x)dx = \frac{1}{8}\pi^2 + \frac{\pi}{2} - 1.$$

48. Proceeding as suggested, we have $u = \ln x, dv = x\,dx; du = \frac{1}{x}\,dx$ and $v = \frac{x^2}{2}$. So

$$\begin{aligned}\int x\ln x\,dx &= \int u\,dv \;=\; uv - \int v\,du \\ &= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx \;=\; \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx \\ &= \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + C.\end{aligned}$$

49. The suggestion is to let $u = \ln x$ and $dv = x^2dx$. So $du = \frac{1}{x}\,dx, v = \frac{x^3}{3}$, and we get

$$\begin{aligned}\int x^2\ln x\,dx &= \int u\,dv \;=\; uv - \int v\,du \\ &= \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx \\ &= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx \\ &= \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C.\end{aligned}$$

50. With $u = x$ and $dv = e^{5x}dx$, we get $du = dx$ and $v = \frac{1}{5}e^{5x}$. So

$$\begin{aligned}\int xe^{5x}dx &= \int u\,dv = uv - \int v\,du \\ &= x \cdot \frac{1}{5}e^{5x} - \int \frac{1}{5}e^{5x}dx \\ &= \frac{x}{5}e^{5x} - \frac{1}{5} \cdot \frac{1}{5}e^{5x} + C \\ &= \frac{x}{5}e^{5x} - \frac{1}{25}e^{5x} + C.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^1 xe^{5x}dx &= \left(\frac{x}{5}e^{5x} - \frac{1}{25}e^{5x}\right)\Big|_0^1 \\ &= \frac{1}{5}e^5 - \frac{1}{25}e^5 - \left(0 - \frac{1}{25}\right) \\ &= \frac{4}{25}e^5 + \frac{1}{25}.\end{aligned}$$

51. The integral should be $\int x^2e^{5x}dx$. Does $u = x^2$ and $dv = e^x dx$ accomplish anything? Let's see. Since $du = 2xdx$ and $v = \frac{1}{5}e^{5x}$, we get

$$\begin{aligned}\int x^2e^{5x}dx &= \int u\,dv = uv - \int v\,du \\ &= x^2 \cdot \frac{1}{5}e^{5x} - \int 2x \cdot \frac{1}{5}e^{5x}dx \\ &= \frac{x^2}{5}e^{5x} - \frac{1}{2}\int xe^{5x}dx.\end{aligned}$$

Notice that the integral has been reduced to the one already solved in Exercise 50. So

$$\begin{aligned}\int x^2e^{5x}\,dx &= \frac{x^2}{5}e^{5x} - \frac{1}{2}\left[\frac{x}{5}e^{5x} - \frac{1}{25}e^{5x} + C'\right] \\ &= \frac{x^2}{5}e^{5x} - \frac{x}{10}e^{5x} + \frac{1}{50}e^{5x} + C.\end{aligned}$$

52. Given the context, we will try to solve $\int e^x \sin x\,dx$ by integration by parts. Let $u = \sin x$ and $dv = e^x dx$. So $du = \cos x\,dx$ and $v = e^x$. Hence

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$$

as required.

53. A quick answer to this question can be obtained by solving the equation derived in Exercise 52 for $\int e^x \cos x\,dx$. But this would not help us to solve Exercise 54. It is therefore more useful to try to solve $\int e^x \cos x\,dx$ by parts. Let $u = \cos x$ and $dv = e^x dx$. So $du = -\sin x\,dx$ and $v = e^x$, and

$$\int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx.$$

Derive this formula by solving the integral of Exercise 52 in a different way (but still by parts).

54. By substituting the equation derived in Exercise 52 into that derived in Exercise 53, we get

$$\int e^x \cos x\,dx = e^x \cos x + e^x \sin x - \int e^x \cos x\,dx.$$

So $2\int e^x \cos x\,dx = e^x \cos x + e^x \sin x + C'$ and therefore,

$$\int e^x \cos x\,dx = \frac{1}{2}e^x \cos x + \frac{1}{2}e^x \sin x + C.$$

Check that similarly,

$$\int e^x \sin x\,dx = \frac{1}{2}e^x \sin x - \frac{1}{2}e^x \cos x + C.$$

55. We begin by trying a substitution on $\displaystyle\int \ln(x+1)dx$. To avoid confusion (with the v, du notation of the integration by parts method that will follow shortly) we let $z = x + 1$. So $dz = dx$ and

$$\int \ln(x+1)dx = \int \ln z\,dz.$$

Now let $u = \ln z$ and $dv = dz$. So $du = \frac{1}{z}dz$ and $v = z$, and

$$\begin{aligned}\int \ln z\,dz &= \int u\,dv \;=\; uv - \int v\,du \\ &= z\ln z - \int z \cdot \frac{1}{z}\,dz \;=\; z\ln z - \int dz \\ &= z\ln z - z + C.\end{aligned}$$

So $\displaystyle\int \ln(x+1)dx = (x+1)\ln(x+1) - (x+1) + C$. Therefore,

$$\begin{aligned}\int_0^1 \ln(x+1)dx &= [(x+1)\ln(x+1) - (x+1)]\,\Big|_0^1 \\ &= (2\ln 2 - 2) - (\ln 1 - 1) \\ &= 2\ln 2 - 1.\end{aligned}$$

56. Let $z = t^{\frac{1}{2}}$. So $dz = \frac{1}{2}t^{-\frac{1}{2}}\,dt$ and $\int \cos t^{\frac{1}{2}}\,dt = \int(\cos z)\cdot 2z\,dz = 2\int z\cos z\,dz$. Now let $u = z$ and $dv = \cos z\,dz$. So $du = dz$ and $v = \sin z$. Therefore,

$$\begin{aligned}\int z\cos z\,dz &= \int u\,dv = uv - \int v\,du \\ &= z\sin z - \int \sin z\,dz \;=\; z\sin z + \cos z + C'.\end{aligned}$$

It follows that

$$\begin{aligned} \int \cos t^{\frac{1}{2}}\, dt &= 2\left[z \sin z + \cos z + C'\right] \\ &= 2t^{\frac{1}{2}} \sin t^{\frac{1}{2}} + 2\cos t^{\frac{1}{2}} + C. \end{aligned}$$

13G. Trigonometric Substitution

57. Let $x = \frac{1}{2}\sin\theta$ with $0 \le \theta \le \frac{\pi}{2}$. Note that $x = 0$ when $\theta = 0$ and $x = \frac{1}{2}$ when $\theta = \frac{\pi}{2}$. Also, $\frac{dx}{d\theta} = \frac{1}{2}\cos\theta$, so $dx = \frac{1}{2}\cos\theta\, d\theta$. With these substitutions, the integral is transformed as follows:

$$\int_0^{\frac{1}{2}} \sqrt{1-4x^2}\, dx = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta} \cdot \frac{1}{2}\cos\theta\, d\theta.$$

Because $1 - \sin^2\theta = \cos^2\theta$ and $\cos\theta \ge 0$ for $0 \le \theta \le \frac{\pi}{2}$,

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta} \cdot \frac{1}{2}\cos\theta\, d\theta &= \int_0^{\frac{\pi}{2}} \frac{1}{2}\cos^2\theta\, d\theta \;=\; \int_0^{\frac{\pi}{2}} \frac{1}{4}(1+\cos 2\theta) d\theta \\ &= \frac{1}{4}(\theta + \frac{1}{2}\sin 2\theta)\Big|_0^{\frac{\pi}{2}} \;=\; \frac{1}{4}\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi - 0\right) \\ &= \frac{\pi}{8}. \end{aligned}$$

The formula $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ is an easy consequence of the addition formula for the cosine (see Exercises 2C) and the equality $\sin^2\theta + \cos^2\theta = 1$. Compute

$$\int_0^{\frac{1}{2}} \sqrt{1-4x^2}\; dx$$

in a totally different way by making use of the circle $x^2 + y^2 = \frac{1}{4}$ of radius $\frac{1}{2}$.

58. Let $x = \tan\theta$ with $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$. Note that $x = -1$ when $\theta = -\frac{\pi}{4}$ and $x = 1$ when $\theta = \frac{\pi}{4}$. Also, $dx = \sec^2\theta\, d\theta$. Therefore

$$\int_{-1}^{1} \frac{1}{(x^2+1)^{\frac{1}{2}}}\, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2\theta\, d\theta}{(\tan^2\theta + 1)^{\frac{1}{2}}}.$$

Recall that $\tan^2\theta + 1 = \sec^2\theta$. Because $\cos\theta \ge 0$ for $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}, \sec\theta \ge 0$ for such θ, and hence

$$\begin{aligned} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2\theta\, d\theta}{(\tan^2\theta+1)^{\frac{1}{2}}} &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec\theta\, d\theta \;=\; \ln(\sec\theta + \tan\theta)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\ &= \ln\left(\frac{2}{\sqrt{2}} + 1\right) - \ln\left(\frac{2}{\sqrt{2}} - 1\right) \end{aligned}$$

$$= \ln\left(\sqrt{2}+1\right) - \ln\ \left(\sqrt{2}-1\right)$$
$$= \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \approx 1.763.$$

59. Taking $x = 3\sec\theta$, we get $x^2 - 9 = 9\sec^2\theta - 9 = 9(sec^2\theta - 1) = 9\tan^2\theta$. Also, $dx = 3\sec\theta\tan\theta d\theta$. So

$$\int \frac{x^2}{\sqrt{x^2-9}}\,dx = \int \frac{9\sec^2\theta}{3\tan\theta} 3\sec\theta\tan\theta\,d\theta = 27\int \sec^3\theta\,d\theta.$$

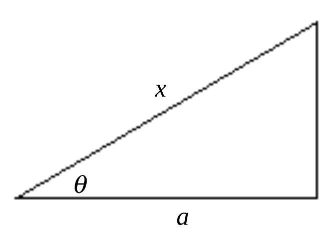
60. Exercise 57 suggests that $\sin\theta$ might play a role. After experimenting a little, let $x = a\sin\theta$. So $a^2 - x^2 = a^2(1-\sin^2\theta) = a^2\cos^2\theta$ and $dx = a\cos\theta\,d\theta$. Assuming that $a \geq 0$ and $\cos\theta \geq 0$, we get

$$\begin{aligned}\int \frac{\sqrt{a^2-x^2}}{x}\,dx &= \int \frac{a\cos\theta}{a\sin\theta}\cdot a\cos\theta\,d\theta = \int \frac{a\cos^2\theta}{\sin\theta}\,d\theta \\ &= a\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta \\ &= a\int \frac{1}{\sin\theta}\,d\theta - a\int \sin\theta\,d\theta. \\ &= a\int \frac{1}{\sin\theta}\,d\theta + a\cos\theta.\end{aligned}$$

The integral $\int \frac{1}{\sin\theta}\,d\theta$ is solved in a way analogous to the solution of

$$\int \frac{1}{\cos\theta}\,d\theta = \int \sec\theta\,d\theta$$

in Section 13.3. The function $\frac{1}{\sin x}$ is the cosecant $\csc x$. Try solving $\int \frac{\sqrt{a^2-x^2}}{x}\,dx$ completely in terms of x by starting with the substitution $x = a\cos\theta$. Make use of Section 10.5 and the right triangle



that depicts the equation $x = a\cos\theta$. You will be able to confirm that you did it correctly by checking your answer against the conclusions of Section 10.4.

13H. Surface Area

61. The surface area is equal to

$$A = \int_0^5 2\pi x^3 \sqrt{1+(3x^2)^2}\,dx = 2\pi \int_0^5 x^3\left(1+9x^4\right)^{\frac{1}{2}}\,dx.$$

This integral can be solved by the substitution $u = 1+9x^4$. Because $du = 36x^3dx$, we get

$$\begin{aligned}\int_0^5 x^3\left(1+9x^4\right)^{\frac{1}{2}}\,dx &= \int_1^{5626} \frac{1}{36}u^{\frac{1}{2}}du = \frac{1}{36}\left[\frac{2}{3}u^{\frac{3}{2}}\Big|_1^{5626}\right] \\ &= \frac{1}{54}\left(5626^{1.5}-1\right) \approx 7814.565.\end{aligned}$$

62. The surface area is equal to

$$A = 2\pi\int_0^1 e^{-x}\sqrt{1+(-e^{-x})^2}\,dx = 2\pi\int_0^1 e^{-x}\sqrt{1+e^{-2x}}\,dx.$$

The substitution $u = e^{-x}$ shows promise (after some trial and error). Since $du = -e^{-x}dx$, we get

$$A = 2\pi\int_1^{e^{-1}} -\sqrt{1+u^2}\,du = 2\pi\int_{e^{-1}}^1 \sqrt{1+u^2}\,du\,.$$

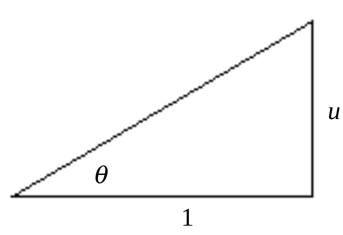
At this point (see the discussion that concludes Section 13.3B), the substitution $u = \tan\theta$ suggests itself. Since $du = \sec^2\theta d\theta$,

$$\int \sqrt{1+u^2}\,du = \int \sqrt{1+\tan^2\theta}\,\sec^2\theta\,d\theta = \int \sec^3\theta\,d\theta.$$

This integral has already been solved in Section 13.3C. Using this solution, we get

$$\int \sqrt{1+u^2}\,du = \frac{1}{2}(\sec\theta)(\tan\theta) + \frac{1}{2}\ln(\sec\theta+\tan\theta) + C.$$

To get from θ back to u consider the "picture" of the equation $\tan\theta = u = \frac{u}{1}$. This is



From this right triangle we see that $\cos\theta = \frac{1}{\sqrt{1+u^2}}$ and hence that $\sec\theta = \sqrt{1+u^2}$. Therefore,

$$\begin{aligned}\int \sqrt{1+u^2}\,du &= \frac{1}{2}(\sec\theta)(\tan\theta) + \frac{1}{2}\ln(\sec\theta+\tan\theta) + C \\ &= \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right) + C.\end{aligned}$$

It follows that

$$\begin{aligned} A &= 2\pi\left[\left(\frac{1}{2}u\sqrt{1+u^2}+\frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right)\Big|_{e^{-1}}^{1}\right] \\ &= \pi\left[1\cdot\sqrt{2}+\ln\left(1+\sqrt{2}\right)-e^{-1}\sqrt{1+e^{-2}}-\ln\left(e^{-1}+\sqrt{1+e^{-2}}\right)\right] \end{aligned}$$

Check that $e^{-1}\sqrt{1+e^{-2}}=\frac{1}{e^2}\sqrt{e^2+1}$ and $e^{-1}+\sqrt{1+e^{-2}}=\frac{1}{e}\left(1+\sqrt{e^2+1}\right)$. Therefore,

$$\begin{aligned} A &= \pi\left[\sqrt{2}+\ln(1+\sqrt{2})-\frac{\sqrt{e^2+1}}{e^2}-\ln\left(\frac{1+\sqrt{e^2+1}}{e}\right)\right] \\ A &= \pi\left[\sqrt{2}+\ln(1+\sqrt{2})-\frac{\sqrt{e^2+1}}{e^2}-\ln\frac{1}{e}-\ln(1+\sqrt{e^2+1})\right] \\ &\approx \pi[1.414+0.881-0.392+1-1.360] \approx 4.848. \end{aligned}$$

63. Because $f'(x)=\frac{1}{2}\left(4-x^2\right)^{-\frac{1}{2}}(-2x)=\frac{-x}{(4-x^2)^{\frac{1}{2}}}$,

$$\begin{aligned} A &= \int_{-\frac{1}{2}}^{\frac{3}{2}} 2\pi\sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}\,dx = \int_{-\frac{1}{2}}^{\frac{3}{2}} 2\pi\sqrt{4-x^2}\sqrt{\frac{4-x^2+x^2}{4-x^2}}\,dx \\ &= \int_{-\frac{1}{2}}^{\frac{3}{2}} 2\pi\sqrt{4}\,dx = \int_{-\frac{1}{2}}^{\frac{3}{2}} 4\pi dx = 4\pi x\Big|_{-\frac{1}{2}}^{\frac{3}{2}} \\ &= 4\pi\left(\frac{3}{2}-\left(-\frac{1}{2}\right)\right) = 8\pi. \end{aligned}$$

13I. Up to the Gills

64. The equation of the circle (a quarter of which is shown in Figure 13.45) is

$$x^2+(y-5.5)^2=2^2.$$

Solving for y, we get $(y-5.5)^2=4-x^2$, hence $y-5.5=\pm\sqrt{4-x^2}$, and therefore, $y=5.5\pm\sqrt{4-x^2}$. Because the arc of Figure 13.45 lies below the line $y=5.5$, the $-$ applies. So the arc is the graph of the function

$$f(x)=5.5-\sqrt{4-x^2},\ 0\le x\le 2.$$

65. Notice that

$$f'(x) = -\frac{1}{2}\left(4-x^2\right)^{-\frac{1}{2}}(-2x) = \frac{x}{(4-x^2)^{\frac{1}{2}}}.$$

Therefore, the surface area is

$$
\begin{aligned}
A &= \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx \\
&= \int_0^2 2\pi\left(5.5-(4-x^2)^{\frac{1}{2}}\right)\sqrt{1+\frac{x^2}{4-x^2}}\,dx \\
&= \int_0^2 2\pi\left(5.5-(4-x^2)^{\frac{1}{2}}\right)\sqrt{\frac{4-x^2+x^2}{4-x^2}}\,dx \\
&= \int_0^2 2\pi\left(5.5-(4-x^2)^{\frac{1}{2}}\right)\frac{2}{(4-x^2)^{\frac{1}{2}}}\,dx \\
&= \int_0^2 4\pi\left[\frac{5.5}{(4-x^2)^{\frac{1}{2}}}-1\right]dx.
\end{aligned}
$$

66. The integral of Exercise 65 is equal to

$$
\begin{aligned}
A &= \int_0^2 \left[\frac{22\pi}{\sqrt{4-x^2}}-4\pi\right]dx = 22\pi\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx - \int_0^2 4\pi dx. \\
&= 22\pi\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx - (4\pi x)\Big|_0^2 \\
&= 22\pi\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx - 8\pi.
\end{aligned}
$$

The remaining integral is solved by the substitution

$$x = 2\sin\theta$$

with $0 \le \theta \le \frac{\pi}{2}$. Note that $x = 0$ when $\theta = 0$ and $x = 2$ when $\theta = \frac{\pi}{2}$. Because $dx = 2\cos\theta d\theta$,

$$
\begin{aligned}
\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \frac{\cos\theta d\theta}{\sqrt{4\cos^2\theta}} \\
&= \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}.
\end{aligned}
$$

Therefore,

$$A = (22\pi)\left(\frac{\pi}{2}\right) - 8\pi = 11\pi^2 - 8\pi = 83.433\ \mu m^2.$$

13J. Points in the Polar Plane

67. We will make use of the transformation equations

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

and elementary facts about the sine and cosine. (See Sections 1.4 and 4.4.)

i. $x = 3\cos\frac{\pi}{4} = 3\cdot\frac{\sqrt{2}}{2} = \frac{3}{2}\sqrt{2}$ and $y = 3\sin\frac{\pi}{4} = 3\cdot\frac{\sqrt{2}}{2} = \frac{3}{2}\sqrt{2}$. So $(x,y) = \left(\frac{3}{2}\sqrt{2}, \frac{3}{2}\sqrt{2}\right)$.

ii. $x = -2\cos\left(-\frac{\pi}{6}\right) = -2\cos\frac{\pi}{6} = -2\frac{\sqrt{3}}{2} = -\sqrt{3}$ $y = -2\sin\left(-\frac{\pi}{6}\right) = (-2)\left(-\sin\frac{\pi}{6}\right) = (-2)\left(-\frac{1}{2}\right) = 1$. So $(x,y) = \left(-\sqrt{3}, 1\right)$.

iii. $x = 3\cos\frac{7\pi}{3} = 3\cos\left(2\pi + \frac{\pi}{3}\right) = 3\cos\frac{\pi}{3} = 3\cdot\frac{1}{2} = \frac{3}{2}$ and $y = 3\sin\frac{7\pi}{3} = 3\sin\left(2\pi + \frac{\pi}{3}\right) = 3\sin\frac{\pi}{3} = 3\cdot\frac{\sqrt{3}}{2} = \frac{3}{2}\sqrt{3}$. So $(x,y) = \left(\frac{3}{2}, \frac{3}{2}\sqrt{3}\right)$.

iv. $x = 5\cos 0 = 5$ and $y = 5\sin 0 = 0$. So $(x,y) = (5,0)$.

v. $x = -2\cos\frac{\pi}{2} = (-2)0 = 0$ and $y = -2\sin\frac{\pi}{2} = (-2)1 = -2$. So $(x,y) = (0,-2)$.

vi. $x = -2\cos\frac{3\pi}{2} = -2\cos\left(\frac{\pi}{2} + \pi\right) = (-2)\left(-\cos\frac{\pi}{2}\right) = 0$ and $y = -2\sin\frac{3\pi}{2} = -2\sin\left(\frac{\pi}{2} + \pi\right) = 2\sin\frac{\pi}{2} = 2$. So $(x,y) = (0,2)$.

vii. Since this problem is virtually identical to (vi), we do $\left(4, -\frac{5\pi}{4}\right)$ instead.
$x = 4\cos\left(\frac{-5\pi}{4}\right) = 4\cos\frac{5\pi}{4} = 4\cos\left(\pi + \frac{\pi}{4}\right) = -4\cos\frac{\pi}{4} = -2\sqrt{2}$ and $y = 4\sin\left(-\frac{5\pi}{4}\right) = -4\sin\frac{5\pi}{4} = -4\sin\left(\pi + \frac{\pi}{4}\right) = 4\sin\frac{\pi}{4} = 2\sqrt{2}$. So $(x,y) = \left(-2\sqrt{2}, 2\sqrt{2}\right)$

viii. $x = 0\cos\frac{6\pi}{7} = 0$ and $y = 0\sin\frac{6\pi}{7} = 0$. So $(x,y) = (0,0)$.

ix. $x = -\cos\left(\frac{-23\pi}{3}\right) = -\cos\left(\frac{23\pi}{3}\right) = -\cos\left(8\pi - \frac{\pi}{3}\right) = -\cos\left(-\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$ and $y = -\sin\left(\frac{-23\pi}{3}\right) = \sin\frac{23\pi}{3} = \sin\left(8\pi - \frac{\pi}{3}\right) = \sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$. So $(x,y) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

x. In view of (ix) we will change this to $\left(5, -\frac{15}{4}\pi\right)$.
$x = 5\cos\left(-\frac{15}{4}\pi\right) = 5\cos\left(4\pi - \frac{15}{4}\pi\right) = 5\cos\frac{\pi}{4} = 5\frac{\sqrt{2}}{2} = \frac{5}{2}\sqrt{2}$ and $y = 5\sin\left(\frac{-15}{4}\pi\right) = 5\sin\left(4\pi - \frac{15}{4}\pi\right) = 5\sin\frac{\pi}{4} = \frac{5}{2}\sqrt{2}$. So $(x,y) = \left(\frac{5}{2}\sqrt{2}, \frac{5}{2}\sqrt{2}\right)$.

xi. In view of (vi), do $\left(1, \frac{3\pi}{2}\right)$ to $\left(-3, \frac{14}{6}\pi\right)$ instead.
$x = -3\cos\left(\frac{14}{6}\pi\right) = -3\cos\left(2\pi + \frac{2}{6}\pi\right) = -3\cos\frac{\pi}{3} = -\frac{3}{2}$ and $y = -3\sin\left(\frac{14}{6}\pi\right) = -3\sin\left(2\pi + \frac{2}{6}\pi\right) = -3\sin\frac{\pi}{3} = -3\cdot\frac{\sqrt{3}}{2}$. So $(x,y) = \left(-\frac{3}{2}, -\frac{3}{2}\sqrt{3}\right)$.

xii. $x = 3\cos\left(-\frac{5\pi}{6}\right) = 3\cos\left(\pi - \frac{\pi}{6}\right) = -3\cos\left(-\frac{\pi}{6}\right) = -3\cos\frac{\pi}{6} = -3\cdot\frac{\sqrt{3}}{2} = -\frac{3}{2}\sqrt{3}$ and $y = 3\sin\left(-\frac{5\pi}{6}\right) = 3\sin\left(\pi - \frac{\pi}{6}\right) = -3\sin\left(-\frac{\pi}{6}\right) = 3\sin\frac{\pi}{6} = \frac{3}{2}$. So $(x,y) = \left(-\frac{3}{2}\sqrt{3}, \frac{3}{2}\right)$.

68. Suppose a point P (other than the origin 0) has polar coordinates (r,θ). Then

$$(r,\theta+2\pi),(r,\theta-2\pi),(r,\theta+4\pi),(r,\theta-4\pi),$$

and more generally, $(r,\theta+2k\pi)$, where k can be any integer (positive or negative), are polar coordinates of P. Observe that any set of polar coordinates of P with first coordinate r has the form $(r,\theta+2k\pi)$. Note that $(-r,\theta+\pi)$ also represents P and hence that any set polar coordinates of P with first coordinate $-r$ has the form $(-r,\theta+\pi+2k\pi) = (-r,\theta+(2k+1)\pi)$. It follows that if a single set of polar coordinates can be determined for P, then all others are given by the "recipe" above. So in the problems that follow it suffices to specify a single set of polar coordinates.

i. The ray $\theta = \frac{\pi}{4}$ goes through the Cartesian point (3,3). Taking $r = \sqrt{x^2+y^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{3}$ provides the rest. So the polar coordinates are $\left(3\sqrt{3}, \frac{\pi}{4}\right)$.

ii. The ray $\theta = -\frac{\pi}{4}$ goes through the Cartesian point $(4, -4)$. Because $r = \sqrt{4^2 + (-4)^2} = \sqrt{32} = 4\sqrt{2}$, the coordinates are $\left(4\sqrt{2}, -\frac{\pi}{4}\right)$.

iii. Since the ray $\theta = \frac{\pi}{2}$ goes through (0, 5), the polar coordinates are $\left(5, \frac{\pi}{2}\right)$.

iv. The ray $\theta = \pi$ goes through $(-4, 0)$ so that the polar coordinates are $(4, \pi)$.

Up to now we were able to determine a set of polar coordinates simply by "inspection". From now on we will be more systematic and use the transformation equations

$$\tan\theta = \frac{y}{x} \quad \text{and} \quad r = \pm\sqrt{x^2 + y^2}.$$

v. Because $x = 3$ and $y = 3\sqrt{3}$, $\frac{y}{x} = \frac{3\sqrt{3}}{3} = \sqrt{3}$. By Table 1.2 of Section 1.4 and Figure 26 of Section 4.4, $\theta = \frac{\pi}{3}$ is the only possible θ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Note that $r = \pm\sqrt{9 + 9 \cdot 3} = \pm\sqrt{36} = \pm 6$. Because the point $(3, 3\sqrt{3})$ is in the first quadrant, $\left(6, \frac{\pi}{3}\right)$ is one answer.

vi. Note that $\tan\theta = \frac{y}{x} = \frac{\frac{\sqrt{3}}{3}}{-\frac{1}{3}} = -3\frac{\sqrt{3}}{3} = -\sqrt{3}$. So $\tan(-\theta) = \sqrt{3}$. Hence $-\theta = \frac{\pi}{3}$, or $\theta = \frac{-\pi}{3}$, is the only possible θ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ (refer to Table 1.2 of Section 1.4 and Figure 26 of Section 4.4). Because $r = \pm\sqrt{\frac{1}{9} + \frac{3}{9}} = \pm\sqrt{\frac{4}{9}} = \pm\frac{2}{3}$ and the point $\left(-\frac{1}{3}, \frac{\sqrt{3}}{3}\right)$ is in the second quadrant, it follows that $\left(-\frac{2}{3}, -\frac{\pi}{3}\right)$ are polar coordinates of the point.

vii. Because $\tan\theta = \frac{\sqrt{3}}{-3} = -\frac{\sqrt{3}}{\sqrt{3}\sqrt{3}} = -\frac{1}{\sqrt{3}}$, $\tan(-\theta) = \frac{1}{\sqrt{3}}$. Again by Table 1.2 of Section 1.4 and Figure 26 of Section 4.4, $-\theta = \frac{\pi}{6}$, or $\theta = -\frac{\pi}{6}$, is the only possible θ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Note that $r = \pm\sqrt{9 + 3} = \pm\sqrt{12} = \pm 2\sqrt{3}$. Since $(-3, \sqrt{3})$ is in the second quadrant, it follows that $\left(-2\sqrt{3}, -\frac{\pi}{6}\right)$ are polar coordinates of the point.

Correction: Change the point in vii. to $(3, -\sqrt{3})$. This puts it into the fourth quadrant.

viii. As in the earlier problem, $\tan\theta = \frac{2}{-2\sqrt{3}} = -\frac{1}{\sqrt{3}}$ implies that $\theta = -\frac{\pi}{6}$ is the only possible θ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Since $r = \pm\sqrt{4 \cdot 3 + 4} = \pm\sqrt{16} = \pm 4$, and $(-2\sqrt{3}, 2)$ is in the second quadrant, $\left(-4, -\frac{\pi}{6}\right)$ are polar coordinates of the point.

ix. To represent the origin 0 in polar coordinates we need $r = 0$. But θ can be arbitrary. So $(0, \theta)$ with any θ is a set of polar coordinates for 0.

x. Because $\tan\theta = \frac{y}{x} = \frac{-5}{-5\sqrt{3}} = \frac{1}{\sqrt{3}}$, we get (as before) that $\theta = \frac{\pi}{6}$ is the only possible θ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Note that $r = \pm\sqrt{25 \cdot 3 + 25} = \pm\sqrt{100} = \pm 10$. Because the point $(-5\sqrt{3}, -5)$ is in the third quadrant, $\left(-10, \frac{\pi}{6}\right)$ is a representation of it in polar coordinates.

13K. Equations in Polar Coordinates

Exercises 69-75 are applications of the transformation equations

$$x = r\cos\theta,\ y = r\sin\theta,\ r = \pm\sqrt{x^2 + y^2},\ \text{and}\ \tan\theta = \frac{y}{x}.$$

69. The equation $2x + 3y = 4$ transforms to

$$2r\cos\theta + 3r\sin\theta = 4 \quad \text{or} \quad r(2\cos\theta + 3\sin\theta) = 4,$$

and hence (since $2\cos\theta + 3\sin\theta$ cannot be zero) to

$$r = \frac{4}{2\cos\theta + 3\sin\theta}\,.$$

70. This equation transforms to

i. $9r^2\cos^2\theta + r^2\sin^2\theta = 4r\sin\theta$. Assume (for the moment that $r \neq 0$). So

$$9r\cos^2\theta + r\sin^2\theta \;=\; 4\sin\theta \quad\text{or}\quad r(9\cos^2\theta + \sin^2\theta) \;=\; 4\sin\theta.$$

Now $9\cos^2\theta + \sin^2\theta$ cannot be zero because if it were, then $\sin\theta = 0$, hence $9\cos^2\theta = 0$ and thus $\cos\theta = 0$. But $\sin\theta$ and $\cos\theta$ cannot both be zero. Why not? So

ii. $r = \frac{4\sin\theta}{9\cos^2\theta+\sin^2\theta}$. Note that equations (i) and (ii) are not algebraically the same. For instance $\left(0, \frac{\pi}{2}\right)$ satisfies (i) but not (ii). But for any (r, θ) with $r \neq 0$, they imply the same relationship between r and θ. Also, the graphs of the two equations are the same because the origin lies on both.

71. This equation transforms to

i. $r^2 = r\cos\theta(r^2\cos^2\theta - 3r^2\sin^2\theta)$. If $r \neq 0$, then

$$1 = r\cos\theta(cos^2\theta - 3\sin^2\theta),$$

and hence, because $\cos\theta(\cos^2\theta - 3\sin^2\theta)$ cannot be zero,

ii. $r = \frac{1}{\cos\theta(\cos^2\theta-3\sin^2\theta)}$. Note that this time the graphs of (i) and (ii) are the same, except at the origin 0 which is on the graph of (i) but not on the graph of (ii).

72. The equation $r = 5$ transforms to $\pm\sqrt{x^2+y^2} = 5$, or $x^2 + y^2 = 25$.

73. Notice that the origin $0 = \left(0, \frac{\pi}{2}\right)$ is on the graph of $r = 3\cos\theta$. It follows that $r^2 = 3r\cos\theta$ has the same graph. This last equation transforms to $x^2 + y^2 = 3x$.

74. $\tan\theta = 6$ becomes $\frac{y}{x} = 6$.

75. Because the origin $0 = (0, 0)$ is on the graph of $r = 2\sin\theta\tan\theta$, the graph of this equation is the same as that of $r^2 = 2r\sin\theta\tan\theta$. This transforms to $x^2 + y^2 = 2y \cdot \frac{y}{x}$. To include the origin on the graph, we rewrite this as $x^3 + xy^2 = 2y^2$.

76. The graph is the set of all (r, θ) with $r = 6$. This is the circle with center 0 and radius 6.

77. The graph is the set of all points of the form $\left(r, -\frac{8\pi}{6}\right)$. Consider the ray $\theta = -\frac{8\pi}{6}$. Since r can be any number (positive, negative, or zero) the graph is the (straight) line through the origin that this ray determines. Because $\tan\left(-\frac{8\pi}{6}\right) = -\tan\left(\frac{4\pi}{3}\right) = -\tan\left(\pi + \frac{\pi}{3}\right) = -\tan\frac{\pi}{3} = -\sqrt{3}$, the Cartesian equation of this line is $y = -\sqrt{3}x$.

78. The graph of $r = \sin\theta$ is the same as that of $r^2 = r\sin\theta$. Its Cartesian equivalent is $x^2 + y^2 = y$. This equation can be analyzed by completing the square on $x^2 + y^2 - y = 0$. Because,

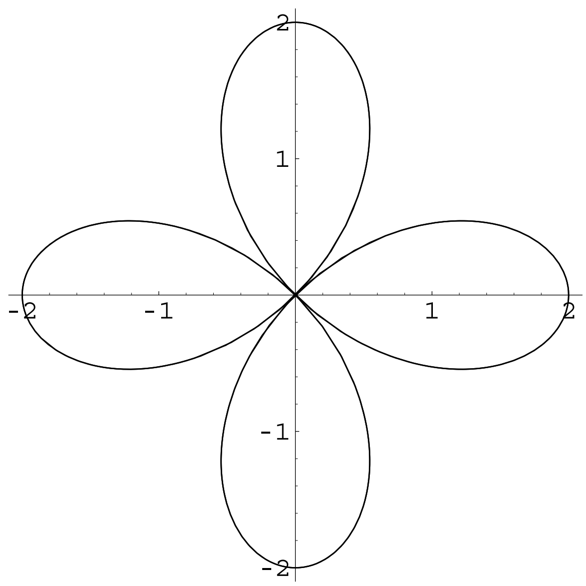
$x^2 + y^2 - y + (\frac{1}{2})^2 = (\frac{1}{2})^2$, we get

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.$$

It follows that the graph of $r = \sin\theta$ is the circle with center the Cartesian point $\left(0, \frac{1}{2}\right)$ and radius $\frac{1}{2}$.

79. This equation transforms to $y + x = 1$, or $y = -x + 1$. This is the line with slope -1 and y-intercept 1.

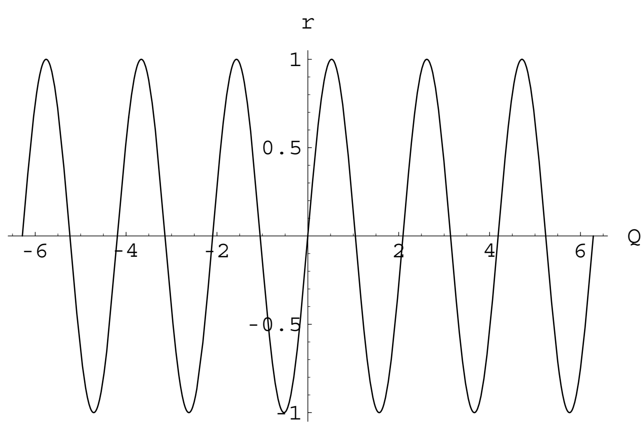
80. The graph of $r = 2\cos 2\theta$ is virtually identical to that of $r = \cos 2\theta$ which is studied in Example 13.23. The only difference is that the petals of the "rose", see Figure 13.35, have length 1 in the example and length 2 in this exercise.



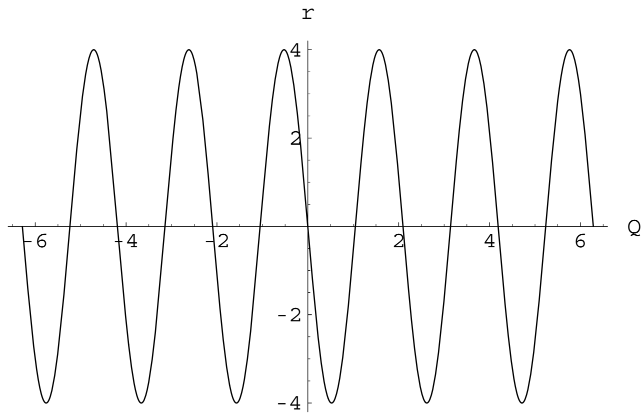
81. Lets start with the graph of $r = -4\sin 3\theta$ in Cartesian coordinates. Since

$$\sin 3\theta = \sin(3\theta + 2\pi) = \sin 3\left(\theta + \frac{2}{3}\pi\right)$$

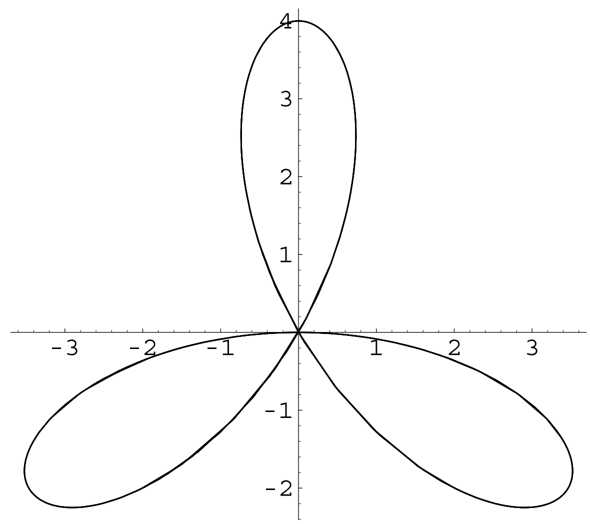
this graph has period $\frac{2}{3}\pi$. So it is the graph of $\sin\theta$ (which has period 2π) compressed by



a factor of 3. The Cartesian graph of $r = \sin 3\theta$ is sketched above. It follows quickly from this that the Cartesian graph of $r = -4\sin 3\theta$ is as drawn below.

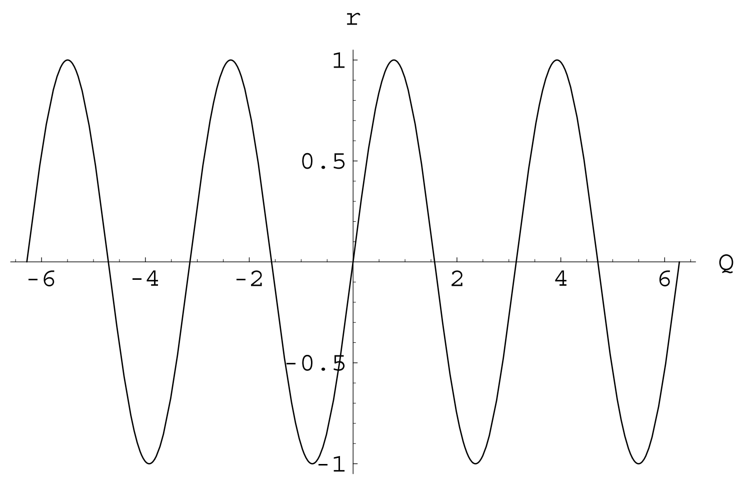


Now let sketch $r = -4\sin 3\theta$ in "polar". As the ray θ rotates from 0 to $\frac{\pi}{6}$, r slides from 0 to -4 and as θ goes from $\frac{\pi}{6}$ to $\frac{\pi}{3}$, r goes from -4 back to 0. The loop on the lower left of the figure below is traced out in the process. Similarly, as the ray θ rotates from $\frac{\pi}{3}$ to $\frac{2\pi}{3}$, r slides from 0 to 4 (at $\theta = \frac{\pi}{2}$) back to 0. So the upper loop is traced out. Next, as the ray θ rotates from $\frac{2\pi}{3}$ to π, r slides from 0 to -4 (at $\theta = \frac{5\pi}{6}$) back to 0. So the loopon the lower right is

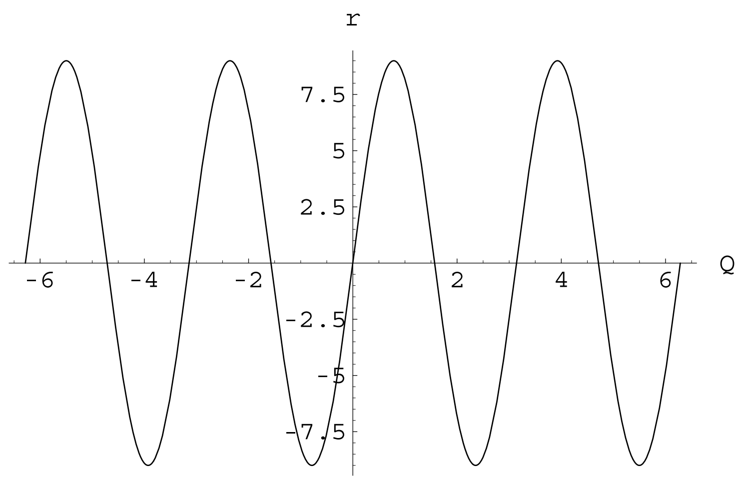


traced out. The rest of the graph is just a repetition of what we already have. For example, as θ moves from 0 to $-\frac{\pi}{3}$, the, loop on the lower right is traced out. As θ moves from π to $\frac{4\pi}{3}$, it is the loop on the lower left.

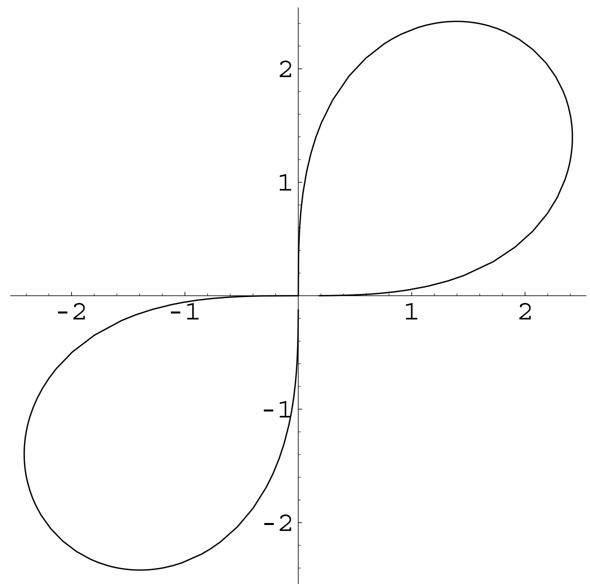
82. The considerations of Example 13.23 show that the Cartesian graph of $\sin 2\theta$ is as sketched below



So the Cartesian graph of $9\sin 2\theta$ is

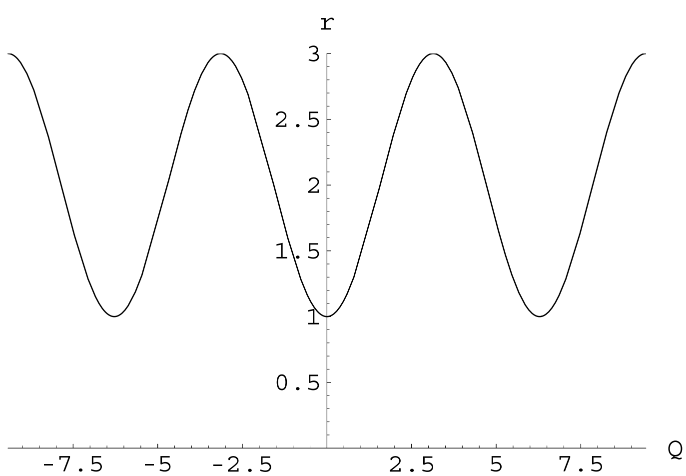


Now to the polar graph of $r^2 = 9\sin 2\theta$. Observe that $9\sin 2\theta \geq 0$ so θ must fall into $\left[0, \frac{\pi}{2}\right], \left[\pi, \frac{3\pi}{2}\right], \left[2\pi, \frac{5\pi}{2}\right], \ldots$ or $\left[-\frac{\pi}{2}, -\pi\right], \left[-\frac{3\pi}{2}, -\frac{5\pi}{2}\right], \ldots$. As the ray θ rotates from 0 to $\frac{\pi}{2}$, $r = +3\sqrt{\sin 2\theta}$ slides from 0 to 3 (at $\theta = \frac{\pi}{4}$) and back to 0. In the process, the loop on the upper right in the graph below is traced out. But $r = -3\sqrt{\sin 2\theta}$ is also possible. This time

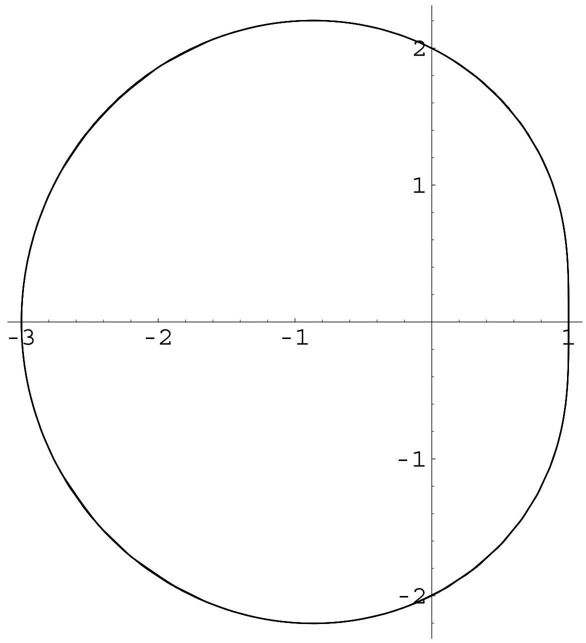


the loop on the lower left is traced out as θ varies from 0 to $\frac{\pi}{2}$. Similar considerations show that as θ varies from π to $\frac{3\pi}{2}$, $r = +3\sqrt{\sin 2\theta}$ traces out the loop on the lower left and $r = -3\sqrt{\sin 2\theta}$ traces out the loop on the lower left. As θ varies from $-\frac{\pi}{2}$ to $-\pi$, $r = +3\sqrt{\sin 2\theta}$ and $r = -3\sqrt{\sin 2\theta}$ traces these loops again. Repeating these considerations shows us that the graph of $r^2 = 9\sin 2\theta$ is complete as sketched.

83. The Cartesian graph of $r = 2 - \cos\theta$ is shown below



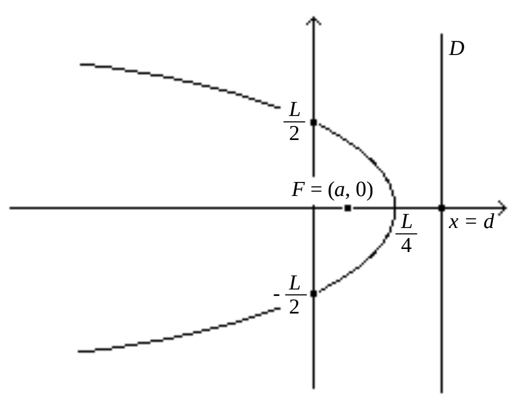
Turning to the polar graph of $r = 2 - \cos\theta$, we see that r: slides from 1 to 2 as θ moves from 0 to $\frac{\pi}{2}$; slides from 2 to 3 as θ moves from $\frac{\pi}{2}$ to π; slides from 3 to 2 as θ moves from π to $\frac{3\pi}{2}$; and slides from 2 to 1 as θ moves from $\frac{3\pi}{2}$ to 2π. This four step process traces out the loop sketched in the graph below.



Because the Cartesian graph $r = 2 - \cos\theta$ is symmetric about the vertical axis, the pattern is exactly the same as θ moves from 0 to $-\frac{3\pi}{2}$. The periodicity of the cosine tells us that the polar graph of $r = 2 - \cos\theta$ is complete as shown. (Incidentally, it is not transparent why this graph is a "snail".)

13L. Parabolas and Hyperbolas in Polar Coordinates

84. Equation $(*)$ becomes $4Lx + 4y^2 = L^2$. Dividing through by $4L$ we get $x + \frac{1}{L}y^2 = \frac{L}{4}$ and hence $x = -\frac{1}{L}y^2 + \frac{L}{4}$. By remarks in Section 4.3, this is a parabola which opens up to the left. The x-intercept is $\frac{L}{4}$. The y-intercepts are gotten by solving $\frac{1}{L}y^2 = \frac{L}{4}$ for y; so they are $y = \pm\frac{L}{2}$. Thus the graph of $x = -\frac{1}{L}y^2 + \frac{L}{4}$ looks as shown below.



Because $V = \left(\frac{L}{4}, 0\right)$ is on the parabola we know that the distance from V to F is equal to the distance from V to D. So $\frac{L}{4} - a = d - \frac{L}{4}$, and hence $\frac{L}{2} = d + a$. Because $P = (a, b)$ is on the parabola, the distance from P to F is equal to the distance from P to D. So $b = d - a = \frac{L}{2} - 2a$.

Since (a, b) is on the parabola, we see that

$$a \;=\; -\frac{1}{L}\left(\frac{L}{2} - 2a\right)^2 + \frac{L}{4}\,.$$

Therefore, $\frac{L}{4} - a \;=\; \frac{1}{L}\left(\frac{L}{2} - 2a\right)^2$ and hence, $\frac{L^2}{4} - aL \;=\; \frac{L^2}{4} - 2aL + 4a^2$. It follows that $aL = 4a^2$. If $a \neq 0$, then $L = 4a$. So $a = \frac{L}{4}$ and hence $V = F$. But this cannot be so. Why? Hence $a = 0$. It follows that the latus rectum is $\frac{L}{2} + \frac{L}{2} = L$. Observe that (3) holds because the axis of the parabola is defined to be the line through F perpendicular to D.

85. Left to the reader.

13M. Areas of Regions Given in Polar Coordinates

86. This is a circle of radius 4, so the area is 16π. Does the area formula

$$A = \int_a^b \frac{1}{2} f(\theta)^2 d\theta$$

provide the same result? Taking $r = f(\theta) = 4,\; a = 0$, and $b = 2\pi$, we get

$$A = \int_0^{2\pi} \frac{1}{2} \cdot 16 d\theta = 8\theta\Big|_0^{2\pi} = 16\pi.$$

87. The region is a loop above the polar axis traced out by the point

$$(r, \theta) = (3\sin\theta, \theta)$$

as θ rotates from 0 to π. (The rotation of θ from π to 2π, or 0 to $-\pi$, etc. simply retraces this loop.) So the area of the region enclosed by the loop is

$$A \;=\; \int_0^\pi \frac{1}{2}(3\sin\theta)^2 d\theta \;=\; \frac{9}{2}\int_0^\pi \sin^2\theta d\theta.$$

To evaluate this integral, recall that $\cos^2\theta = \frac{1+\cos 2\theta}{2}$. So

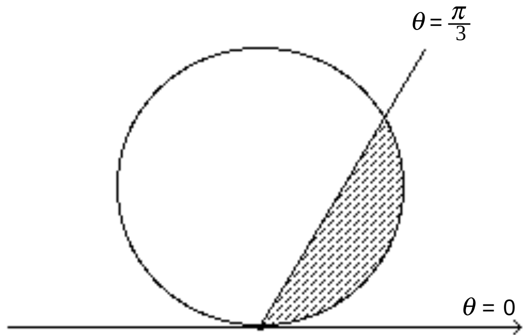
$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{1+\cos 2\theta}{2} = \frac{1-\cos 2\theta}{2}.$$

It follows that

$$\begin{aligned} A \;&=\; \frac{9}{2}\int_0^\pi \frac{1-\cos 2\theta}{2} d\theta \;=\; \frac{9}{4}\int_0^\pi (1-\cos 2\theta) d\theta \\ &=\; \frac{9}{4}\left[\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^\pi\right] \;=\; \frac{9}{4}\pi. \end{aligned}$$

By proceeding as in the solution of Exercise 78, one can show that the graph of $r = 3\sin\theta$ is the circle with equation $x^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$ of radius $\frac{3}{2}$. So the area is $\pi\left(\frac{3}{2}\right)^2 = \frac{9}{4}\pi$.

88. The region in question



has area $A = \int_0^{\frac{\pi}{3}} \frac{1}{2}(3\sin\theta)^2 d\theta$. Using facts from Exercise 87 (and using Examples 4.11 and 4.12), we find that the value of this definite integral is

$$\begin{aligned} A &= \frac{9}{4}\left[\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{3}}\right] = \frac{9}{4}\left(\frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3}\right) \\ &= \frac{9}{4}\left(\frac{\pi}{3} + \frac{1}{2}\sin\left(-\frac{2\pi}{3}\right)\right) = \frac{9}{4}\left(\frac{\pi}{3} - \frac{1}{2}\sin\frac{\pi}{3}\right) \\ &= \frac{9}{4}\left(\frac{\pi}{3} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) = \frac{3}{4}\pi - \frac{9\sqrt{3}}{16}. \end{aligned}$$

89. The graph of $r = 2 + 2\cos\theta = 2(1 + \cos\theta)$ is in essence the same as that discussed in Example 13.22. (What is the difference?) So the entire graph is sketched out by letting the ray determined by θ rotate from $\theta = 0$ to $\theta = 2\pi$. Therefore the area A of the cardioid is equal to

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} f(\theta)^2 d\theta = \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) d\theta \\ &= \int_0^{2\pi} (2 + 4\cos\theta + 1 + \cos 2\theta) d\theta \\ &= \left(3\theta + 4\sin\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = 6\pi. \end{aligned}$$

13N. Differential Equations

90. For $y = \frac{1}{3}e^{3x}$, we get $\frac{dy}{dx} = \frac{1}{3}e^{3x} \cdot 3 = e^{3x}$. So $y = \frac{1}{3}e^{3x}$ does satisfy $\frac{dy}{dx} = e^{3x}$.

91. With $y = \tan x + \sec x$, $\frac{dy}{dx} = \sec^2 x + \sec x \tan x$. So

$$\begin{aligned} 2\frac{dy}{dx} - y^2 &= 2(\sec^2 x + \sec x \tan x) - (\tan^2 x + 2\sec x \tan x + \sec^2 x) \\ &= \sec^2 x - \tan^2 x = 1. \end{aligned}$$

Why the restriction $0 < x < \frac{\pi}{2}$? Because $\cos\frac{\pi}{2} = 0$ means that $\sec x$ and hence $\frac{dy}{dx}$ is not defined for $x = \frac{\pi}{2}$. Note that the less restrictive condition $-\frac{\pi}{2} < x < \frac{\pi}{2}$ would have been sufficient.

92. With $y = \sin 2x - \cos 2x$, we get $\frac{dy}{dx} = 2\cos 2x + 2\sin 2x$, and $\frac{d^2y}{dx^2} = -4\sin 2x + 4\cos 2x$. So $\frac{d^2y}{dx^2} + 4y = 0$.

93. For $y = xe^{-2x}$, we get

$$\begin{aligned}\frac{dy}{dx} &= 1\cdot e^{-2x} + x(-2e^{-2x}) \;=\; e^{-2x} - 2xe^{-2x}, \text{ and}\\ \frac{d^2y}{dx^2} &= -2e^{-2x} - (2e^{-2x} + 2x(-2e^{-2x})) = -4e^{-2x} + 4xe^{-2x}\,.\end{aligned}$$

Therefore, $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = (-4e^{-2x} + 4xe^{-2x}) + (4e^{-2x} - 8xe^{-2x}) + 4xe^{-2x} = 0$.

13O. The Method of Separation of Variables

94. After separating variables, we get

$$\int y dy = \int x dx,$$

and therefore, $\frac{y^2}{2} = \frac{x^2}{2} + C$. So $y^2 = x^2 + 2C$ and hence (after changing C) the general solution is $y = \pm\sqrt{x^2 + C}$. The requirement $y(0) = 4$, forces $4 = \pm\sqrt{C}$. So $C = 16$ and the $-$ option is impossible. So the particular solution is

$$y = (x^2 + 16)^{\frac{1}{2}}.$$

95. After separating variables,

$$\int \frac{dy}{y} = \int x dx.$$

Hence $\ln y = \frac{x^2}{2} + C$, and therefore the general solution is

$$y = e^{\ln y} = e^{\frac{x^2}{2} + C}.$$

Because $y(1) = 3$, we get $3 = e^{\frac{1}{2} + C}$. So $\ln 3 = \frac{1}{2} + C$ and $C = \ln 3 - \frac{1}{2}$. The particular solution is $y = e^{\frac{x^2}{2} + \ln 3 - \frac{1}{2}}$.

96. By separating variables,

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

and hence $\ln y = \ln x + C$. So $y = e^{\ln y} = e^{\ln x + C} = e^{\ln x}\cdot e^C = Ax$, where $A = e^C$. Therefore the general solution is

$$y = Ax \text{ with } A > 0.$$

Note that the initial condition $y(2) = 0$, cannot be met because $y = 0$ forces $x = 0$. So there is no particular solution that satisfies the given initial condition.

97. Correction: It should be $(y-3)\frac{dy}{dt} = 1$ instead of $(y^2-3)\frac{dy}{dt} = 1$. By separating variables,

$$\int (y-3)dy = \int dt.$$

So $\frac{y^2}{2} - 3y = t + C$ and (changing C's),

$$y^2 - 6y - (2t + C) = 0.$$

By the quadratic formula,

$$y = \frac{6 \pm \sqrt{36 + 4(1)(2t + C)}}{2} = \frac{6 \pm 2\sqrt{9 + 2t + C}}{2}.$$

So $y = 3 \pm \sqrt{2t + 9 + C}$. To get $y = 3$ when $t = 0$, we need $9 + C = 0$ or $C = -9$. So there are two particular solutions:

$$y = 3 + \sqrt{2t} \quad \text{and} \quad y = 3 - \sqrt{2t}.$$

98. By separating variables,

$$\int (1 + y)dy = \int (\sin x - \cos x)dx$$

and hence

$$y + \frac{y^2}{2} = -\cos x - \sin x + C.$$

So $y^2 + 2y + (2\sin x + 2\cos x - C) = 0$. By the quadratic formula, the general solution is

$$\begin{aligned} y &= \frac{-2 \pm \sqrt{4 - 4(2\sin x + 2\cos x - C)}}{2} \\ y &= -1 \pm \sqrt{1 - 2\sin x - 2\cos x + C}. \end{aligned}$$

To get a particular solution, we need

$$\begin{aligned} 0 &= -1 \pm \sqrt{1 - 2\sin\pi - 2\cos\pi + C} \\ 1 &= \pm\sqrt{1 - 0 - 2(-1) + C}. \end{aligned}$$

So the + option prevails, and $C = -2$. Hence the particular solution is

$$y = -1 + \sqrt{-1 - \sin x - \cos x}.$$

Notice that $\sin x + \cos x$ needs to be negative or 0 for this to make sense. Compare the graphs in Figures 10.28 and 10.29 and determine an interval between $\frac{\pi}{2}$ and 2π for which this is the case.

99. After separating variables,

$$\int \frac{(\ln y)^2}{y} dy = \int x^2 dx$$

To solve the integral on the left, let $u = \ln y$. So $\frac{du}{dy} = \frac{1}{y}$ and hence

$$\begin{aligned} \int \frac{(\ln y)^2}{y} dy &= \int u^2 du = \frac{u^3}{3} + C_1 \\ &= (\ln y)^3 + C_1. \end{aligned}$$

Therefore $(\ln y)^3 + C_1 = \frac{x^3}{3} + C_2$ and hence $(\ln y)^3 = \frac{x^3}{3} + C$. So $\ln y = \left(\frac{x^3}{3} + C\right)^{\frac{1}{3}}$ and $y = e^{\ln y} = e^{(\frac{x^3}{3}+C)^{\frac{1}{3}}}$. For the particular solution, we need $1 = e^{(\frac{8}{3}+C)^{\frac{1}{3}}}$ and hence that $\left(\frac{8}{3} + C\right)^{\frac{1}{3}} = 0$. So $C = -\frac{8}{3}$. So the particular solution is

$$y = e^{\left(\frac{x^3}{3} - \frac{8}{3}\right)^{\frac{1}{3}}}.$$

13P. Chemical Reactions

100. **i.** Turning to the case $a = b$ first, we get

$$\int \frac{dy}{(a-y)^2} = \int k dt\,.$$

For the integral on the left, we turn to the substitution $u = a - y$. So $dy = -du$ and

$$\begin{aligned} \int \frac{dy}{(a-y)^2} &= -\int u^{-2} du = -\frac{u^{-1}}{-1} + C_1 \\ &= (a-y)^{-1} + C_1. \end{aligned}$$

Therefore,

$$(a-y)^{-1} + C_1 = kt + C_2$$

and hence $(a-y)^{-1} = kt + C$. So $a - y = \frac{1}{kt+C}$ and hence $y = a - \frac{1}{kt+C}$. Refer back to the narrative of the problem, and notice that $y(0) = 0$. Therefore, $\frac{1}{c} = a$ and $C = \frac{1}{a}$. Hence

$$\begin{aligned} y &= a - \frac{1}{kt + \frac{1}{a}} = a - \frac{1}{\frac{akt+1}{a}} \\ &= a - \frac{a}{akt+1} = \frac{a(akt+1) - a}{akt+1}\,. \end{aligned}$$

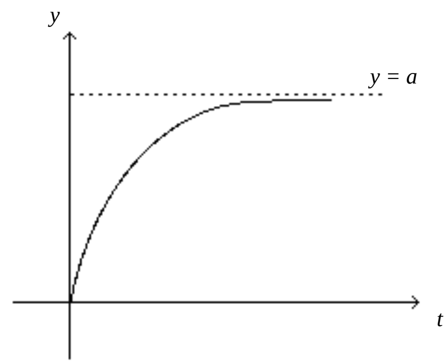
So $y(t) = \frac{a^2kt}{akt+1}$.

Observe that $\lim\limits_{t\to\infty} y(t) = a$. By the quotient rule,

$$\frac{dy}{dt} = \frac{a^2k(akt+1) - (a^2kt)(ak)}{(akt+1)^2}$$

$$
\begin{aligned}
&= \frac{a^2k}{(akt+1)^2} = a^2k(akt+1)^{-2}, \text{ and} \\
\frac{d^2y}{dt^2} &= -2a^2k(akt+1)^{-3}(ak) \\
&= \frac{-2a^3k^2}{(akt+1)^3}.
\end{aligned}
$$

We now have the following information about the graph of $y(t)$: it is increasing, concave down, and has $y = a$ as horizontal asymptote. So the graph has the shape



Note that the number of molecules produced at time t is always less that a. Suppose that this were a commercial production run and that the manufacturer of the compound is satisfied with a production of $0.99a$ molecules. How long should the reaction be allowed to run? This is simple: put $\frac{a^2kt}{akt+1} = 0.99a = \frac{99}{100}a$ and solve for t:

$$100a^2kt = 99a(akt+1), \text{ so } 100a^2kt - 99a^2kt = 99a, \text{hence } a^2kt = 99a \text{ and } \; t = \frac{99}{ak}.$$

So if a and k are both large, then the run will end relatively soon.

ii. Now the case $a \neq b$. As before,

$$\int \frac{dy}{(a-y)(b-y)} = \int kdt.$$

By "reversing" common denominators, we get

$$
\begin{aligned}
\frac{1}{(a-y)(b-y)} &= \frac{A}{a-y} + \frac{B}{b-y} \\
&= \frac{A(b-y) + B(a-y)}{(a-y)(b-y)} \\
&= \frac{-(A+B)y + Ab + Ba}{(a-y)(b-y)}.
\end{aligned}
$$

So $A + B = 0$ and $Ab + Ba = 1$. Since $B = -A$, we get $Ab - Aa = 1$ and hence $A = \frac{1}{b-a}$ and $B = \frac{-1}{b-a}$. (Notice that $a \neq b$ is needed here.) Therefore,

$$\frac{1}{(a-y)(b-y)} = \frac{\frac{1}{b-a}}{a-y} + \frac{-\frac{1}{b-a}}{b-y} = \left(\frac{1}{b-a}\right)\frac{1}{a-y} - \left(\frac{1}{b-a}\right)\frac{1}{b-y}.$$

So

$$\begin{aligned}
\int \frac{dy}{(a-y)(b-y)} &= \frac{1}{b-a}\int \frac{dy}{a-y} - \frac{1}{b-a}\int \frac{dy}{b-y} \\
&= \frac{1}{b-a}\left[-\ln(a-y) + \ln(b-y) + C_1\right] \\
&= \frac{1}{b-a}\left[\ln\left(\frac{b-y}{a-y}\right) + C_1\right].
\end{aligned}$$

[Refer to the narrative of the problem and notice that $a-y>0$ and $b-y>0$ throughout the reaction. Why is this relevant in the computations above?] Because

$$\int k dt = kt + C_2,$$

we now get

$$\frac{1}{b-a}\left[\ln\left(\frac{b-y}{a-y}\right) + C_1\right] = kt + C_2$$

and, after combining constants, that

$$\ln\left(\frac{b-y}{a-y}\right) = (b-a)kt + C.$$

Therefore, $\frac{b-y}{a-y} = e^{(b-a)kt+C} = De^{(b-a)kt}$, where $D = e^C$. Because $y=0$ when $t=0$ (see the narrative of the problem), we see that $D = \frac{b}{a}$. So

$$\frac{b-y}{a-y} = \frac{b}{a}e^{(b-a)kt}.$$

Finally, solve for y. After multiplying through and simplifying,

$$\begin{aligned}
b - y &= (a-y)\frac{b}{a}\ e^{(b-a)kt} \\
y - y\frac{b}{a}\ e^{(b-a)kt} &= b - be^{(b-a)kt} \\
y\left(1 - \frac{b}{a}\ e^{(b-a)kt}\right) &= b - be^{(b-a)kt} \\
y &= \frac{b(1-e^{(b-a)kt})}{1-\frac{b}{a}\ e^{(b-a)kt}}
\end{aligned}$$

This is the expression we needed to find.

We will now assume that $a > b$. This is no restriction, for as $a \neq b$, we could have arranged this at the beginning of the narrative of the problem. What about $\lim_{t\to\infty} y(t) = ?$ Since one molecule of the compound is formed by a combination of one molecule of each of the reacting chemicals, it is clear that no more than b molecules of the compound can be formed. Because $a > b$, $\lim_{t\to\infty} e^{(b-a)kt} = 0$, and it follows that

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{b(1-e^{(b-a)kt})}{1-\frac{b}{a}e^{(b-a)kt)}} \;=\; b.$$

Because $\frac{dy}{dt} = k(a-y)(b-y) > 0, y$ is an increasing function of t. What about the matter of concavity? The second derivative $\frac{d^2y}{dt^2}$ is best computed with the chain rule as follows: Let r (for rate) be the function $\frac{dy}{dt}$ of t. By the chain rule,

$$\frac{d^2y}{dt^2} = \frac{dr}{dt} = \frac{dr}{dy}\cdot\frac{dy}{dt}.$$

Because $r = k(ab-(a+b)y+y^2)$, we get $\frac{dr}{dy} = k(2y-(a+b))$. So

$$\begin{aligned}\frac{dr}{dt} &= -k((a-y)+(b-y))[k(a-y)(b-y)] \\ &= -k^2((a-y)+(b-y))(a-y)(b-y).\end{aligned}$$

Because $a-y > 0$ and $b-y > 0$, it follows that $\frac{dr}{dt} = \frac{d^2y}{dt^2}$ is negative throughout, and hence that the graph of $y(t)$ is concave down. So the graph of $y(t)$ has the same shape as that sketched earlier for the case $a = b$.

How long will it take for $0.99b$ molecules of the compound to be formed? Set

$$y(t) = \frac{b(1-e^{(b-a)kt})}{1-\frac{b}{a}e^{(b-a)kt}} = \frac{99}{100}b$$

and solve for t. Doing this, we get

$$\begin{aligned}100(1-e^{(b-a)kt}) &= 99(1-\frac{b}{a}\,e^{(b-a)kt}) \\ -100e^{(b-a)kt} &= -1-99\frac{b}{a}\,e^{(b-a)kt}.\end{aligned}$$

Therefore,

$$\left(100-99\frac{b}{a}\right)e^{(b-a)kt} \;=\; 1, \text{ and hence } \left(\frac{100a-99b}{a}\right)e^{(b-a)kt} \;=\; 1.$$

So $e^{(b-a)kt} \;=\; \dfrac{a}{100a-99b}$ and $(b-a)kt \;=\; \ln\left(\dfrac{a}{100a-99b}\right)$. Now solving for t we get

$$\begin{aligned}t &= \frac{1}{(b-a)k}\ln\left(\frac{a}{100a-99b}\right) = \frac{1}{-(a-b)k}\ln\left(\frac{a}{100a-99b}\right) \\ &= \frac{1}{(a-b)k}\ln\left(\frac{a}{100a-99b}\right)^{-1} = \frac{1}{(a-b)k}\ln\left(\frac{100a-99b}{a}\right) \\ &= \frac{1}{(a-b)k}\ln\left(100-\frac{99b}{a}\right).\end{aligned}$$

This will happen quickly if $a \approx b$ and if k is large.

13Q. First-Order Linear Differential Equations

101. The equation $y' + y = t$ fits into the scheme of equation (2) with $p(t) = 1$ and $q(t) = t$. Taking $P(t) = t$, we need to solve

$$\int te^t dt.$$

This integral succumbs to integration by parts after taking $u = t$ and $dv = e^t dt$. Since $du = dt$ and $v = e^t$, we get that

$$\begin{aligned} \int te^t dt &= \int udv = uv - \int vdu \\ &= te^t - \int e^t dt \ = \ te^t - e^t + C. \end{aligned}$$

Therefore, the general solution that we are looking for is

$$\begin{aligned} y(t) &= e^{-t}\left[te^t - e^t + C\right] \\ &= t - 1 + Ce^{-t}. \end{aligned}$$

Because $y'(t) = 1 - Ce^{-t}$,

$$y'(t) + y(t) = t$$

as required. To find the particular solution $f(t) = t - 1 + Ce^{-t}$ with $f(0) = 2$ we need $2 = 0 - 1 + C$ and hence $C = 3$. So $f(t) = t - 1 + 3e^{-t}$ is the required particular solution.

102. The scheme applies to this equation with $p(t) = 2t$ and $q(t) = 2t$. Taking $P(t) = t^2$, we get

$$\begin{aligned} y(t) &= e^{-t^2}\left[\int 2te^{t^2} dt + C\right] \\ &= e^{-t^2}\left[e^{t^2} + C\right] \ = \ 1 + Ce^{-t^2}. \end{aligned}$$

To get the particular solution, we solve

$$0 = y(0) = 1 + Ce^{\circ} = 1 + C$$

for C, to get $C = -1$. So $y(t) = 1 - e^{-t^2}$.

103. Take $p(t) = \frac{1}{t}$ and $q(t) = \cos t$. With $P(t) = \ln t$, we get

$$\begin{aligned} y(t) &= e^{-\ln t}\left[\int \cos t \ e^{\ln t} dt + C\right] \\ &= t^{-1}\left[\int t \cos t \ dt + C\right]. \end{aligned}$$

An integration by parts handles $\int t \cos t \ dt$. Let $u = t$ and $dv = \cos t \ dt$. So $du = dt$, $v = \sin t$, and hence

$$\begin{aligned} \int t \cos t dt &= \int udv \ = \ uv - \int vdu \\ &= t \sin t - \int \sin t \, dt \\ &= t \sin t + \cos t + C. \end{aligned}$$

So

$$y(t) = t^{-1}[t\sin t + \cos t + C] = \sin t + t^{-1}\cos t + Ct^{-1}.$$

To get $y(\pi) = 0$, we need $0 = 0 + \pi^{-1}(-1) + C\pi^{-1}$; hence $C\pi^{-1} = \pi^{-1}$ and $C = 1$. So the particular solution is $y(t) = \sin t + t^{-1}\cos t + t^{-1}$.

104. **i.** We rewrite the equation as

$$\frac{dy}{dx} - x^{-1} + x^{-2} = 0.$$

So $\frac{dy}{dx} = x^{-1} - x^{-2}$, and $y = \ln x + x^{-1} + C$. To get $y(1) = 3$, we need $3 = \ln 1 + 1 + C = 1 + C$. So $C = 2$ and the answer is

$$y(x) = \ln x + x^{-1} + 2.$$

ii. This time,

$$\frac{dy}{dx} - x^{-1}y = -x^{-2}.$$

This fits into the scheme discussed earlier with $p(x) = -x^{-1}$ and $q(x) = -x^{-2}$. Taking $P(x) = -\ln x$, we get

$$\begin{aligned} y(x) &= e^{\ln x}\left[\int -x^{-2}e^{-\ln x}dx + C\right] = x\left[\int -x^{-2}e^{\ln x^{-1}}dx + C\right] \\ &= x\left[\int -x^{-3}dx + C\right] = x\left[\frac{1}{2}x^{-2} + C\right] \\ &= \frac{1}{2}x^{-1} + Cx. \end{aligned}$$

To get $y(1) = 3$, we solve $3 = y(1) = \frac{1}{2} + C$ for C. So $C = \frac{5}{2}$ and $y(x) = \frac{1}{2}x^{-1} + \frac{5}{2}x$ is the particular solution.

iii. This time,

$$\frac{dy}{dx} + x^{-2}y = -x^{-2}$$

and we are "in the scheme" by taking $p(x) = x^{-2}$ and $q(x) = -x^{-2}$. Take $P(x) = -x^{-1}$ to get

$$\begin{aligned} y(x) &= e^{x^{-1}}\left[\int -x^{-2}e^{-x^{-1}}dx + C\right] \\ &= e^{-x^{-1}}\left[-e^{-x^{-1}} + C\right] \\ &= -e^{-2x^{-1}} + Ce^{-x^{-1}}. \end{aligned}$$

To get the particular solution, we need $3 = -e^{-2} + Ce^{-1}$; hence $Ce^{-1} = 3 + e^{-2}$ and $C = 3e + e^{-1}$. Therefore,

$$y(x) = -e^{-2x^{-1}} + (3e + e^{-1})e^{-x^{-1}}.$$

13R. Application to Radioactive Decay

105. Notice that there are $x_0 = 0.9928$ grams of ${}^{238}_{92}\mathrm{U}$ in the sample at time $t = 0$. Since the half-life of ${}^{238}_{92}\mathrm{U}$ is $\lambda_1 = 4.5 \times 10^9$ years, we convert that of ${}^{234}_{90}\mathrm{Th}$ into years as well. This is $\lambda_2 = \frac{24.1}{365.25} = 0.066 = 6.6 \times 10^{-2}$. Since we are dealing with a sample of pure uranium at time $t = 0$, the amount of ${}^{234}_{90}\mathrm{Th}$ at $t = 0$ is $y_0 = 0$. Therefore by the formula, the amount (in grams) of ${}^{234}_{90}\mathrm{Th}$ at any time $t \geq 0$, (in years) is

$$\begin{aligned} y(t) &\approx \frac{(0.9928)(4.5 \times 10^9)}{(-4.5 \times 10^9)}(0 - e^{-0.066t}) \\ &\approx 0.9928e^{-0.066t}. \end{aligned}$$

With $t = 10^9$ this is approximately

$$0.9928e^{-66\times 10^6} \approx 0.$$

So the amount is negligible. Think of it this way: Given the long half-life of ${}^{238}_{92}\mathrm{U}$, the daughter ${}^{234}_{90}\mathrm{Th}$ is produced very slowly; given the relatively short half-life of ${}^{234}_{90}\mathrm{Th}$, the small amounts of thorium produced vanish rapidly.

106. This problem is virtually identical to the previous problem and will be omitted.