

Solutions to Exercises of Chapter 12

Note: Differences in calculator accuracy and rounding strategies may lead to answers that differ slightly from those given here.

12A. Basic Interest

1. The operative formula for (i) - (v) is $A_{nt} = A_0 \left(1 + \frac{r}{n}\right)^{nt}$ and $A(t) = A_0 e^{rt}$ for (vi). In the current situation, $A_0 = \$5,000$, $t = 7$ years, $r = 0.08$, and n varies as required.

i. Annually means $n = 1$. So

$$\begin{aligned} A_0 \left(1 + \frac{r}{n}\right)^{nt} &= 5000 \left(1 + \frac{0.08}{1}\right)^{1 \cdot 7} = 5000(1.08)^7 \\ &= \$8,569.12 \end{aligned}$$

ii. Semiannually means $n = 2$. So

$$\begin{aligned} A_0 \left(1 + \frac{r}{n}\right)^{nt} &= 5000 \left(1 + \frac{0.08}{2}\right)^{2 \cdot 7} = 5000(1.04)^{14} \\ &= \$8,658.38 \end{aligned}$$

iii. Quarterly means $n = 4$. So

$$\begin{aligned} A_0 \left(1 + \frac{r}{n}\right)^{nt} &= 5000 \left(1 + \frac{0.08}{4}\right)^{4 \cdot 7} = 5000(1.02)^{28} \\ &= \$8,705.12 \end{aligned}$$

iv. Monthly means $n = 12$. So

$$\begin{aligned} A_0 \left(1 + \frac{r}{n}\right)^{nt} &= 5000 \left(1 + \frac{0.08}{12}\right)^{12 \cdot 7} = 5000(1.00666\dots)^{84} \\ &= \$8,737.11 \end{aligned}$$

v. Daily means $n = 365$. So

$$\begin{aligned} A_0 \left(1 + \frac{r}{n}\right)^{nt} &= 5000 \left(1 + \frac{0.08}{365}\right)^{365 \cdot 7} = 5000(1.000219178\dots)^{2555} \\ &= \$8,752.82 \end{aligned}$$

vi. $A(7) = 5000e^{(0.08)7} = 5000e^{0.56} = 8,753.36$.

2. After t years the account will have $A(t) = A_0 e^{rt} = A_0 e^{0.07t}$. The question is what t gives $A(t) = 2A_0$. So we solve $A_0 e^{0.07t} = 2A_0$ or $e^{0.07t} = 2$ for t . Taking natural logs of both sides, we get $0.07t = \ln e^{0.07t} = \ln 2$ and hence

$$t = \frac{\ln 2}{0.07} = 9.9 \text{ years.}$$

3. Proceeding as above, we get

$$t = \frac{\ln 2}{r} = \frac{100 \ln 2}{100r} = \frac{69.3}{i}.$$

4. The “Rule of 70” says that the doubling time d is approximately $d \approx \frac{70}{i}$. For $d = 8$, $8 \approx \frac{70}{i}$, so $i \approx \frac{70}{8} = 8\frac{3}{4}\%$.

5. The formula that applies is $S_p = \frac{12}{r}A_0 \left(1 + \frac{r}{12}\right) \left(\left(1 + \frac{r}{12}\right)^p - 1\right)$ where $A_0 = \$250$ is the fixed monthly payment, r the annual interest rate, and p the number of monthly payments. At $r = 0.07$ we get the amount

$$\begin{aligned} S_{120} &= \frac{12}{0.07}(250) \left(1 + \frac{0.07}{12}\right) \left(\left(1 + \frac{0.07}{12}\right)^{120} - 1\right) \\ &= \$43,523.62 \end{aligned}$$

after 10 years, and

$$\begin{aligned} S_{180} &= \frac{12}{0.07}(250) \left(1 + \frac{0.07}{12}\right) \left(\left(1 + \frac{0.07}{12}\right)^{180} - 1\right) \\ &= \$79,702.81 \end{aligned}$$

after 15 years. With an interest rate of $r = 0.08$, we get

$$\begin{aligned} S_{120} &= \frac{12}{0.08}(250) \left(1 + \frac{0.08}{12}\right) \left(\left(1 + \frac{0.08}{12}\right)^{120} - 1\right) \\ &= \$46,041.42 \end{aligned}$$

after 10 years and

$$\begin{aligned} S_{180} &= \frac{12}{0.08}(250) \left(1 + \frac{0.08}{12}\right) \left(\left(1 + \frac{0.08}{12}\right)^{180} - 1\right) \\ &= \$87,086.29 \end{aligned}$$

after 15 years.

6. Here too, the formula is

$$S_p = \frac{12}{r}A_0 \left(1 + \frac{r}{12}\right) \left(\left(1 + \frac{r}{12}\right)^p - 1\right).$$

In the current case, $A_0 = \$20$, $r = 0.06$ and $p = (21)(12) = 252$. Plugging these data into the formula, we get

$$\begin{aligned} S_{252} &= \frac{12}{0.06}(20) \left(1 + \frac{0.06}{12}\right) \left(\left(1 + \frac{0.06}{12}\right)^{252} - 1\right) \\ &= \$10,107.77 \end{aligned}$$

7. Recall that the present value of B dollars p months from now is $B \left(1 + \frac{r}{12}\right)^{-p}$, where r is the annual interest rate. So with $B = \$100$ and $r = 0.04$, the present value is $\$98.02$ for 6 months, $\$92.32$ for 24 months, and $\$81.90$ for 60 months.

12B. Annuities, Mortgages, and Bonds

8. After $(35)(12) = 420$ monthly payments of $\$1000$ at an annual interest rate of $r = 0.06$, she will have

$$\begin{aligned} S_{420} &= \frac{12}{0.06}(1000) \left(1 + \frac{0.06}{12}\right) \left(\left(1 + \frac{0.06}{12}\right)^{420} - 1 \right) \\ &= \$1,431,833.85. \end{aligned}$$

This will become present value PV_p of the income stream of B dollars per month for $p = (15)(12) = 180$ months at an annual interest rate of $r = 0.08$. So

$$\begin{aligned} \$1,425,710.30 &= \frac{12B}{0.08} \left[1 - \left(1 + \frac{0.08}{12}\right)^{-180} \right] \\ &= 104.64B. \end{aligned}$$

So $B = \$13,683.35$.

9. After $(38)(12) = 456$ monthly payments of $\$800$ each at an annual interest rate of $r = 0.04$, she will have

$$\begin{aligned} S_{456} &= \frac{12}{0.04}(800) \left(1 + \frac{0.04}{12}\right) \left(\left(1 + \frac{0.04}{12}\right)^{456} - 1 \right) \\ &= \$857,412.34. \end{aligned}$$

This will become the present value PV_p of an income stream of B dollars per month for $p = (10)(12) = 120$ months at an annual interest rate of $r = 0.05$. So

$$\begin{aligned} \$855,363.78 &= \frac{12B}{0.05} \left[1 - \left(1 + \frac{0.05}{12}\right)^{-120} \right] \\ &= 94.28B. \end{aligned}$$

So $B = \$9,094.19$.

10. The present value PV_p of the income stream of B dollars per month for $p = (30)(12) = 360$ months at an annual interest rate of $r = 0.07$ is

$$\begin{aligned} PV_{360} &= \frac{12B}{0.07} \left[1 - \left(1 + \frac{0.07}{12}\right)^{-360} \right] \\ &= 150.31B. \end{aligned}$$

With $PV_{360} = \$100,000.00$, we get $B = \$665.30$.

11. Proceeding as in Exercise 10, we get

$$\begin{aligned}PV_{360} &= \frac{12B}{0.09} \left[1 - \left(1 + \frac{0.09}{12} \right)^{-360} \right] \\ &= 124.28B.\end{aligned}$$

So $B = \frac{100,000.00}{124.28} = \804.62 .

12. Let's consider the AT&T bond first. From February 27, 1996 until its maturity on August 1, 2000, the bond generates 9 coupon payments of \$30 each. The present value of an AT&T bond on February 27, 1996 was therefore

$$\frac{(2)(30)}{r} \left[1 - \left(1 + \frac{r}{2} \right)^{-9} \right] + 1000 \left(1 + \frac{r}{2} \right)^{-9}.$$

With $r = 0.06$, this comes to $\$233.58 + \$766.42 = \$1000.00$. This is very nearly equal to the $10 \left(100\frac{1}{8} \right) = \1001.25 indicated on the bond page.

Turn to the IBM bond next. From February 27, 1996 until its maturity on June 15, 2000 an IBM bond will pay out 9 coupon payments of \$31.875 each. It follows that the present value of an IBM bond on February 27, 1996 was

$$\frac{63.75}{r} \left[1 - \left(1 + \frac{r}{2} \right)^{-9} \right] + 1000 \left(1 + \frac{r}{2} \right)^{-9}.$$

With $r = 0.06$ this comes to $\$248.18 + 766.42 = \1014.60 . This is close to the \$1020 that one can read off the bond page.

Finally, the Texaco bond. From February 27, 1996 to the date of maturity on February 15, 2003, the bond pays 14 coupon payments of \$42.50 each. It follows that the present value of a Texaco bond on February 27, 1996 was

$$\frac{85}{r} \left[1 - \left(1 + \frac{r}{2} \right)^{-14} \right] + 1000 \left(1 + \frac{r}{2} \right)^{-14}.$$

Taking $r = 0.06$, this equals

$$\$480.08 + 661.12 = \$1141.20.$$

This is fairly close to the \$1120 listed on the bond page.

Answer for each of the three bonds: Was the actual interest rate with which the closing price was computed greater or less than 6%?

13. The expression $A_{nt} = A_0 \left(1 + \frac{r}{n} \right)^{nt}$ with $A_0 = \$1120$, $r = 0.06$, $n = 12$, and $t = 5$, tells us that the certificate of deposit will be worth $(\$1120) \left(1 + \frac{0.06}{12} \right)^{60} = \1510.71 .

The Texaco bond will generate 10 coupon payments of \$42.50 each from August 15, 1996 to February 15, 2001. What will the account that receives these coupon payments semi-annually paying interest at an annual rate of $r = 0.06$ be worth on February 27, 2001? To compute this, use the formula for S_p in Section 12.1B but modify it from the monthly (12) situation there to the current semi-annual (2) context. This gives the value of this account as

$$\begin{aligned} S_{10} &= \frac{2}{0.06}(42.50) \left(1 + \frac{0.06}{2}\right) \left(\left(1 + \frac{0.06}{2}\right)^{10} - 1\right) \\ &= \$529.71. \end{aligned}$$

It remains to compute the present value of the Texaco bond on February 27, 2001, at the interest rate of $r = 0.04$. From February 27, 2001, until the day of maturity on February 15, 2003, exactly 4 coupon payments remain. So this present value is

$$\begin{aligned} \frac{85}{0.04} &\left[1 - \left(1 + \frac{0.04}{2}\right)^{-4}\right] + 1000 \left(1 + \frac{0.04}{2}\right)^{-4} \\ &= \$161.83 + 923.85. \end{aligned}$$

So the total return from the bond is $\$529.71 + 161.83 + 923.85 = \1615.39 .

CORRECTION: Exercise 13, line n-6, should be "compounded" (not "compunded") and the last line should read "certificate" (and not "verificate").

12C. The Consumer Price Index

14. It follows from Section 12.2, especially Example 12.7, that the average rate of inflation over a time period $0 \leq t \leq t_1$ is given by

$$k = \frac{1}{t_1} \ln \left(\frac{p(t_1)}{p(0)} \right)$$

where $p(0)$ and $p(t_1)$ are the CPI's at $t = 0$ and $t = t_1$ respectively. Taking July 1965 as $t = 0$, then July 1995 is $t_1 = 30$, so that

$$\begin{aligned} k &= \frac{1}{30} \ln \left(\frac{152.5}{31.6} \right) \\ &= 0.52. \end{aligned}$$

This means that the average annual inflation rate over this thirty year period was 5.2%. Doing a similar thing over the period December 1978 to December 1980, we get

$$\begin{aligned} k &= \frac{1}{2} \ln \left(\frac{86.3}{67.7} \right) \\ &= 0.121. \end{aligned}$$

This tells us that the average annual inflation over this two year period was 12.1%.

15. As suggested in the problem, we will assume that the inflation rate from September 1995 until September 2000 will be $k = 0.04$. Taking September 1995 as $t = 0$ we know from Section 12.2 that the CPI $p(t)$ is given by

$$p(t) = p(0)e^{kt}$$

for any $t \geq 0$, where $p(0)$ is the CPI at time 0. So the CPI in September 2000 will be

$$\begin{aligned} p(25) &= 153.2e^{(0.04)(25)} \\ &= 153.2e \\ &= 416.4. \end{aligned}$$

If the cost of the same education in the year 2000 is C , then you would expect $\frac{C}{25,000} \approx \frac{416.4}{153.2}$. So $C \approx \$25,000 (2.718) = 67,950$.

16. Because CPI_{83} was 16.7 in 1930 and 152.5 in 1995 (see Exercise 14) we find that if C is the cost of the bridge in 1995, in millions of dollars, then $\frac{C}{60} \approx \frac{152.5}{16.7} \approx 9.13$. So $C \approx (60)(9.13) \approx 550$.
17. The increase in the real price from 1973 to 1974 is $35.48 - 29.17 = 6.31$ cents; from 1979 to 1980 it is $48.26 - 40.38 = 7.88$ cents. So the percentage increases are $\frac{6.31}{29.17} = 0.216$ or 21.6% and $\frac{7.88}{40.38} = 0.195$ or 19.5%. The decline from 1981 to 1987 is $48.13 - 26.35 = 21.78$ cents. The percentage of this decline is $\frac{21.78}{48.13} = 0.453$ or 45.3%. The average decline over the six years is $\frac{45.3}{6} \approx 7.6\%$.

CORRECTION: Reword the last sentence of Exercise 17 as follows: Compute the percentage of this decline and the average of this percentage per year.

12D. Supply and Demand

18. Because the short-run supply and demand functions $S(p)$ and $D(p)$ are both linear, we can set $D(p) = a + bp$ and $S(p) = c + dp$, where a, b, c , and d are constants. We know that $D(7) = S(7) = 220,000$ units per month. We also know that the price elasticities $e_D(7)$ and $e_S(7)$ are equal to $e_D(7) = -0.12$ and $e_S(7) = 0.20$ respectively. We now get

$$D(7) = a + 7b = 220,000 \text{ and } e_D(7) = 7 \frac{D'(7)}{D(7)} = \frac{7b}{a + 7b} = -0.12.$$

So $\frac{7b}{220,000} = -0.12$ and $7b = -26,400$. Hence $a - 26,400 = 220,000$. It follows that $b = -3771$ and $a = 246,400$. So $D(p) = 246,400 - 3771p$.

In the same way, $S(7) = c + 7d = 220,000$ and $e_S(7) = 7 \frac{S'(7)}{S(7)} = \frac{7d}{c + 7d} = 0.20$. So $\frac{7d}{220,000} = 0.20$ and $7d = 44,000$. Thus $c + 44,000 = 220,000$. It follows that $c = 176,000$ and $d = 6286$. So $S(p) = 176,000 + 6286p$.

After the production cut, the supply function is

$$\begin{aligned} S_1(p) &= S(p) - 40,000 \\ &= 136,000 + 6286p. \end{aligned}$$

The new equilibrium price p^* satisfies $S_1(p^*) = D(p^*)$. So $136,000 + 6286p^* = 246,400 - 3771p^*$. Solving for p^* gives us $10,057p^* = 110,400$, or $p^* \approx \$11$.

CORRECTION: In Exercise 18 add “per month” after “40,000 units”

19. Under this assumption, the world’s supply of oil would have decreased to

$$S_2(p) = S_1(p) - 2 = 15.4 + 0.15p,$$

and the new equilibrium price p^* would have satisfied $S_2(p^*) = D(p^*)$, or $15.4 + 0.15p^* = 18.9 - 0.225p^*$. This would have resulted in $p^* \approx \$9.33$ dollars per barrel.

20. The fact that the demand is more sensitive to price change, should mean that there will be less demand (than in the earlier situation) for oil as the price increases. So there should be less upward pressure on the price, and hence the expectation would be for the equilibrium price to be less than \$12 per barrel. Let’s check this analysis against the theory. Replacing $e_D(4) = -0.05$ by $e_D(4) = -0.08$ in Section 12.3B gives us $a + 4b = 18$ and $\frac{4b}{a+4b} = -0.08$. So $4b = -1.44$. So $a - 1.44 = 18$. Hence $a = 19.44$ and $b = -0.36$. So the world’s demand for oil is $D(p) = 19.44 - 0.36p$. As before, the equilibrium price p^* is determined by $D(p^*) = S_2(p^*)$. So $19.44 - 0.36p^* = 14.4 + 0.15p^*$. Hence $p^* \approx \$10$.

12E. Least-Square Fit

21. i.

$$\begin{aligned} E(f) &= (7 - f(1))^2 + (5 - f(2))^2 + (9 - f(3))^2 + (11 - f(4))^2 \\ &= (7 - 6)^2 + (5 - 4)^2 + (9 - 10)^2 + (11 - 18)^2 \\ &= 1 + 1 + 1 + 49 = 52. \end{aligned}$$

- ii.

$$\begin{aligned} E(g) &= (7 - g(1))^2 + (5 - g(2))^2 + (9 - g(3))^2 + (11 - g(4))^2 \\ &= (7 - 7)^2 + (5 - 7)^2 + (9 - 17)^2 + (11 - 31)^2 \\ &= 0 + 4 + 64 + 400 = 468. \end{aligned}$$

- iii.

$$\begin{aligned} E(h) &= (7 - h(1))^2 + (5 - h(2))^2 + (9 - h(3))^3 + (11 - h(4))^4 \\ &= (7 - 7)^2 + (5 - 5)^2 + (9 - 9)^2 + (11 - 11)^2 \\ &= 0. \end{aligned}$$

22. i.

$$\begin{aligned} E(f) &= (3 - f(2))^2 + (6 - f(5))^2 + (2 - f(7))^2 + (10 - f(9))^2 \\ &= \left(3 - 3\frac{2}{3}\right)^2 + \left(6 - 3\frac{2}{3}\right)^2 + (2 - 4)^2 + \left(10 - 18\frac{2}{3}\right)^2 \\ &= \frac{4}{9} + \frac{49}{9} + 4 + \frac{676}{9} = \frac{765}{9} = 85. \end{aligned}$$

ii.

$$\begin{aligned} E(g) &= (3 - g(2))^2 + (6 - g(5))^2 + (2 - g(7))^2 + (10 - g(9))^2 \\ &= (3 - 3)^2 + (6 - 6)^2 + (2 - 2)^2 + (10 - 10)^2 \\ &= 0. \end{aligned}$$

12F. The Hy-Tech Toy Company

23. The profit function is

$$\begin{aligned} \Pi(x) &= R(x) - C(x) \\ &= x - (0.985x - 97.231) \\ &= 0.015x + 97.231. \end{aligned}$$

Since $\Pi(x)$ is a linear function whose graph is a line with positive slope, the (short-term) suggestion is that an increase in the company's production will increase profit.

12G. Economies and Diseconomies of Scale

24.

$$\begin{aligned} \frac{d}{dx}AC(x) &= \frac{MC(x)x - C(x) \cdot 1}{x^2} \\ &= \frac{MC(x)}{x} - \frac{C(x)}{x} \\ &= \frac{MC(x) - AC(x)}{x}. \end{aligned}$$

If $MC(x) < AC(x)$ over an interval of outputs I , then $\frac{d}{dx}AC(x) < 0$ for all x in I , and hence $AC(x)$ is a decreasing function over I . If $MC(x) > AC(x)$ over an interval of outputs I , then $\frac{d}{dx}AC(x) > 0$ for all x in I , and hence $AC(x)$ is an increasing function over I .

Suppose $MC(x_1) < AC(x_1)$ at an output of x_1 units (per year, say). Then $AC(x)$ has a negative derivative at x_1 and hence the tangent of the graph of $AC(x)$ at the point $(x_1, AC(x_1))$ has negative slope. It follows that $AC(x)$ is a decreasing function, at least in a small interval containing x_1 . So a slight increase in output from x_1 will decrease the average total cost

and a slight decrease will increase the average total cost. A similar discussion shows that if $MC(x_1) > AC(x_1)$ at an output of x_1 , then a slight increase in output from x_1 will increase the average total cost, and a slight decrease will decrease the average total cost. It follows that if $AC(x)$ is a minimum at x_1 , then $MC(x_1)$ and $AC(x_1)$ must be equal.

25. $e_C(x) = x \frac{C'(x)}{C(x)} = \frac{MC(x)}{\frac{C(x)}{x}} = \frac{MC(x)}{AC(x)}$. If $e_C(x_1) < 1$ at an output of x_1 , then $MC(x_1) < AC(x_1)$; so by the solution of Exercise 24, an increase in output will decrease the average total cost. This is advantageous for the firm. If $e_C(x_1) > 1$ at an output of x_1 , then $MC(x_1) > AC(x_1)$ so, again by the solution of Exercise 24, a decrease in output will decrease the average total cost.

12H. More About the Electric Utility Industry

26. The actual average variable costs are:

$$\text{Brayton Point: } \frac{272,272}{9203.2} = \$29.58.$$

$$\text{Merrimack: } \frac{73,988}{3272.7} = \$22.61.$$

$$\text{Walter F. Wyman: } \frac{119,165}{2029.5} = \$58.72.$$

Plugging into $AVC(x) = -0.0000001x^2 - 0.00143x + 51.0007 + 4159.45x^{-1}$, we get $AVC(9203.2) = \$29.82$ for the Brayton Point plant, $AVC(3272.7) = \$46.52$ for the Merrimack plant, and $AVC(2029.5) = \$49.74$ for the Walter F. Wyman plant.

27. We need to show that $e_C(x) < 1$ for the output x of each of the plants. Because $e_C(x) = \frac{MC(x)}{AC(x)}$ by Exercise 25, it suffices to show that $MC(x) < AC(x)$, or $AC(x) - MC(x) > 0$, for all x . Since $C(x) = VC(x) + FC$, $AC(x) = AVC(x) + \frac{FC}{x}$. So in the current situation,

$$\begin{aligned} AC(x) &= -0.0000001x^2 - 0.00143x + 51.0007 + 4159.45x^{-1} + \frac{42,000}{x} \\ &= -0.0000001x^2 - 0.00143x + 51.0007 + 46,159.45x^{-1}. \end{aligned}$$

We already know from Section 12.4C that $MC(x) = -0.0000003x^2 - 0.00286x + 51.0007$.

Therefore,

$$AC(x) - MC(x) = 0.0000002x^2 + 0.00143x + 46,159.45x^{-1}.$$

Since output x is always positive, $AC(x) - MC(x) > 0$ for all x .

NOTE: This solution has assumed that the average total cost and marginal cost for each plant are given by the functions $AC(x)$ and $MC(x)$ that were derived by collecting the data for all plants. Is this a valid assumption?

12I. Cost and Profit

28. We need to set $MC(x) = 26.50$ and solve for x . A look at Figure 12.17 shows that the output x must lie in the interval $10,700 \leq x \leq 12,000$. So the equation $MC(x) = 0.0093x - 73.91$ applies. Solving $0.0093x - 73.91 = 26.50$ for x , we get $x = 10,797$. This is the profit-maximizing daily output at the market price of \$26.50 per barrel. The daily revenue at this output is $R(10,797) = (26.50)(10,797) = \$286,121$. To determine $\Pi(10,797)$ it remains to compute $C(10,797)$. To do this, we determine the cost function $C(x)$ for $10,700 \leq x \leq 12,000$. Because $C'(x) = MC(x) = 0.0093x - 73.91$, we get that $C(x) = 0.00465x^2 - 73.91x + k$ for some constant k . Because $C(x) = 0.0000002x^3 - 0.0058x^2 + 81.07x - 194,192$ for $9,700 \leq x \leq 10,700$, we get $C(10,700) = 254,224$. Using this value to compute k , we see that

$$\begin{aligned} k &= 254,224 - 0.00465(10,700)^2 + 73.91(10,700) \\ &= 512,683. \end{aligned}$$

So $C(x) = 0.00465x^2 - 73.91x + 512,683$. Hence $C(10,797) = \$256,751$. Therefore

$$\begin{aligned} \Pi(10,797) &= R(10,797) - C(10,797) \\ &= \$286,121 - \$256,751 \\ &= \$29,370. \end{aligned}$$

29. Exercise 25 tells us that it suffices to show that $MC(x) - AC(x) > 0$ for all x in the interval $8,000 \leq x \leq 9,700$. Because $MC(x) = 0.00009x + 22.93$ and $AC(x) = \frac{C(x)}{x} = .000045x + 22.93 + \frac{2280}{x}$, we get

$$MC(x) - AC(x) = 0.000045x - \frac{2280}{x}.$$

Let $f(x) = 0.000045x - 2280x^{-1}$. Because $f'(x) = 0.000045 + 2280x^{-2}$, $f'(x) > 0$ for all x . So $f(x)$ is an increasing function. The fact that

$$f(8,000) = (0.000045)(8000) - \frac{2280}{8000} = 0.36 - 0.28570$$

means that $f(x) > 0$ for all $x \geq 8,000$. Therefore

$$MC(x) - AC(x) > 0 \text{ for } 8000 \leq x \leq 9,700.$$

Over the interval $9,700 \leq x \leq 10,700$,

$$\begin{aligned} MC(x) - AC(x) &= MC(x) - \frac{C(x)}{x} \\ &= 0.0000006x^2 - 0.0166x + 81.07 \\ &\quad - \left(0.0000002x^2 - 0.0058x + 81.07 - \frac{194,192}{x} \right) \\ &= 0.0000004x^2 - 0.0058x + \frac{194,192}{x} \\ &= \frac{0.0000004x^3 - 0.0058x^2 + 194,192}{x} \end{aligned}$$

To show that $MC(x) - AC(x) > 0$ for $9,700 \leq x \leq 10,700$, we need to show that

$$f(x) = 0.0000004x^3 - 0.0058x^2 + 194,192 > 0$$

for all x in this interval. We will show that the minimum value of $f(x)$ is positive. Because $f'(x) = 0.0000012x^2 - 0.0116x$, we find that $f'(x) = 0$ for $x = 0$ and $x = \frac{0.0116}{0.0000012} = 9666\frac{2}{3}$. Check that $f(9666\frac{2}{3}) > 0$ and conclude that $f(x) > 0$ for all x with $9,700 \leq x \leq 10,700$.

- 30.** The fact that the firm suffers from diseconomies of scale over the outputs $8,000 \leq x \leq 10,700$, tells us (see the solution of Exercise 25) that the average cost per unit of output increases with increasing output. So the higher the output of the refinery, the less cost-efficient its production is. The reason that this does not contradict the conclusions about the refinery's profit in Section 12.4C is that profit depends on revenue and cost and not on cost alone.
- 31.** The output at which the profit is a maximum is obtained by solving $MC(x) = 5760$ for x . Doing so, we get $0.00003x^2 - 0.48x - 3260 = 0$. Hence by the quadratic formula,

$$x = \frac{0.48 \pm \sqrt{(0.48)^2 + 4(0.00003)(3260)}}{0.00006}.$$

Because the $-$ in \pm is not possible (Why?), we get that the profit-maximizing output is $x = 21,140$ units per quarter. The average total cost at an output of 15,000 units per quarter is $\frac{36,300,000}{15,000} = \2420 . Because $C'(x) = 0.00003x^2 - 0.48x + 2500$, we get that $C(x) = 0.00001x^3 - 0.24x^2 + 2500x + k$, for some constant k . Using the fact that $C(15,000) = 36,300,000$, tells us that $k = 19,050,000$. Therefore

$$\begin{aligned} \Pi(x) &= R(x) - C(x) = 5760x - 0.00001x^3 + 0.24x^2 - 2500x - 19,050,000 \\ &= -0.00001x^3 + 0.24x^2 + 3260x - 19,050,000. \end{aligned}$$

Check that $\Pi(21,140) = \$62,650,000$. This is the maximal profit per quarter.

12J. A Cost Function for Freight Trucking

CORRECTION: The average variable cost function $AVC(x)$ is incorrect as stated because it is based on the wrong data: Namely the output/variable cost data was inadvertently interchanged. Note that the data (102,813, 0.509), (176,163, 0.517), etc., should have read (201,953, 0.509), (340,608, 0.517), etc. The corrections, as well as suitably adjusted Exercises follow below:

Consider the twelve data points (output, variable cost) in order of increasing output:

$$\begin{aligned} (201,953, 102,813), & \quad (340,608, 176,163), \\ (377,940, 196,121), & \quad (571,714, 226,356), \\ (979,267, 296,416), & \quad (1,367,596, 378,446), \\ (1,413,807, 450,666), & \quad (1,466,024, 222,885), \end{aligned}$$

$$(1,670,195, 636,133), \quad (1,701,125, 579,696),$$

$$(2,071,861, 607,082), \quad (2,195,352, 624,680).$$

A least-squares analysis using Maple shows that the cubic polynomial that fits these points (x, y) best is

$$-(0.4694552007 \times 10^{-13})x^3 + (0.2141959821 \times 10^{-6})x^2 - (0.01330171200)x + 142744.0019.$$

We will simplify this a little and take

$$VC(x) = -(0.4694552 \times 10^{-13})x^3 + (0.2141960 \times 10^{-6})x^2 - (0.0133017)x + 142744$$

as the cubic variable cost function for the (output, variable cost) data of the twelve trucking firms.

- 32.** Compare the actual variable cost with that given by the function $VC(x)$ for firms 3, 4, 5, 6, 7 and 10. In terms of cost effectiveness, which of the firms operate as expected, better than expected, and worse than expected?

Firm 3. Output 979,267, Actual VC = 296,416

$$VC(x) = -44,086 + 205,406 - 13,026 + 142,744 \approx 291,038.$$

Firm 4. Output 571,714, Actual VC = 226,356

$$VC(x) = -8773 + 70,011 - 7605 + 142,744 \approx 196,377.$$

Firm 5. Output 340,608, Actual VC = 176,163

$$VC(x) = -1855 + 24,850 - 4531 + 142,744 \approx 161,208.$$

Firm 6. Output 1,413,807, Actual VC = 450,666

$$VC(x) = -132,667 + 428,146 - 18,806 + 142,744 \approx 419,417.$$

Firm 7. Output 2,071,861, Actual VC = 607,082

$$VC(x) = -417,519 + 919,459 - 27,559 + 142,744 \approx 617,125.$$

Firm 10. Output 1,466,024, Actual VC = 222,885

$$VC(x) = -147,917 + 460,356 - 19,501 + 142,744 \approx 435,682.$$

Consider a variable cost function of the form

$$VC(x) = ax^3 + bx^2 + cx + d$$

where $a, b, c,$ and d are constants with a negative, d positive, and b and c arbitrary. Note that the variable cost function for the trucking firms as well as that for the New England utility companies have this form.

The average variable cost and the marginal cost are given by

$$AVC(x) = ax^2 + bx + c + \frac{d}{x} \text{ and } MC(x) = 3ax^2 + 2bx + c.$$

So $MC(x) - AVC(x) =$

$$2ax^2 + bx - \frac{d}{x} = \frac{2ax^3 + bx^2 - d}{x}$$

and

$$\begin{aligned} \frac{dAVC(x)}{dx} &= \frac{MC(x) - AVC(x)}{x} \\ &= \frac{2ax^3 + bx^2 - d}{x^2}. \end{aligned}$$

33. Consider the polynomial

$$f(x) = 2ax^3 + bx^2 - d.$$

Show that

- i. If $\frac{b^3}{27a^2} < d$, then $f(x) < 0$ for all $x \geq 0$.
- ii. If $\frac{b^3}{27a^2} > d$, then $f(x)$ has two positive roots r_1 and r_2 , such that $r_1 < \frac{-b}{3a} < r_2$, $f(x) < 0$ for $0 \leq x < r_1$ and $x > r_2$, and $f(x) > 0$ for $r_1 < x < r_2$.

Deduce that in case (i) $AVC(x)$ is decreasing for all $x \geq 0$. Check that in case (ii) $AVC(x)$ is decreasing for $0 \leq x < r_1$, increasing for $r_1 < x < r_2$, and decreasing for $x > r_2$.

[Hints: Verify that the graph of $f(x)$ has two horizontal tangents and that they occur at the points $(0, -d)$ and $(-\frac{b}{3a}, \frac{b^3}{27a^2} - d)$. Use this fact together with information about $\lim_{x \rightarrow -\infty} f(x)$ and $\lim_{x \rightarrow \infty} f(x)$ to draw relevant conclusions about the graph of $f(x)$.]

34. In the situation of the trucking firms, $a = -0.4694552 \times 10^{-13}$, $b = 0.2141960 \times 10^{-6}$ and $d = 142,744$. Check that the polynomial $f(x) = 2ax^3 + bx^2 - d$ satisfies condition (ii) of Exercise 33 and that $-\frac{b}{3a} \approx 1,521,000$.

- i. Use Newton's method with $c_1 = 1,000,000$ to show that $c_3 \approx r_1 \approx 1,169,000$.
- ii. Use Newton's method with $c_1 = 2,000,000$ to show that $c_4 \approx r_2 \approx 1,825,000$.

Explain the relevance of these conclusions for the trucking companies.

By substituting all the constants, we get

$$AVC(x) = -(0.4694552 \times 10^{-13})x^2 + (0.214196 \times 10^{-6})x - 0.0133017 + \frac{142,744}{x}.$$

35. Check that the actual average variable costs in dollars per ton-mile for the twelve trucking companies are, in order of increasing output (from 201953 to 2195352 ton-miles):

0.509, 0.517, 0.519, 0.396,
 0.303, 0.277, 0.319, 0.152,
 0.381, 0.341, 0.293, 0.285.

Next check that

AVC (200,000) = 0.741
 AVC (400,000) = 0.422
 AVC (600,000) = 0.336
 AVC (1,000,000) = 0.297
 AVC (1,169,000) = 0.295 (local minimum)
 AVC (1,825,000) = 0.299 (local maximum)
 AVC (2,200,000) = 0.296
 AVC (2,500,000) = 0.285

What general features do the two sets of data have in common? What are some differences? What correlation between output and average variable cost does the function $AVC(x)$ indicate for outputs less than 1,000,000 ton-miles; between 1,000,000 and 2,200,000 ton-miles? What does the actual average cost tell us about these two questions?

12K. Producer Surplus

36. Consider the prices

$$p_1 < p_2 < p_3 < \cdots < p_{n-1} < p_n = p^*$$

where p_1 is the very lowest price at which at least one producer is willing to sell and the differences between these prices is very small, say one cent. Group the x^* units of the product that were actually sold as follows: A first group consisting of the (comparatively) small number of units, let's label it Δx_1 , that were supplied by producers who would have been willing to sell the product at the price p_1 ; a second group of Δx_2 units that were supplied by producers who would have sold at p_2 but not p_1 ; a third group of Δx_3 units supplied by producers willing to sell at p_3 but not less than p_3 ; and so on, down to a last group Δx_n of units supplied by the producers who would not have been willing to sell for less than p^* .

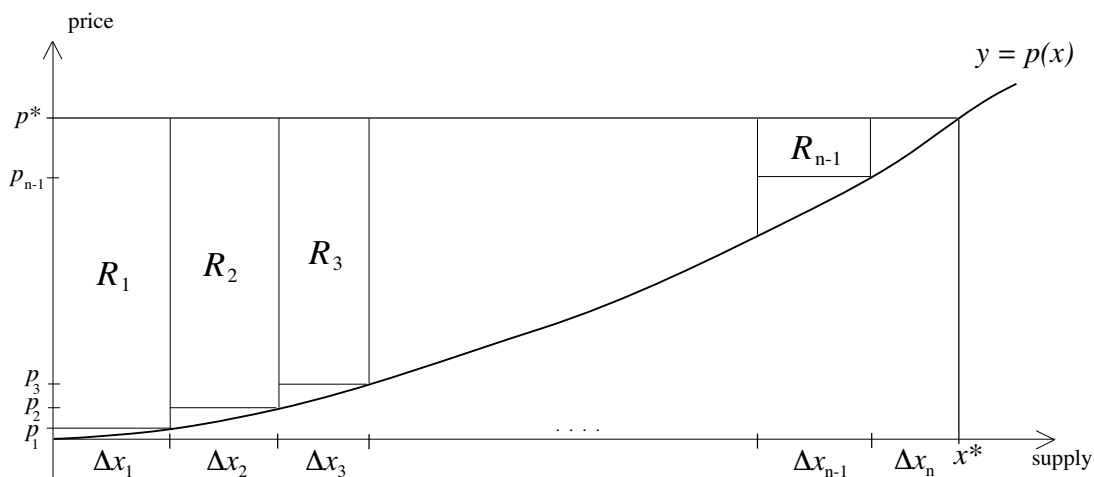
Let x_1 be the supply generated by the price p_1 , x_2 the supply generated by the price p_2 , x_3 the supply generated by p_3, \dots , and x_{n-1} the supply generated by p_{n-1} . Let $x_n = x^*$. We already know that the market equilibrium price $p_n = p^*$ generated the supply $x_n = x^*$. Notice

that

$$\begin{aligned}x_1 &= \Delta x_1 \\x_2 &= \Delta x_1 + \Delta x_2 \\x_3 &= \Delta x_1 + \Delta x_2 + \Delta x_3 \\&\vdots \\x^* &= \Delta x_1 + \Delta x_2 + \cdots + \Delta x_n.\end{aligned}$$

Now consider the price function $p(x)$ introduced earlier. This function relates price and supply, so it differs from that of Section 12.6 which relates price and demand. Notice that $p(x_1) = p_1, p(x_2) = p_2, \dots, p(x_{n-1}) = p_{n-1}$ and $p(x^*) = p^*$. Turn to the producers who would have supplied the Δx_1 units at the price p_1 . Since these Δx_1 units were in fact sold at p^* , these suppliers benefitted (implicitly) from earnings of $(p^* - p_1)\Delta x_1$. This is the *producer surplus* for these suppliers. The producers who would have supplied Δx_2 units at the price p_2 benefitted from a producer surplus of $(p^* - p_2)\Delta x_2$. Continuing in this way, we see that the total producer surplus is equal to

$$(p^* - p_1)\Delta x_1 + (p^* - p_2)\Delta x_2 + \cdots + (p^* - p_{n-1})\Delta x_{n-1} + (p^* - p_n)\Delta x_n.$$



The term $(p^* - p_1)\Delta x_1$ is equal to the area of R_1 , $(p^* - p_2)\Delta x_2$ is equal to the area of R_2 , and so on. Because $\Delta x_1, \dots, \Delta x_n$ are all relatively small, it follows that the total producer surplus is in essence equal to the area above the graph of $y = p(x)$ and below the line $y = p^*$ as required.

- 37.** The area just discussed is equal to the area of the rectangle with base x^* and height p^* minus the area under the graph of $y = p(x)$ from 0 to x^* . A look back at the calculus of Leibniz or Newton shows that this area is also equal to

$$p^* x^* - \int_0^{x^*} p(x) dx.$$

38. Solving $x = 17.4 + 0.15p$ for p , we get $0.15p = x - 17.4$. Because $0.15 = \frac{15}{100} = \frac{3}{20}$, we get $p = \frac{20}{3}x - 116$. Since p is not negative, we find that

$$p(x) = \begin{cases} 0 & \text{when } x \leq 17.4, \text{ and} \\ \frac{20}{3}x - 116 & \text{when } x \geq 17.4. \end{cases}$$

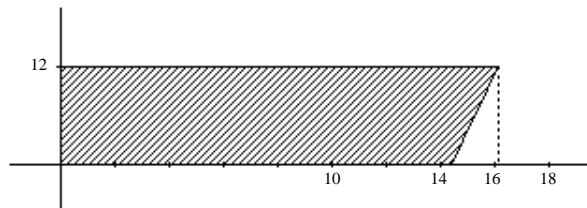
Because $p(x^*) = p^*$, $p(18) = 4$. It follows that the graph of $y = p(x)$ is as shown in Figure 12.23. The symbol S in the Figure reminds us that the issue is supply. The producer surplus is equal to the shaded area. This equals $(17.4)(4) + \frac{1}{2}(0.6)4 = 69.6 + 1.2 = 70.8$. Because the supply is in billion barrels per year and the price per barrel is in dollars, the producer surplus is 70.8 billion dollars per year.

CORRECTION: In Exercise 38, replace 70.8 by 70.88

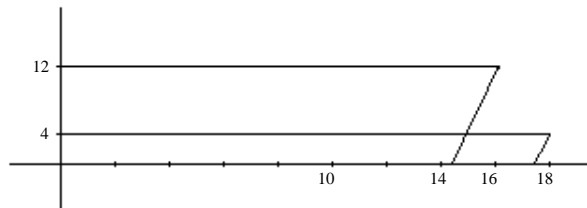
39. After OPEC's production cuts, the supply function for the short-run supply was $x = 14.4 + 0.15p$. Now

$$p(x) = \begin{cases} 0 & \text{for } x \leq 14.4 \\ \frac{20}{3}x - 96 & \text{for } x \geq 14.4. \end{cases}$$

Because $p(x^*) = p^*$ now becomes $p(16.2) = 12$, the producer surplus is now given as the shaded area in the figure below. It is thus equal to $(14.4)(12) + \frac{1}{2}(1.8)12 = 172.8 + 10.8 = 183.6$ bil-



lion dollars. The difference between this sum and the 70.8 billion dollars is explained by the figure



The gain is: $(15)(8) + \frac{1}{2}(1.2)(8) = 120 + 4.8 = \124.8 billion due to price increase from 4 to 12. The loss is $(17.4 - 14.4)4 = 12$, due to a drop in demand from 17.4 to 16.2. The net gain is \$112.8 billion.

12L. An Excerpt from Cournot's Mathematics of Value and Demand

40. Figure 12.24 is to be understood as follows: $0, q,$ and t are numbers on the price axis. The horizontal coordinate of the point n is q . The vertical coordinate of n is the value $D(q)$. Notice that $D'(q)$ is the slope of the tangent at n . So $D'(q) = \frac{D(q)}{q-t}$. If q satisfies $D(q) + qD'(q) = 0$, then $D'(q) = \frac{-D(q)}{q}$. In this case, $-\frac{1}{q} = \frac{1}{q-t}$. So $-q = q - t$ and hence $t = 2q$. So $t - q = q$. It follows that the legs nq and nt of the triangle are equal. For any price p , the revenue $pD(p)$ is the area of a rectangle with base of length p and height $D(p)$. So at a price q for which the revenue is a maximum, this rectangle has its maximal area.

Addition to Exercise 40. Explain the assertion: If $\frac{\Delta D}{\Delta p} < \frac{D}{p}$, then an increase in price Δp will increase the revenue $pD(p)$, and if $\frac{\Delta D}{\Delta p} > \frac{D}{p}$, then an increase in price Δp will decrease the revenue $pD(p)$.

Solution: $D(p + \Delta p) = D(p) - \Delta D$ (the demand decreases with an increase in price). So

$$D(p + \Delta p) - D(p) = -\Delta D. \text{ So } \frac{\Delta D}{\Delta p} = -\frac{D(p + \Delta p) - D(p)}{\Delta p} \approx -D'(p). \text{ Now } \frac{\Delta D}{\Delta p} < \frac{D(p)}{p}$$

translates to

$$\begin{aligned} -D'(p) &< \frac{D(p)}{p} \\ -pD'(p) &< D(p) \\ D(p) + pD'(p) &> 0. \end{aligned}$$

So $pD(p)$ increases, etc.

12M. The Lerner Index

41.

$$\begin{aligned} \Pi(x) &= R(x) - D(x) \\ \Pi(D(p)) &= R(D(p)) - C(D(p)) \\ &= pD(p) - C(D(p)). \end{aligned}$$

This difference expresses profit in terms of price.

42. At a price p that maximizes profit, the derivative of the profit as function of price must be zero, i.e. $\Pi(D(p))' = 0$. Because

$$\Pi(D(p))' = D(p) + pD'(p) - C'(D(p))D'(p),$$

we see that at a profit-maximizing price p , $-D(p) = (p - MC(D(p)))D'(p)$, and hence that

$$-\frac{D(p)}{D'(p)} = p - MC(D(p)).$$

So,

$$i(p) = -\frac{D(p)}{pD'(p)} = \frac{p - MC(D(p))}{p}.$$

We have verified (i) and (ii).

In a competitive market, the profit-maximizing output x_{\max} is found as the solution of $MC(x_{\max}) = p^*$. So under this profit-maximizing assumption, $x_{\max} = D(p^*)$. So

$$i(p^*) = \frac{p^* - MC(x_{\max})}{p^*} = 0.$$

To verify (iv), use (ii) to get $MC(D(p)) = p - i(p)p = (1 - i(p))p$. So

$$p = \frac{MC(D(p))}{1 - i(p)}.$$

12N. The Demand for Domestic Automobiles

NOTE: Constant dollars = inflation adjusted dollars. The prime interest rate is the interest rate that banks charge to good customers (eg. typical businesses).

43. Certainly a_0 has to be positive, otherwise the demand D is zero or negative. Suppose, if possible, that a_1 were positive. Consider RY_D and i to be constant. Under this assumption,

$$D = a_0 C p^{a_1}$$

with C a constant. Because $a_1 > 0$, an increase in the price p would bring about an increase in the demand D . Since this does not correspond to market realities, the assumption $a_1 > 0$ is not realistic. If $a_1 = 0$, then $p^{a_1} = p^0 = 1$ for any p , so in this case the demand D does not depend on p . This is also unrealistic. So the expectation is that $a_1 < 0$. In the same way, one would expect that $a_3 > 0$. What would an analysis of the type undertaken above tell us about a_2 ?

44. Substituting the data of Table 12.6 into the formula

$$D = 257.934p^{-1.184}RY_D^{2.183}i^{-0.191}$$

gives us

$$D = 9,598,432 \text{ for } 1973,$$

$$D = 8,324,574 \text{ for } 1975,$$

$$D = 8,350,179 \text{ for } 1978, \text{ and}$$

$$D = 6,604,006 \text{ for } 1982.$$

Note that the formula approximates the actual demand for 1973 very well, overestimates the actual demands for 1975 and 1982 by about one million, and underestimates the actual 1978 demand by about one million. Note that the function for D overestimates low actual demands

and underestimates high actual demands. This parallels a common characteristic: functions arising from numerical data by least squares approximation tend to “smooth out” the given numerical reality. In other words, the function tends to level the peaks and fill in the valleys.

Addition to Exercise 44: Predict the demand for new automobiles for 1988 under the assumption that p , RY_D , and i increase/decrease in the same way they did from 1986 to 1987.

Solution: The concern is the computation of D with the formula using the data $p = 240.6$, $RY_D = 2707.1$, and $i = 8.11$. This gives $D = 8,137,124$. In the same way, the demand predicted by the formula can be shown to be $D = 8,380,027$ for 1986 and $D = 8,263,440$ for 1987.

45. Taking both RY_D and i as constants, we get $D(p) = cp^{-1.184}$. So the price elasticity of demand is

$$\begin{aligned} p \frac{D'(p)}{D(p)} &= \frac{p(-1.184c)p^{-2.184}}{cp^{-1.184}} \\ &= \frac{-1.184p^{-1.184}}{p^{-1.184}} \\ &= -1.184. \end{aligned}$$

The same analysis shows that the interest elasticity of demand is -0.191 . Refer to Figure 12.6. Given that the index (it is -0.184) that measures the sensitivity of demand with respect to price is greater (in absolute value) than the index (-0.191) that measures the sensitivity of demand with respect to interest, one would expect the demand to be more sensitive to change in price than to change in interest.

To see that this expectation is affirmed by the data in the table, compare the years 1973 and 1975. The interest drops by a little and the disposable income increases by a little. Both of these facts would suggest an increase (perhaps small) in the demand. But the price index increases by 16.5 points and this drives the demand down by over 2.5 million cars. A comparison of the years 1981 and 1982 yields a similar (but less dramatic) conclusion. The years 1986 and 1987 provide another such example.

46. The data of the table give evidence that there are other parameters that play a role. Compare, for example, the data for the year 1986 with those for 1987. The disposable income rose slightly and the interest rate dropped a little. These two facts put an upward pressure on the demand. The increase in the price index of $232.5 - 224.4 = 8.1$ was a modest $\frac{8.1}{224.4} \times 100 = 3.6\%$. The expectation is that the influence of these changes in price, disposable income, and interest should not affect the demand for new cars to a great extent. The table, however, shows that the demand dropped by over one million cars. This is a clear indication that the demand is influenced by other factors. What might they be?

One is consumer confidence overall. If the economy is perceived to be going well, for example, if the stock markets are going up in value, inflation is under control, and unemployment is in

check, then consumers will in general feel positive about the financial aspects of their future and will not be afraid to spend. So they might be moved to purchase a new car.

New product lines can also have an expanding effect on demand. The very popular new vans and sport utility vehicles introduced in the 1980s and 1990s had the effect of driving up the demand for new cars overall.

CORRECTION: Pages 433 and 434. The year of publication of Evan Douglas, *Managerial Economics: Analysis and Strategy*, is 1992 and not 1979.

120. The Demand for Gasoline

47. The expectation is that increases in either RP_R or MPG will lead to decreases in the demand, and hence that a_1 and a_4 will be negative. On the other hand, increases in either RP_T or N or RY_D should result in increased demand, so that a_2 , a_3 , and a_5 should all be positive. Assume for the sake of argument that RP_T , N , and RY_D are all relatively small, and that RP_R and MPG are large. Given the signs of a_1 , a_2 , a_3 , a_4 and a_5 as already discussed, this assumption would lead to a negative demand (an impossibility) unless a_0 is taken to be positive.
48. The expectation as pointed out in the solution to Exercise 47 was that a_5 would be positive. This was based on the argument that an increase in disposable income would make it possible to spend more for gasoline and thus lead to increased demand. How then do we explain a_5 being negative? Consider the following: An increase in disposable income will make possible more purchases of new, generally more fuel efficient, cars thus decreasing the demand on gasoline. This connection suggests that a_5 should be negative. In reference to the data of Table 12.7, the downward influence on the demand of gasoline of this factor outweighs the upward influence on the demand of the effect described earlier.
49. Substituting all the relevant data into the equation for D gives $D = 78,263$ for 1973, and $D = 76,211$ for 1974. So the equation does predict a drop in the demand. However, it predicts the drop to be about 2000 million gallons; this is much less than the actual drop of $78,011 - 74,217$ or about 3,800 million gallons. This reflects a phenomenon encountered before: the function based on the data tends to “fill in the valleys” and “level the peaks.”
50. In 1987 the tax per gallon of gasoline was $P_R - P_S = 89.7 - 66.33 = 23.37$ cents per gallon. Given that the CPI was 100 in 1967 and 340.4 in 1987 tells us that this tax was

$$23.37 \times \frac{100}{340.4} = 6.87$$

cents per gallon in 1967 dollars. At the 1987 demand of 70,984 million gallons, this brought a tax revenue of

$$\begin{aligned} (70,984 \times 10^6) & \quad (6.87 \times 10^{-2}) \\ & = 4877 \times 10^6 \\ & \approx 5 \text{ billion dollars} \end{aligned}$$

in 1967 dollars. This equals $5 \times \frac{340.4}{100} \approx 17$ billion in 1987 dollars.

51. Refer to the solution of Exercise 50 and notice that the tax is now $6.87 + 5 = 11.87$ cents (in 1967 dollars) on one gallon of gasoline. The new retail price RP_R will be $26.35 + 5 = 31.35$ cents. Using 1987 data for all the other variables of the equation for D we get a demand projection of $D = 68,289$ million gallons. So the projected tax revenue for 1988 is

$$\begin{aligned} (68,289 \times 10^6) & \quad (11.87 \times 10^{-2}) \\ & = 8106 \times 10^6 \\ & \approx 8 \text{ billion dollars.} \end{aligned}$$

In 1987 dollars this equals $8 \times \frac{340.4}{100} \approx 27$ billion dollars. So the 5 cent tax increase would raise an additional $27 - 17 = 10$ billion dollars.