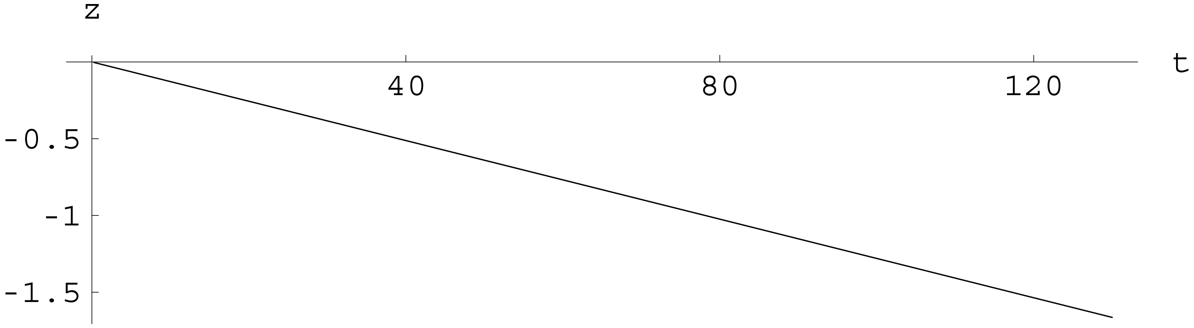
Solutions to the Exercises of Chapter 11

11A. Radioactive Decay

1. First use a calculator to show that $\ln 0.60 = -0.511, \ln 0.36 = -1.022$, and $\ln 0.22 = -1.514$. Then check that the points $(0,0), (40, \ln 0.60), (80, \ln 0.36)$, and $(120, \ln 0.22)$ do fall on a straight line shown. The negative slope $-\lambda$ of this line can be computed by using $(0,0)$ and each of the other points. So



$$\begin{aligned} -\lambda &= \frac{\ln 0.60 - 0}{40 - 0} = \frac{-0.511}{40} = -0.0128, \\ -\lambda &= \frac{\ln 0.36 - 0}{80 - 0} = \frac{-1.022}{80} = -0.0128, \\ -\lambda &= \frac{\ln 0.22 - 0}{120 - 0} = \frac{-1.514}{120} = -0.0126. \end{aligned}$$

The conclusion $\lambda = 0.0128$ is certainly reasonable. The half-life of radium-220 is

$$h = \frac{0.693}{0.0128} = 54.$$

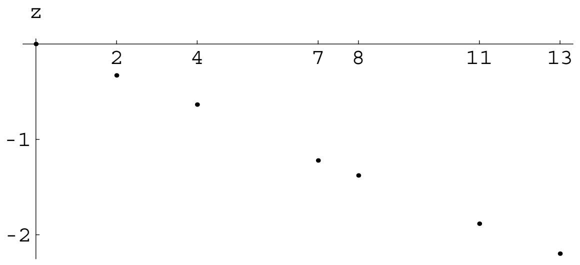
What are the units? Seconds!

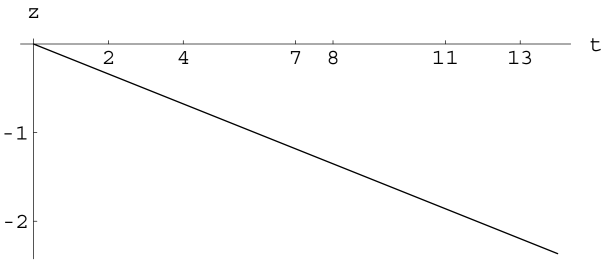
2. 5 minutes = 300 seconds. So there are $y(300) = y_0 e^{-0.0128(300)}$ atoms left. So

$$\frac{y(300)}{y_0} \approx 0.021 \text{ or } 2.1\%.$$

Taking $y_0 = 10^9$, we get $y(t) = 10^9 e^{-0.0128t}$. Taking $t = 2$, we see that $y(2) = 10^9 e^{-0.0256} \approx 0.975 \times 10^9 = 9.75 \times 10^8$. So $10 \times 10^8 - 9.75 \times 10^8 = 0.25 \times 10^8 = 2.5 \times 10^7$ atoms decayed during the first two seconds. They decayed at an average rate of $\frac{2.5 \times 10^7}{2} \approx 1.25 \times 10^7$ atoms/second. The rate of decay at $t = 2$ seconds was $y'(2)$. Because $y'(t) = 10^9 e^{-0.0128t} \cdot (-0.0128), y'(2) = -(1.28 \times 10^7)(0.975) \approx 1.25 \times 10^7$.

3. Take a calculator and check that $\ln 0.72 = -0.329, \ln 0.53 = -0.635, \ln 0.295 = -1.221$, $\ln 0.252 = -1.378, \ln 0.152 = -1.884, \ln 0.11 = -2.198$. The graphs below show the required points as well as the line determined by $(0,0)$ and $(13, -2.20)$.





Because $-\lambda$ is the slope of this line, Rutherford deduced that

$$-\lambda = \frac{-2.20 - 0}{13 - 0} = -\frac{2.20}{13} = -0.169.$$

So $\lambda = 0.169 \frac{1}{\text{days}}$.

4. **i.** The equation $y(t) = y_0 e^{-0.18t}$ is based on $\lambda = 0.18$ days^{-1}. So $h = \frac{0.693}{0.18} = 3.85$ days.

ii. $y(t) = 0.90 y_0$, implies that $0.90 y_0 = y_0 e^{-0.18t}$. So $e^{-0.18t} = 0.90$. Hence $-0.18t = \ln 0.90 = -0.105$ and hence $t = \frac{0.105}{0.18} \approx 0.58$ days.

iii. $\frac{1}{3} y_0 = y_0 e^{-0.18t}$ implies that $-0.18t = \ln \frac{1}{3} = -1.099$. So $t = 6.1$ days.

5. **i.** Using the equation $\lambda = \frac{0.693}{h}$, we get $\lambda = \frac{0.693}{138} = 0.0050 \frac{1}{\text{days}}$.

ii. By Avogadro's number, 30 milligrams contain

$$\left(\frac{30}{1000}\right) \frac{1}{210} \left(6.02 \times 10^{23}\right) = \left(8.60 \times 10^{-4}\right) \times 10^{23} = 8.60 \times 10^{19} \text{ atoms.}$$

iii. $y(t) = (8.60 \times 10^{19}) e^{-0.005t}$.

iv. 4 weeks is 28 days, so $y(28) = (8.60 \times 10^{19}) e^{-0.14} = 7.48 \times 10^{19}$ atoms.

6. **i.** The decay constant is $\lambda = \frac{0.693}{1.83 \times 10^{-3}} = 0.379 \times 10^3 \frac{1}{\text{secs}}$. So $y(t) = 75 e^{-(0.379 \times 10^3)t}$ milligrams.

ii. A reformulation of the question is: $y(?) = \frac{1}{10} \cdot 75$. Set $y(t) = 75 e^{-(0.379 \times 10^3)t} = \frac{75}{10}$, to get $e^{-(0.379 \times 10^3)t} = \frac{1}{10}$, and then $e^{(0.379 \times 10^3)t} = 10$. So $(0.379 \times 10^3)t = \ln 10$, and hence

$$t = \frac{\ln 10}{0.379} \times 10^{-3} = 6.075 \times 10^{-3} \text{ seconds.}$$

7. Because 3.7×10^{10} refers to the number of atoms that decay in one second, we start by converting 1 gram to a number of atoms. By Avogadro's number this is $\frac{1}{226}(6.02 \times 10^{23}) = 2.66 \times 10^{21}$. If $y(t)$ is the number of radium-226 atoms at any time t, then if we consider the measurement to have been made at $t = 0$, we see that $y(0) = 2.66 \times 10^{21}$ and $y'(0) = -3.7 \times 10^{10}$. Applying equation (11b) with $t = 0$ we get $-3.7 \times 10^{10} = -\lambda(2.66 \times 10^{21})$. So $\lambda = \frac{3.7 \times 10^{10}}{2.66 \times 10^{21}} = 1.39 \times 10^{-11} \frac{1}{\text{secs}}$. Therefore, the half-life is $h = \frac{0.693}{1.39} \times 10^{11} = 4.99 \times 10^{10}$ sec. Converting to years, with 1 year $= 3.16 \times 10^7$ seconds, we get

$$h = 4.99 \times 10^{10} \text{secs} \cdot \frac{1 \text{ year}}{3.16 \times 10^7 \text{secs}} = 1.58 \times 10^3 = 1580 \text{ years.}$$

Compare this with the results $\lambda = 1.37 \times 10^{-11}\frac{1}{\text{secs}}$ and $h = 1600$ years on pages 340-41.

8. Take 8 A.M. as time $t = 0$, and let $y(t)$ be the number of atoms in the sample at any time $t \geq 0$ in hours. What do we know? We know that $y'(0) = -3200$ and $y'(9) = -900$, both in atoms per minute. By Section 11.1B, we know that $\ln \frac{y'(9)}{y'(0)} = -\lambda \cdot 9$, where λ is the disintegration constant of the substance. So $\lambda = -\frac{1}{9} \ln \frac{900}{3200} = -\frac{1}{9}(-1.27) \approx 0.14\,\text{hours}^{-1}$. Hence $h = \frac{0.693}{0.14} = 4.95$ hours. Be aware that we "mixed" the units minutes and hours units in this problem. This did not bring about an error because min^{-1} canceled in $\frac{y'(9)}{y'(0)}$.

9. The strategy is the same as that in the problem above. Let $t = 0$ be the moment the first observation is made. Let $y(t)$ be the number of atoms in the sample being tested at any time $t \geq 0$ in minutes. We know that

$$y'(0) \;=\; -8.67 \times 10^{13} \;\text{ and }\; y'(6) \;=\; -7.67 \times 10^{12}.$$

So $-6\lambda \;=\; \ln \frac{y'(6)}{y'(0)} \;=\; \ln \frac{7.67 \times 10^{12}}{8.67 \times 10^{13}} \;=\; \ln 0.088 \;=\; -2.43$. So $\lambda \;=\; -\frac{1}{6}(-2.43) \;\approx\; 0.41\;\text{mins}^{-1}$. Therefore the half-life of the substance is $h = \frac{0.693}{0.41} = 1.69$ minutes.

10. Let $y(t)$ be the number of atoms in the sample at any time $t \geq 0$ in hours, where $t = 0$ is the instant the measurement is taken. So $y'(0) = -6.57 \times 10^{15}$. By equation (11b),

$$y'(0) = -\lambda y(0),$$

where λ is the disintegration constant. If we can determine $y(0)$, we will be able to solve for λ. Since we know $y(0)$ in milligrams we need only to convert to atoms. By Avogadro's number, $y(0) \;=\; \frac{1}{252} \cdot (25 \times 10^{-3})(6.02 \times 10^{23}) \;=\; 0.597 \times 10^{20} \;=\; 5.97 \times 10^{19}$. So

$$\lambda \;=\; -\frac{y'(0)}{y(0)} = \frac{6.57 \times 10^{15}}{5.97 \times 10^{19}} \;=\; 1.10 \times 10^{-4}\,\text{hours}^{-1}.$$

11. Because 0.012% corresponds to 0.00012, there are $(0.48)(0.00012) = 5.76 \times 10^{-5}$ grams of ${}^{40}_{19}\text{K}$ in one liter of seawater. By Avogadro's number this corresponds to

$$(5.76 \times 10^{-5})\frac{1}{40}(6.02 \times 10^{23}) = 8.67 \times 10^{17} \text{ atoms.}$$

Take this measurement as having been made at time $t = 0$. So $y(0) = 8.67 \times 10^{17}$. The rate at which these atoms are disintegrating is $y'(0)$. By formula (11b), $y'(0) = -\lambda y(0)$, where λ is the disintegration constant. Once this is determined in sec^{-1}, a simple plug in will determine $y'(0)$. Because $\lambda = \frac{0.693}{1.3 \times 10^{9}} = 5.33 \times 10^{-10}\frac{1}{\text{years}}$,

$$\lambda \;=\; \frac{5.33 \times 10^{-10}}{\text{year}} \cdot \frac{1 \text{ year}}{3.16 \times 10^{7}\text{ sec}} \;=\; 1.69 \times 10^{-1}\,\text{sec}^{-1}\,.$$

So $y'(0) \;=\; -(1.69 \times 10^{-17})(8.67 \times 10^{17}) \;=\; 14.65$ atoms/sec.

12. **i.** Because 0.35% corresponds to the fraction 0.0035, the mass of the potassium in the person is $(0.0035)(60{,}000) = 210$ grams.

ii. Radioactive potassium ${}^{40}_{19}\mathrm{K}$ constitutes 0.012% or the fraction 0.00012 of all naturally occuring potassium. So the mount of ${}^{40}_{19}\mathrm{K}$ in the person is $(0.00012)(210) = 0.0252$ grams.

iii. The estimate for the number of ${}^{40}_{19}\mathrm{K}$ atoms in the person is provided by Avogadro's number and is $(0.0252)\frac{1}{40}(6.02 \times 10^{23}) = 3.79 \times 10^{20}$.

iv. The decay constant is $\lambda = \frac{0.693}{1.3\times 10^9} = 5.33 \times 10^{-10}\text{years}^{-1}$.

v. Taking the time of the measurement as $t = 0$, we get $y'(0) = -\lambda y(0)$, where (see formula 11b) $y'(0)$ is the rate of disintegrations and $y(0) = 3.79 \times 10^{20}$ is the number of atoms at that time. It remains to convert λ into sec^{-1} and to compute $y'(0)$. Because 1 year $\approx 3.16 \times 10^7$ seconds, $\lambda = \frac{5.33\times 10^{-10}}{\text{year}} \times \frac{1\text{ year}}{3.16\times 10^7\text{ sec}} = 1.69 \times 10^{-17}\ \text{sec}^{-1}$. So $y'(0) = -(1.69 \times 10^{-17})(3.79 \times 10^{20}) = -6.41 \times 10^3$. This is the number of ${}^{40}_{19}\mathrm{K}$ atoms that disintegrate per second.

11B. Matter and Energy

13. Taking the 92 protons and $235 - 92 = 143$ neutrons of a ${}^{235}_{92}\mathrm{U}$ nucleus separately, we get a mass of $92(1.0076) + 143(1.0090) = 92.6992 + 144.2870 = 236.9862$.

14. The difference between the total mass of the individual particles in the nucleus and the nucleus as a whole is $236.9862 - 235.1175 = 1.8687$ amu. Because 1 amu corresponds (see Exercises 11B) to 933 MeV and 14.94×10^{-11} joules, we see that the binding energy of a ${}^{235}_{92}\mathrm{U}$ nucleus is $(1.8687)(933) = 1743.5$ MeV and hence $(1.867)(14.94 \times 10^{-11}) = 2.79 \times 10^{-10}$ joules.

11C. Nuclear Fission

15. The mass before the reaction is $235.118 + 1.009 = 236.127$ amu. After the reaction the mass is $138.950 + 94.936 + 2(1.009) = 235.904$ amu. By Exercises 11B, 1 amu corresponds to 933 MeV. So 0.223 amu corresponds to $(0.223)(933) = 208$ MeV.

11D. Chain Reactions

16. Using Avogadro's number, we see that 1 gram of pure uranium-235 has

$$\frac{1}{235}(6.02 \times 10^{23}) \;=\; 2.56 \times 10^{21}\text{atoms}.$$

So 1 pound of uranium-235 has $(2.56 \times 10^{21})(453.6) = 1.16 \times 10^{24}$ atoms. Hence if all the atoms in one pound of uranium-235 were to undergo fission,

$$(200)(1.16 \times 10^{24}) \;=\; 2.32 \times 10^{26}\text{ MeV}$$

of energy would be produced. By Exercises 11B, this is equal to $(2.32 \times 10^{26})(1.6 \times 10^{-13}) = 3.71 \times 10^{13}$ joules.

17. Note that $rE_{m\text{-total}} = rEN + r^2EN + \cdots + r^{m+1}EN$. So

$$rE_{m\text{-total}} - E_{m\text{-total}} = r^{m+1}EN - EN = (r^{m+1} - 1)EN.$$

Hence $(r-1)E_{m\text{-total}} = (r^{m+1} - 1)EN$ and therefore,

$$E_{m\text{-total}} = \left(\frac{r^{m+1} - 1}{r - 1}\right) EN.$$

18. During the first 10^{-8} seconds, the chain reaction will undergo approximately $\frac{10^{-8}}{10^{-12}} = 10^4$ steps. We will therefore take $m = 10^4$ in the formula. So the energy produced is approximately

$$\begin{aligned} \frac{1.005^{10{,}001} - 1}{1.005 - 1}(200)(100) &= \frac{20000}{0.005}(4.6 \times 10^{21})\text{MeV} = \frac{20 \times 10^3}{5 \times 10^{-3}}(4.6 \times 10^{21}) \\ &= 18.4 \times 10^{27} = 1.84 \times 10^{28}\text{MeV}. \end{aligned}$$

By Exercises 11B, this is equal to $(1.84 \times 10^{28})(1.6 \times 10^{-13}) = 2.9 \times 10^{15}$ joules. By Exercise 16, 3.71×10^{13} joules corresponds to the energy in 2.5 million pounds of coal. So the energy produced by the chain reaction corresponds to the energy in

$$(2.5 \times 10^6)\frac{2.9 \times 10^{15}}{3.71 \times 10^{13}} = (2.5 \times 10^6)(0.78 \times 10^2) = 1.95 \times 10^8$$

pounds of coal.

19. Because $h = 25$ years, the decay constant is $\lambda = \frac{0.693}{25} = 0.0277$ in $(\text{years})^{-1}$. Let $t \geq 0$ be time in years. Taking $t = 0$ to be now, and letting $y(t)$ be the amount of strontium-90 in the sample in milligrams at any time t, we get

$$y(t) = y_0e^{-\lambda t} \approx 20e^{-0.0277t}.$$

So $y(15) \approx 13.2$ milligrams. Solving $5 = 20e^{-0.0277t}$ for t, we get $e^{-0.0277t} = \frac{1}{4} = 0.25$ and hence $-0.0277t = \ln 0.25 = -1.386$. So $t = \frac{1.386}{0.0277} = 50$ years.

11E. Critical Mass

20. Let n_e be the number of neutrons that escape and let n_f be the number of neutrons that can produce a fission. Then there are constants a and b such that $n_e = a(4\pi R^2)$ and $n_f = b(\frac{4}{3}\pi R^3)$. So

$$\frac{n_e}{n_f} = \frac{a \cdot 4\pi R^2}{b \cdot \frac{4}{3}\pi R^3} = \frac{a}{b \cdot \frac{1}{3} \cdot R} = \frac{3a}{b}\,\frac{1}{R}.$$

Therefore, $\frac{n_e}{n_f}$ is proportional to $\frac{1}{R}$.

11G. About the Moon

21. The formula that applies is $t = (6.8 \times 10^{10}) \ln\left(\frac{z(t)}{y(t)} + 1\right)$ where $\frac{z(t)}{y(t)}$ is the ratio of strontium-87 to rubidium-87 in the sample. Taking $t = 4.53 \times 10^9$, we get

$$\ln\left(\frac{z(t)}{y(t)} + 1\right) = \frac{4.53 \times 10^9}{6.8 \times 10^{10}} = 0.067.$$

So $\frac{z(t)}{y(t)} = e^{0.067} - 1 = 0.069$.

22. The formula that applies is $t = (1.89 \times 10^{10}) \ln\left(\frac{9.07z(t)}{y(z)} + 1\right)$ where $\frac{z(t)}{y(t)}$ is the ratio of argon-40 to potassium-40 in the sample. Taking $t = 4.19 \times 10^9$, we get

$$\ln\left(\frac{9.07z(t)}{y(t)} + 1\right) = \frac{4.19 \times 10^9}{1.89 \times 10^{10}} = 0.22.$$

So $\frac{9.07z(t)}{y(t)} = e^{0.22} - 1 \approx 0.246$ and $\frac{z(t)}{y(t)} \approx 0.027$.

11H. Geology and Anthropology

23. At the rate of $\frac{1}{2}$ inch per year, the Red Sea will separate by

$$0.5 \times 100 \times 10^6 \text{ inches} \times \frac{1 \text{ ft.}}{12 \text{ inches}} \times \frac{1 \text{ mile}}{5280 \text{ ft.}} \approx 790 \text{ miles}$$

in 100 million years. So its width would be between 890 and 990 miles.

24. The appropriate formula is $t = (6.8 \times 10^{10}) \ln\left(\frac{z(t)}{y(t)} + 1\right)$ where $\frac{z(t)}{y(t)}$ is the ratio of strontium-87 atoms to rubidium-87 atoms in the sample after time t. So

$$t = (6.8 \times 10^{10}) \ln\left(\frac{4.67}{305} + 1\right) = (6.8 \times 10^{10}) \ln(1.0153) = 1.0 \times 10^9 = 1 \text{ billion years.}$$

25. The operative formula is $t = (6.8 \times 10^{10}) \ln\left(\frac{z(t)}{y(t)} + 1\right)$ where $\frac{z(t)}{y(t)}$ is the ratio of strontium-87 atoms to rubidium-87 atoms in the sample after time t. Because $\frac{z(t)}{y(t)} = \frac{5.3}{420} = 0.0126$, we get

$$t = (6.8 \times 10^{10}) \ln(1.013) = 8.78 \times 10^8 \text{ years.}$$

26. The formula that applies is $t = (1.89 \times 10^9) \ln\left(\frac{9.07z(t)}{y(t)} + 1\right)$ where $\frac{z(t)}{y(t)}$ is the ratio of argon-40 atoms to potassium-40 atoms in the sample after time t. Because

$$\frac{z(t)}{y(t)} = \frac{1.1739 \times 10^{12}}{1.7368 \times 10^{16}} = 6.759 \times 10^{-5},$$

$t = (1.89 \times 10^9) \ln(6.13 \times 10^{-4} + 1) = (1.89 \times 10^9)(6.13 \times 10^{-4}) = 1.16 \times 10^6$ years. So the mineral grain formed 1.16 million years ago.

27. The operative formula is $t = (1.89 \times 10^9)\ln\left(\frac{9.07z(t)}{y(t)} + 1\right)$ where $\frac{z(t)}{y(t)}$ is the ratio of argon-40 atoms to potassium-40 atoms in the sample after time t. Starting with the given inequality

$$0.00011 \le \frac{z(t)}{y(t)} \le 0.00018$$

and using the formula as a guide, we will work "toward t". So

$$\begin{aligned}
9.977 \times 10^{-4} &\le \frac{9.07z(t)}{y(t)} \le 16.326 \times 10^{-4}\\
1.0009977 &\le \frac{9.07z(t)}{y(t)} + 1 \le 1.0016326\\
9.972 \times 10^{-4} &\le \ln\left(\frac{9.07z(t)}{y(t)} + 1\right) \le 16.313 \times 10^{-4}\\
18.847 \times 10^5 &\le (1.89 \times 10^9)\ln\left(\frac{9.07z(t)}{y(t)} + 1\right) \le 30.832 \times 10^5.
\end{aligned}$$

It follows that t is between 1.88 and 3.1 million years old.

28. We are given that the ratio of radioactive carbon-14 atoms to stable atoms is $\frac{y(t)}{k} = \frac{1}{1.573 \times 1012}$. So

$$\begin{aligned}
t &= (8.26 \times 10^3)\ln\left(r_0 \frac{k}{y(t)}\right) = (8.26 \times 10^3)\ln\left(\frac{1.573 \times 10^{12}}{6.463 \times 10^{11}}\right)\\
&= (8.26 \times 10^3)\ln 2.434 = 7.35 \times 10^3 \text{ years.}
\end{aligned}$$

29. **i.** $r_0 = \frac{1}{6.463 \times 10^{11}}$ grams.

ii. By Avogadro's number there are $\frac{1}{14}(6.02 \times 10^{23})$ ${}^{14}_{6}\mathrm{C}$ atoms in one gram of carbon-14. So

$$y_0 = \frac{1}{6.463 \times 10^{11}} \cdot \frac{1}{14} \cdot (6.02 \times 10^{23}) = \frac{602}{(6.463)(14)} \times 10^{10} = 6.65 \times 10^{10} \text{atoms.}$$

iii. Observe that $y'(0) = -15.3$ atoms per minute. So $-15.3 = -\lambda(6.65 \times 10^{10})$ and hence $\lambda = \frac{15.3}{6.65} \times 10^{-10} = 2.30 \times 10^{-10}\ \text{min}^{-1}$.

iv. $h = \frac{0.693}{\lambda} = \frac{0.693}{2.30} \times 10^{10} = 3.01 \times 10^9$ minutes $\approx \frac{3.01 \times 10^9}{5.26 \times 10^5} \approx 5772$ years.

30. This problem calls for the use of the formula $t = (8.26 \times 10^3)\ln\left(r_0 \frac{k}{y(t)}\right)$. We are told that t is between 14.3 thousand and 15 thousand years and asked to say something about $\frac{y(t)}{k}$. Proceeding as suggested by the formula, we get

$$14.3 \times 10^3 \le (8.26 \times 10^3)\ln\left(r_0 \frac{k}{y(t)}\right) \le 15 \times 10^3$$

$$\begin{aligned}
1.73 &\leq \ln\left(r_0 \frac{k}{y(t)}\right) \leq 1.82 \\
5.64 &\leq e^{1.73} \leq \left(r_0 \frac{k}{y(t)}\right) \leq e^{1.82} \leq 6.18 \\
\frac{5.64}{r_0} &\leq \frac{k}{y(t)} \leq \frac{6.18}{r_0} \\
\frac{r_0}{5.64} &\geq \frac{y(t)}{k} \geq \frac{r_0}{6.18} \\
\frac{1}{3.65 \times 10^{12}} &\geq \frac{y(t)}{k} \geq \frac{1}{4.0 \times 10^{12}}
\end{aligned}$$

31. The applicable formula is $t = (8.26 \times 10^3) \ln\left(r_0 \frac{k}{y(t)}\right)$. Taking $t = 1.75 \times 10^6$ and solving for $\frac{y(t)}{k}$ we get,

$$\begin{aligned}
\ln\left(r_0 \frac{k}{y(t)}\right) &= \frac{1.75 \times 10^6}{8.26 \times 10^3} = 2.12 \times 10^2 \\
r_0 \frac{k}{y(t)} &= e^{2.12 \times 10^2} \approx 1.176 \times 10^{92} \\
\frac{y(t)}{k} &\approx \frac{r_0}{1.176 \times 10^{92}} \approx \frac{1}{(1.18 \times 10^{92})(6.46 \times 10^{11})} \\
&\approx \frac{1}{7.62 \times 10^{103}}.
\end{aligned}$$

This number is as close to 0 as any of you will ever encounter. So effectively all the ${}^{14}_{6}\mathrm{C}$ atoms will have disintegrated.

11I. Integrals and Equations involving Derivatives

32. **i.** Set $\frac{1}{x(x-1)} = \frac{C}{x} + \frac{D}{x-1}$ and solve for C and D. By taking common denonimators,

$$\frac{1}{x(x-1)} = \frac{C}{x} + \frac{D}{x-1} = \frac{C(x-1) + Dx}{x(x-1)} = \frac{(C+D)x - C}{x(x-1)}.$$

A comparison of numerators tells us that $C + D = O$ and $-C = 1$. So $C = -1$ and $D = +1$. Therefore, $\frac{1}{x(x-1)} = \frac{-1}{x} + \frac{1}{x-1}$. So

$$\int \frac{dx}{x(x-1)} = -\int \frac{1}{x} dx + \int \frac{1}{x-1} dx.$$

By using the formula $\frac{d}{dx} \ln g(x) = \frac{g'(x)}{g(x)}$ in Section 10.3, we get

$$\int \frac{dx}{x(x-1)} = -\ln x + \ln(x-1) + C.$$

ii. Again, set $\frac{1}{(x-2)(x-3)} = \frac{C}{x-2} + \frac{D}{x-3}$ and solve for C and D. Doing so, we get

$$\frac{1}{(x-2)(x-3)} = \frac{C}{x-2} + \frac{D}{x-3} = \frac{C(x-3)+D(x-2)}{(x-2)(x-3)} = \frac{(C+D)x-3C-2D}{(x-2)(x-3)}\,.$$

By comparing numerators, $C+D=0$ and $-3C-2D=1$. So $D=-C$ and $-3C+2C=1$. So. $C=-1$ and $D=1$. Hence $\frac{1}{(x-2)(x-3)} = \frac{-1}{x-2} + \frac{1}{x-3}$. So

$$\int \frac{1}{(x-2)(x-3)}\,dx = -\int \frac{1}{x-2}\,dx + \int \frac{1}{x-3}dx.$$

By using the formula $\frac{d}{dx}\ln g(x) = \frac{g'(x)}{g(x)}$, we get

$$\int \frac{1}{(x-2)(x-3)}\,dx = -\ln(x-2) + \ln(x-3) + C\,.$$

iii. Set $\frac{x}{(x-2)(x+3)} = \frac{C}{x-2} + \frac{D}{x+3}$ and solve for C and D. Doing this, we get

$$\begin{aligned}\frac{x}{(x-2)(x+3)} &= \frac{C}{x-2} + \frac{D}{x+3} = \frac{C(x+3)+D(x-2)}{(x-2)(x+3)}\\ &= \frac{(C+D)x+3C-2D}{(x-2)(x+3)}.\end{aligned}$$

Comparing numerators tells us that $C+D=1$ and $3C-2D=0$. So $C=\frac{2}{3}D$ and hence $\frac{2}{3}D+D=1$. So $\frac{5}{3}D=1$ and $D=\frac{3}{5}$. Hence $C=1-\frac{3}{5}=\frac{2}{5}$. Therefore, $\frac{x}{(x-2)(x+3)} = \frac{\frac{2}{5}}{x-2} + \frac{\frac{3}{5}}{x+3}$. So

$$\begin{aligned}\int \frac{x}{(x-2)(x+3)}\,dx &= \int \frac{\frac{2}{5}}{x-2}\,dx + \int \frac{\frac{3}{5}}{x+3}\,dx\\ &= \frac{2}{5}\ln(x-2) + \frac{3}{5}\ln(x+3) + C.\end{aligned}$$

iv. Doing the same thing once more, we see that

$$\begin{aligned}\frac{x+1}{(x+2)(x-3)} &= \frac{C}{x+2} + \frac{D}{x-3} = \frac{C(x-3)+D(x+2)}{(x+2)(x-3)}\\ &= \frac{(C+D)x-3C+2D}{(x+2)(x-3)}\,.\end{aligned}$$

So $C+D=1$ and $-3C+2D=1$. Because $D=1-C$, we get $-3C+2(1-C)=1$; hence $-3C-2C=-1$ and $C=\frac{1}{5}$. So $D=\frac{4}{5}$. Therefore,

$$\begin{aligned}\int \frac{x+1}{(x+2)(x-3)}\,dx &= \int \frac{\frac{1}{5}}{x+2}\,dx + \int \frac{\frac{4}{5}}{x-3}\,dx\\ &= \frac{1}{5}\ln(x+2) + \frac{4}{5}\ln(x-3) + C\,.\end{aligned}$$

33. **i.** By separating variables we get $\frac{dy}{(y-2)(y+4)} = 3dt$. So

$$\int \frac{dy}{(y-2)(y+4)} = \int 3dt.$$

To solve the integral on the left, put $\frac{1}{(y-2)(y+4)} = \frac{C}{y-2} + \frac{D}{y+4}$. Taking common denominators, we get

$$\begin{aligned} \frac{C}{y-2} + \frac{D}{y+4} &= \frac{C(y+4) + D(y-2)}{(y-2)(y+4)} \\ &= \frac{(C+D)y + 4C - 2D}{(y-2)(y+4)}. \end{aligned}$$

It follows by comparing coefficients that $C + D = 0$ and $4C - 2D = 1$. So $D = -C$ and $4C + 2C = 1$. Hence $C = \frac{1}{6}$ and $D = -\frac{1}{6}$. We have shown that

$$\frac{1}{(y-2)(y+4)} = \frac{\frac{1}{6}}{y-2} - \frac{\frac{1}{6}}{y+4}.$$

Because $\ln(y-2)$ and $\ln(y+4)$ are anti-derivatives of $\frac{1}{y-2}$ and $\frac{1}{y+4}$ respectively, we get

$$\begin{aligned} \int \frac{dy}{(y-2)(y+4)} &= \frac{1}{6}\int \frac{dy}{y-2} - \frac{1}{6}\int \frac{dy}{y+4} \\ &= \frac{1}{6}\ln(y-2) - \frac{1}{6}\ln(y+4) + C_1. \end{aligned}$$

Because $\int 3\,dt = 3t + C_2$, we get, after setting $C = C_2 - C_1$, that $\frac{1}{6}\left[\ln(y-2) - \ln(y+4)\right] = 3t + C$. We need to solve for y in terms of t. By basic properties of the log function $\ln\left(\frac{y-2}{y+4}\right) = 18t + 6C$ and $\frac{y-2}{y+4} = e^{18t+6C} = Ae^{18t}$, where A is the constant $A = e^{6C}$. Using the fact that $y = 4$ when $t = 0$, we get $\frac{4-2}{4+4} = Ae^0 = A$. So $A = \frac{1}{4}$, and hence

$$\frac{y-2}{y+4} = \frac{1}{4}e^{18t}.$$

Solving for y we get

$$y - 2 = \frac{1}{4}e^{18t}(y+4) = \frac{1}{4}e^{18t}y + e^{18t}.$$

Therefore, $y - \frac{1}{4}e^{18t}y = 2 + e^{18t}$, $y(1 - \frac{1}{4}e^{18t}) = 2 + e^{18t}$, and finally, $y = \frac{2+e^{18t}}{1-\frac{1}{4}e^{18t}}$. So

$$y = y(t) = \frac{8 + 4\,e^{18t}}{4 - e^{18t}}.$$

ii. After separating variables, we get

$$\int \frac{(y+1)}{(y-2)(y+4)}dy = \int 2dt.$$

Proceeding as in Section 11.5B, we get

$$\begin{aligned}\frac{y+1}{(y-2)(y+4)} &= \frac{C}{y-2} = \frac{D}{y+4}\\ &= \frac{C(y+4)+D(y-2)}{(y-2)(y+4)}\\ &= \frac{(C+D)y+4C-2D}{(y-2)(y+4)}.\end{aligned}$$

By comparing numerators, we see that $C+D=1$ and $4C-2D=1$. To solve for C and D, add $2C+2D=2$ to $4C-2D=1$ to get $6C=3$. So $C=\frac{1}{2}$ and $D=1-\frac{1}{2}=\frac{1}{2}$. Therefore,

$$\frac{y+1}{(y-2)(y+4)} = \frac{\frac{1}{2}}{y-2} + \frac{\frac{1}{2}}{y+4}.$$

So

$$\begin{aligned}\int \frac{y+1}{(y-2)(y+4)}\,dy &= \int \frac{\frac{1}{2}}{y-2}\,dy + \int \frac{\frac{1}{2}}{y+4}dy\\ &= \frac{1}{2}\ln(y-2) + \frac{1}{2}\ln(y+4) + C_1.\end{aligned}$$

Because $\int 2dt = 2t + C_2$, we get $\frac{1}{2}\ln(y-2)+\frac{1}{2}\ln(y+4) = 2t + C_2 - C_1$. So

$$\ln(y-2)+\ln(y+4) = 4t + C,$$

where $C = 2(C_2 - C_1)$. Using properties of the natural log and solving for y we get $\ln\left[(y-2)(y+4)\right] = 4t + C$, and hence $(y-2)(y+4) = e^{4t+C} = e^C e^{4t} = A$. Because $y=3$ when $t=0$, we see that $(3-2)(3+4) = Ae^0 = A$. So $A=7$, and hence

$$(y-2)(y+4) = 7e^{4t}.$$

How do we solve this for y? By using the quadratic formula! Because

$$(y-2)(y+4) = y^2 + 2y - 8,$$

we see that $y^2 + 2y - (7e^{4t}+8) = 0$. Hence

$$y = \frac{-2 \pm \sqrt{4 + 4(7e^{4t}+8)}}{2} = \frac{-2 \pm 2\sqrt{1+(7e^{4t}+8)}}{2} = -1 \pm \sqrt{9+7e^{4t}}.$$

So the requirements are satisfied by both

$$y(t) = -1 + \sqrt{9+7e^{4t}} \quad\text{and}\quad y(t) = -1 - \sqrt{9+7e^{4t}}\,.$$

iii. By separating variables,

$$\int \frac{dy}{(y-2)(y+4)} = \int t\,dt.$$

The integral on the left was already solved in (**i**) as

$$\int \frac{dy}{(y-2)(y+4)} = \frac{1}{6}\ln(y-2) - \frac{1}{6}\ln(y+4) + C_1.$$

The one on the right is equal to $\frac{t^2}{2} + C_2$ and hence

$$\frac{1}{6}\ln(y-2) - \frac{1}{6}\ln(y+4) = \frac{t^2}{2} + C_2 - C_1.$$

Letting $C = C_2 - C_1$ and solving for y by using properties of the log gives us

$$\frac{1}{6}\ln\left(\frac{y-2}{y+4}\right) \;=\; \frac{t^2}{2} + C \quad\text{and}\quad \ln\left(\frac{y-2}{y+4}\right) \;=\; 3t^2 + 6C,$$

and hence

$$\frac{y-2}{y+4} \;=\; e^{3t^2+6C} = e^{6C}e^{3t^2} \;=\; Ae^{3t^2}.$$

Because $y(0) = 6$, we find that $\frac{6-2}{6+4} = Ae^0 = A$. So $A = \frac{4}{10} = 0.4$. Therefore, $\frac{y-2}{y+4} = (0.4)e^{3t^2}$, so $y - 2 = (0.4)e^{3t^2}(y+4)$, and hence $y - 0.4e^{3t^2}y = 2 + 1.6e^{3t^2}$. Therefore, $y(1 - 0.4e^{3t^2}) = 2 + 1.6e^{3t^2}$ and $y = \frac{2+1.6e^{3t^2}}{1-0.4e^{3t^2}}$. It follows that

$$y(t) \;=\; \frac{2 + 1.6e^{3t^2}}{1 - 0.4e^{3t^2}}$$

is the required function.

iv. Separation of the variables gives us

$$\int \frac{y+1}{(y-2)(y+4)}\; dy = \int 5t\; dt.$$

The integral on the left was already solved in (**ii**) as

$$\int \frac{y+1}{(y-2)(y+4)}\; dy = \frac{1}{2}\ln(y-2) + \frac{1}{2}\ln(y+4) + C_1.$$

Because the one on the right is equal to $\frac{5}{2}t^2 + C_2$, we get $\frac{1}{2}\ln(y-2) + \frac{1}{2}\ln(y+4) = \frac{5}{2}t^2 + C_2 - C_1$. Solving for y and rewriting the constants, we get

$$\begin{aligned} \frac{1}{2}\ln\left[(y-2)(y+4)\right] &= \frac{5}{2}t^2 + C_2 - C_1 \\ \ln\left[(y-2)(y+4)\right] &= 5t^2 + 2\;(C_2 - C_1) \\ (y-2)(y+4) &= e^{5t^2+C} = e^C e^{5t^2} = Ae^{5t^2}. \end{aligned}$$

Using the fact that $y(0) = 6$, we get $(6-2)(6+4) = Ae^0 = A$. So $A = 40$, and $(y-2)(y+4) = 40e^{5t^2}$. Solving for y as in (**ii**), we get $y^2 + 2y - 8 = 40e^{5t^2}$ and hence $y^2 + 2y - (40e^{5t^2} + 8) = 0$. It follows by the quadratic formula that

$$y \;=\; \frac{-2 \pm \sqrt{4 + 4(40e^{5t^2} + 8)}}{2} \;=\; \frac{-2 \pm 2\sqrt{1 + (40e^{5t^2} + 8)}}{2} \;=\; -1 \pm \sqrt{40e^{5t^2} + 9}.$$

So the requirements are met by both

$$y(t) = -1 + \sqrt{40e^{5t^2} + 9} \;\text{ and }\; y(t) = -1 - \sqrt{40e^{5t^2} + 9}\,.$$

v. By separating variables,

$$\int \frac{dy}{(y-2)^2} = \int 2dt\,.$$

Notice that $-x^{-1}$ is an antiderivative of $\frac{1}{x^2} = x^{-2}$. Check in the same way that $-(y-2)^{-1}$ is an antiderivative of $\frac{1}{(y-2)^2}$. Therefore,

$$\int \frac{dy}{(y-2)^2} = -(y-2)^{-1} + C_1.$$

Because $\int 2\; dt = 2t + C_2$, we get $-(y-2)^{-1} \;=\; 2t + C_2 - C_1 = 2t + C$. So $y - 2 \;=\; \frac{-1}{2t+C}$, and therefore, $y \;=\; 2 - \frac{1}{2t+C}$. Using the condition $y(0) \;=\; 1$, we get $1 \;=\; 2 - \frac{1}{C}$. So $\frac{1}{C} = 2 - 1 = 1$, and $C = 1$. Therefore, $y(t) = 2 - \frac{1}{2t+1}$.

11J. The Logistics Model

34. **i.** A look at equation (11j) shows that $\mu \;=\; 0.08$ and $k \;=\; 0.02$. So $M \;=\; \frac{0.08}{0.02} \;=\; 4$. By equation (11n),

$$y(t) \;=\; \frac{4}{1 + (\frac{4}{2.5} - 1)e^{-0.08t}} \;=\; \frac{4}{1 + 0.6e^{-0.08t}}.$$

ii. By equation (11j), $\mu = 0.02$ and $k = 0.08$. So $M = \frac{0.02}{0.08} = 0.25$. By (11n),

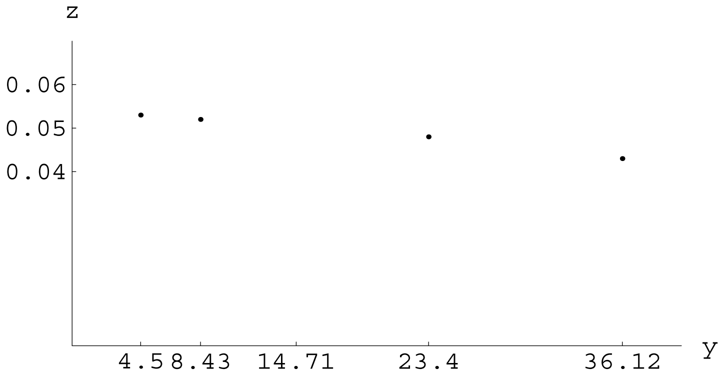
$$y(t) \;=\; \frac{0.25}{1 + \left(\frac{0.25}{2.5} - 1\right) e^{-0.02t}} \;=\; \frac{0.25}{1 - 0.9e^{-0.02t}}.$$

35. From $\frac{y'}{y} \;=\; 0.0261 - 0.0018y$ we get $\mu \;=\; 0.0261$ and $k \;=\; 0.0018$. So $M \;=\; \frac{\mu}{k} \;=\; \frac{0.0261}{0.0018} \;=\; 14.5$ billion. Taking $t = 0$ in 1965 we have $y_0 = 3.34$. Substituting these constants into equation (11n), we get

$$y(t) \;=\; \frac{14.5}{1 + \left(\frac{14.5}{3.34} - 1\right) e^{-0.0261t}} \;=\; \frac{14.5}{1 + 3.34e^{-0.0261t}}.$$

For the year 2000, we get a population of $y(35) = 6.20$ billion. For the year 2010, $y(45) = 7.14$ billion. For 2020, $y(55) = 8.08$ billion. For 2050, $y(85) = 10.64$ billion, and for the year 2095, we get $y(130) = 13.04$ billion.

36. The data fall close to the line through (8.43, 0.052) and (36.12, 0.043). This line's slope is $\frac{0.052-0.043}{8.43-36.12} = -\frac{0.009}{27.69} \approx -0.00033$. So an equation for the line is $z - 0.052 = -0.00033(y - 8.43)$

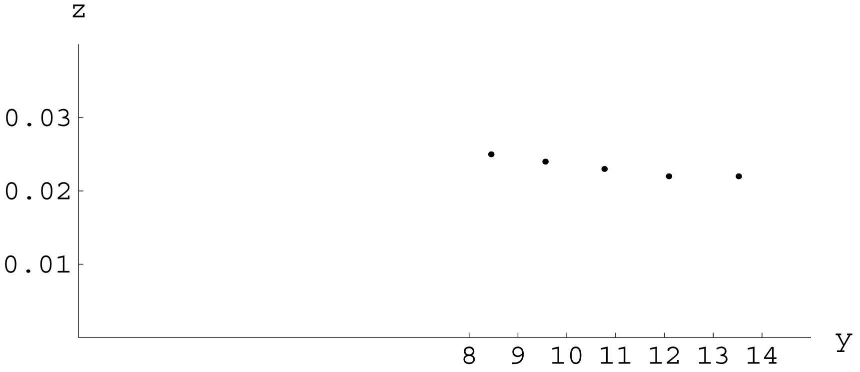


or $z = 0.055 - 0.00033y$. Hence $k = 0.00033$ and $\mu = 0.055$, so $M = \frac{0.055}{0.00033} = 166.67$. Letting $t = 0$ correspond to 1950, we get $y_0 = 4.50$. So

$$y(t) \;=\; \frac{166.67}{1 + \left(\frac{166.67}{4.50} - 1\right) e^{-0.055t}} \;=\; \frac{166.67}{1 + 36.04e^{-0.055t}}.$$

The limit on the population is 166.67 million.

37. **i.** The data do fall close to a line, namely the line through (8.45, 0.025) and (12.09, 0.022).



The line's slope is $\frac{0.025-0.022}{8.45-12.09} = -\frac{0.003}{3.64} \approx -0.00082$, and its equation is

$$z - 0.025 = -0.00082(y - 8.45) \quad \text{or} \quad z = 0.032 - 0.00082y.$$

Hence $k = 0.00082$ and $\mu = 0.032$.

ii. $M = \frac{0.032}{0.00082} \approx 39.02$.

iii. Take $t = 0$ in 1975, so $y_0 = 8.4$. Subsitituting $M = 39.02, y_0 = 8.45$, and $\mu = 0.032$ into equation (11n), we get

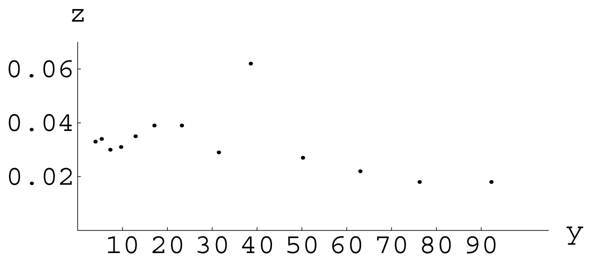
$$y(t) \;=\; \frac{39.02}{1 + \left(\frac{39.02}{8.45} - 1\right) e^{-0.032t}} \;=\; \frac{39.02}{1 + 3.62e^{-0.032t}}.$$

iv. The year 2020 corresponds to $t = 45$, and $y(45) = 21$ million.

38. **i.**

	1790	1800	1810	1820	1830	1840	1850	1860	1870	1880	1890	1900	1910
y	3.93	5.31	7.24	9.64	12.87	17.07	23.19	31.44	38.56	50.19	62.98	76.21	92.23
$\frac{y'}{y}$	0.033	0.034	0.030	0.031	0.035	0.039	0.039	0.029	0.062	0.027	0.022	0.018	0.018

ii. The table provides the plot



iii. The line determined by the points (12.87, 0.035) and (92.23, 0.018) seems to fit the data fairly well. The slope of this line is

$$\frac{0.035 - 0.018}{12.87 - 92.23} = -\frac{0.017}{79.36} \approx -0.00021.$$

Since (12.87, 0.035) is on the line, we get the equation $z - 0.035 = -0.00021(y - 12.87)$. So

$$\frac{y'}{y} = z = 0.038 - 0.00021 = 0.038 - 0.00021.$$

Therefore, $\mu = 0.038$ and $k = 0.00021$. The predicted limiting value of the population is $M = \mu k^{-1} = \frac{0.038}{0.00021} = 180.95$ million.

iv. Plugging $y_0 = 3.93, \mu = 0.038$ and $M = 180.95$ into equation (11n) we get

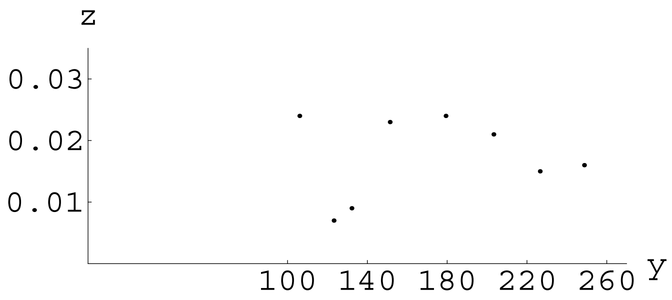
$$y(t) = \frac{180.95}{1 + \left(\frac{180.95}{3.93} - 1\right) e^{-0.038t}} = \frac{180.95}{1 + 45.04e^{-0.038t}}.$$

v. Note that 1930 corresponds to $t = 140$, 1950 to $t = 160$, 1970 to $t = 180$, and 1990 to $t = 200$. The U.S. population data (in millions) for these years are 123.20, 151.33, 203.30 and 248.71, and this corresponds respectively to the values $y(140) = 148.27, y(160) = 164.04, y(180) = 172.63$, and $y(200) = 176.96$.

vi.

	1920	1930	1940	1950	1960	1970	1980	1990
y	106.02	123.20	132.16	151.33	179.32	203.30	226.54	248.71
$\frac{y'}{y}$	0.024	0.007	0.009	0.023	0.024	0.021	0.015	0.016

The corresponding plot with $z = \frac{y'}{y}$ is



The case for the assertion that these points fall along a straight line is not completely convincing.

39. We will assume that the court of 4000 was observed at time $t = 0$. So $y(t) = 4000e^{\mu t}$. Because $6000 = 4000e^{12\mu}$, we get $e^{12\mu} = \frac{3}{2}$. So $12\mu = \ln\frac{3}{2} \approx 0.405$. Hence $\mu \approx 0.034$. Finally, $d = \frac{0.693}{\mu} \approx 20.38$ hours.

40. Starting with $y(t) = y_0e^{\mu t}$, we get $y(t) = 1000e^{\mu t}$ and $y(2) = 4000 = 1000e^{2\mu}$.

- **i.** We get $e^{2\mu} = 4$, so $2\mu = \ln 4 \approx 1.386$ and $\mu \approx 0.693$. Therefore $y(t) = 1000e^{0.693t}$.
- **ii.** $y(5) \approx 1000e^{(0.693)5} \approx 32,000$.
- **iii.** Setting $y(t) = 30,000$, we get $e^{0.693t} = 30$. Hence $0.693t = \ln 30 \approx 3.401$, and $t \approx 4.91$ hours.

41. Using $y(t) = y_0e^{\mu t}$, we get $y(t) = 10{,}500e^{\mu t}$ and $y(2) = 23{,}000 = 10{,}500\ e^{2\mu}$.

- **i.** So $e^{2\mu} = \frac{23{,}000}{10{,}500} \approx 2.190$. Hence $2\mu \approx \ln 2.190 \approx 0.784$ and $\mu \approx 0.392$. Therefore, $y(t) \approx 10,500e^{0.392t}$.
- **ii.** $y(6) \approx 10,500e^{(0.392)6} \approx 110{,}000$.
- **iii.** Setting $y(t) = 130{,}000$, we get $e^{0.392t} = \frac{130{,}000}{10{,}500} \approx 12.381$ and hence $0.392t \approx 2.516$. So $t = 6.42$ hours.

42. The simulation $y(t) = y_0e^{\mu t}$ applies. Setting $y(2) = 5,000$ and $y(7) = 256,000$, we get $5,000 = y_0e^{2\mu}$ and $256,000 = y_0e^{7\mu}$.

- **i.** We need to solve for y_0. Because $e^{2\mu} = \frac{5000}{y_0}$, we have $e^{\mu} = \frac{5000^{\frac{1}{2}}}{(y_0)^{\frac{1}{2}}}$. Therefore,

$$y_0 = \frac{256{,}000}{e^{7\mu}} = 256{,}000(e^{-\mu})^7 = 256{,}000\frac{y_0^{\frac{7}{2}}}{5000^{\frac{7}{2}}}.$$

Therefore $y_0^{-\frac{5}{2}} = \frac{256{,}000}{5000^{\frac{7}{2}}}$, hence $y_0^{\frac{5}{2}} = \frac{5000^{\frac{7}{2}}}{256{,}000} \approx 34.5$ million. So $y_0 \approx 1035$.

- **ii.** Since $5000 = 1035e^{2\mu}$, we get $e^{2\mu} \approx 4.83$. So $2\mu \approx \ln 4.831 \approx 1.575$ and $\mu \approx 0.79$. So $y(t) \approx 1035\ e^{0.79t}$.
- **iii.** Because $\mu = \frac{0.693}{d}$, we get $d = \frac{0.693}{0.79} = 0.88$ hours or about 53 minutes.

43. As in Section 11.6B, we will start with the equation $y(t) = \frac{M}{1+(\frac{M}{y_0-1})e^{-\mu t}}$ and use the data of the experiment to determine the parameters y_0, M, and μ. A look at Table 11.6 tells us that $y_0 = 64,250$ and that M can taken to be 200,000,000. So

$$y(t) = \frac{2 \times 10^8}{1 + 3,110e^{-\mu t}}.$$

Using the cell counts at $t = 0$ and $t = 3$, we get

$$\begin{aligned} m &= \frac{\log_{10}\ y(3) - \log_{10}\ y(0)}{3-0} \\ &= \frac{\log_{10}\ 16,250,000 - \log_{10}\ 64,250}{3} \\ &= \frac{2.40}{3} = 0.80. \end{aligned}$$

Therefore $\mu = (2.3)(0.8) = 1.8$ as in Section 11.6B. Hence

$$y(t) = \frac{2 \times 10^8}{1 + 3,110e^{-1.8t}}.$$

Substituting $t = \frac{1}{2}, 1, 2, \ldots, 7$, we get $y(\frac{1}{2}) = 158,000; y(1) = 388,000; y(2) = 2,330,000; y(3) = 13,300,000; y(4) = 60,200,000; y(5) = 144,500,000; y(6) = 188,000,000; y(7) = 198,000,000$.

44. Compare Gause's equation with equation (11n). Notice that if Gause's equation is rewritten as

$$y(t) = \frac{375}{1 + e^{5.169} \cdot e^{-2.309t}}$$

then it has the same form as (11n) with $M = 375, \mu = 2.309$, and $\frac{M}{y_0} - 1 = e^{5.169} \approx 176$. So $\frac{M}{y_0} \approx 177$, and $y_0 \approx \frac{M}{177} \approx 2$. The fact that $y_0 \approx 2$ and not 5 is some indication of the discrepancy between the mathematical model and the observed reality.

45. Consider the function $f(y) = y(a\mu + b - \frac{a\mu}{M}y)$ where y is the number of microbes per milliliter of medium. Note that

$$f'(y) = -\frac{2a\mu}{M}y + a\mu + b$$

So $f'(y) = 0$ when $\frac{2a\mu}{M}y = a\mu + b$, so when

$$y = \frac{M}{2a\mu}(a\mu + b) = \frac{M}{2}(1 + \frac{b}{a\mu}).$$

When $y > \frac{M}{2a\mu}(a\mu+b)$, then $\frac{2a\mu}{M}y > a\mu+b$ and it follows that $f'(y) < 0$. When $y < \frac{M}{2a\mu}(a\mu+b)$, then $\frac{2a\mu}{M}y < a\mu + b$ and $f'(y) > 0$. So $f(y)$ attains its maximum value when $y = \frac{M}{2a\mu}(a\mu + b)$.

11L. Gompertz's Model

46. **i.** Because $\frac{d}{dt}\ln y(t) = \frac{1}{y(t)}y^{\prime(t)}$ and $\frac{d}{dt}\left(-me^{-kt}\right) = mke^{kt}$, we see that the functions $\ln y(t)$ and $-me^{-kt}$ have the same derivative. They therefore differ by a constant. (See Section 5.5B for instance.)

ii. From $\ln y(t) = -me^{-kt}+C$, we get $y(t) = e^{\ln y(t)} = e^{-me^{-kt}+C} = e^{C}e^{-me^{-kt}}$. By taking $t = 0$, we get $y_0 = y(0) = e^{C}e^{-m}$. So $e^{C} = y_0e^{m}$ and hence $y(t) = y_0e^{m}e^{-me^{-kt}} = y_0e^{m-me^{-kt}}$.

iii. When t is pushed to infinity, $e^{-kt} = \frac{1}{e^{kt}}$ goes to zero. So $y(t) = y_0e^{m}e^{-me^{-kt}}$ goes to y_0e^{m}.

iv. Since $y'(t) = y(t)mke^{-kt}$, we get

$$\begin{aligned} y'' &= y'(t)mke^{-kt} - y(t)mk^2e^{-kt} \\ &= y(t)mke^{-kt}mke^{-kt} - y(t)mk^2e^{-kt} \\ &= y(t)mk^2e^{-kt}\left[me^{-kt} - 1\right]. \end{aligned}$$

v. After setting $y'' = 0$ we get $me^{-kt} = 1$ and hence $e^{kt} = m$. So $t = \frac{1}{k}\ln m$. For $t > \frac{1}{k}\ln m$ we get $kt > \ln m$. Because e^{x} is an increasing function, $e^{kt} > e^{\ln m} = m$ and hence $1 > e^{-kt}m$. It follows from (iv) that $y(t)$ is concave down for $t > \frac{1}{k}\ln m$. In the same way, $y(t)$ is concave up for $t < \frac{1}{k}\ln m$. Therefore $y(t)$ has an inflection point when $t = \frac{1}{k}\ln m$. When $m \leq 1$, $\frac{1}{k}\ln m \leq 0$. So $y(t)$ is concave up for $t \leq 0$. But by assumption $t \geq 0$. So y is never concave up; hence only concave down; and there is no point of inflection.

47. **i.** Using $y_0 = 2850$ and (see Exercise 46 *iii*) $100,000,000 = y_0e^{m} = 2850e^{m}$, we get $m = \ln\frac{100{,}000{,}000}{2850} = 10$. So $y(t) = 2850e^{10.47-10.47e^{-kt}}$. With $t = 3$, we get $625,000 = 2850e^{10.47-10.47e^{-3k}}$. So $10.47 - 10.47e^{-3k} = \ln\frac{625{,}000}{2850} = 5.39$. Hence $-3k = \ln\frac{5.39-10.47}{-10.47} = -0.72$ and $k = 0.24$.

ii. From (i) we see that $y(t) = 2850e^{10.47-10.47^{-0.24t}}$. For $t = 2, 5$, and 6 we get the values 154,000, 42,900,000 and 84,000,000. A look at Table 11.7 shows that the fit of the logistics model is considerably better.

48. **i.** We are given that $\frac{y'(t)}{y(t)} = mke^{-kt}$ for some constants m and k and $t \geq 0$. For $t = 0$, we get $mk = \frac{y'(0)}{y(0)} = 0.021$. So $\frac{y'(t)}{y(t)} = 0.021e^{-kt}$. So when $t = 30$, $0.016 = \frac{y'(30)}{y(30)} = 0.021e^{-30k}$. Thus $e^{30k} = \frac{0.021}{0.016}$, so $30k = \ln\frac{0.021}{0.016}$ and hence $k = \frac{1}{30}\ln\frac{21}{16} \approx 0.009$. Hence $m \approx \frac{0.021}{0.009} \approx 2.317$. So by Exercise 46$(ii)$, $y(t) = 3.34e^{2.317-2.317e^{-0.009t}}$ in billions of people.

ii. Observe from (i) that $\frac{y'(t)}{y(t)} = 0.021e^{-0.009t}$. Taking $t = 0, 5, \ldots, 30$ we get that the specific growth rates for 1965, 1970, ... , 1995 are 0.021, 0.020, 0.019, 0.018, 0.018, 0.017 and 0.016 respectively. The approximation $\frac{y'}{y} \approx 0.028 - 0.002y$ and Table 11.4 provides the specific growth rates 0.021, 0.021, 0.020, 0.019, 0.018, 0.017 and 0.016 for the same sequence of years. A look at Table 11.4 shows that Gompertz's model fits a bit better.

iii. Return to the formula $y(t) = 3.34e^{2.317-2.317e^{-0.009t}}$. Taking $t = 35, 55, 85$, and 130 gives the projections (in billions) 6.25, 8.25, 11.53, and 16.51, respectively, for the years 2000, 2020, 2050, and 2095.

iv. Pushing t to infinity in the expression for $y(t)$ gives $3.34e^{2.317} = 33.88$ billion.

11J. Monod's Equation

49. i. Monod's equation specializes to

$$\frac{y'}{y} = \frac{0.6s(t)}{9+s(t)}.$$

ii. With $s(t) = \frac{50}{t^2}+1$, we get

$$\frac{y'}{y} = \frac{0.6\left[\frac{50}{t^2}+1\right]}{9+\frac{50}{t^2}+1} = \frac{0.6\left[50+t^2\right]}{10t^2+50} = \frac{30}{10t^2+50} + \frac{0.6t^2}{10t^2+50},$$

so that

$$\frac{y'}{y} = \frac{3}{t^2+5} + \frac{0.06t^2}{t^2+5}.$$

How do we solve this equation for $y(t)$? Start with a fact from Section 10.3. Because

$$\frac{d}{dt}\ln y(t) = \frac{1}{y(t)}\cdot y'(t) = \frac{y'(t)}{y(t)}$$

we see that $\ln y(t)$ is an antiderivative of $\frac{y'}{y}$. If we can find an antiderivative of the right side of the equation, we can use the exponential function to solve for $y(t)$. As just asserted,

$$\ln y(t) + C = \int \frac{3}{t^2+5}dt + \int \frac{0.06t^2}{t^2+5}dt.$$

Because (see Section 10.5) $\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1}$, we get that

$$\frac{d}{dt}a\sqrt{b}\tan^{-1}\left(\frac{t}{\sqrt{b}}\right) = a\sqrt{b}\;\frac{1}{\frac{t^2}{b}+1}\cdot\frac{1}{\sqrt{b}} = \frac{a}{\frac{t^2+b}{b}} = \frac{ab}{t^2+b}.$$

Taking $b=5$ and $a=\frac{3}{5}$, we see that $\frac{d}{dt}\frac{3}{5}\sqrt{5}\tan^{-1}\left(\frac{t}{\sqrt{5}}\right) = \frac{3}{t^2+5}$, so

$$\int \frac{3}{t^2+5}dt = 0.6\sqrt{5}\;\tan^{-1}\left(\frac{t}{\sqrt{5}}\right) + C.$$

Next, observe that

$$\frac{d}{dt}\left[at - a\sqrt{b}\;\tan^{-1}\left(\frac{t}{\sqrt{b}}\right)\right] = a - \frac{ab}{t^2+b} = \frac{a(t^2+b)-ab}{t^2+b} = \frac{at^2}{t^2+b}.$$

Taking $b=5$ and $a=0.06$,

$$\int \frac{0.06}{t^2+5} dt = 0.06t - 0.06\sqrt{5}\ \tan^{-1}\left(\frac{t}{\sqrt{5}}\right) + C.$$

Combining the above conclusions,

$$\ln y(t) = 0.06t - 0.06\sqrt{5}\ \tan^{-1}\left(\frac{t}{\sqrt{5}}\right) + 0.6\sqrt{5}\ \tan^{-1}\left(\frac{t}{\sqrt{5}}\right) + C.$$

Therefore,

$$\ln y(t) \ = \ 0.06t - 0.54\sqrt{5}\tan^{-1}\left(\frac{t}{\sqrt{5}}\right) + C.$$

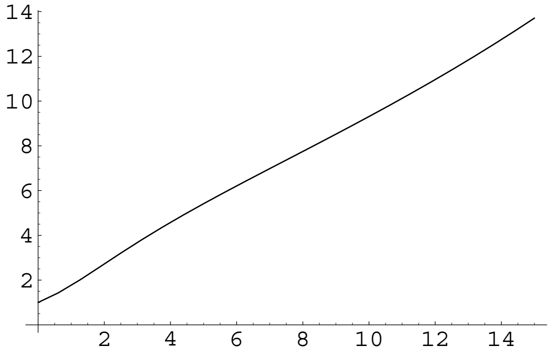
By properties of the exponential function, we get

$$\begin{aligned} y(t) &= e^{\ln y(t)} = e^{0.06t - 0.54\sqrt{5}\ \tan^{-1}\left(\frac{t}{\sqrt{5}}\right) + C} \\ &= e^{0.06t - 0.54\sqrt{5}\ \tan^{-1}\left(\frac{t}{\sqrt{5}}\right)} \cdot e^C. \end{aligned}$$

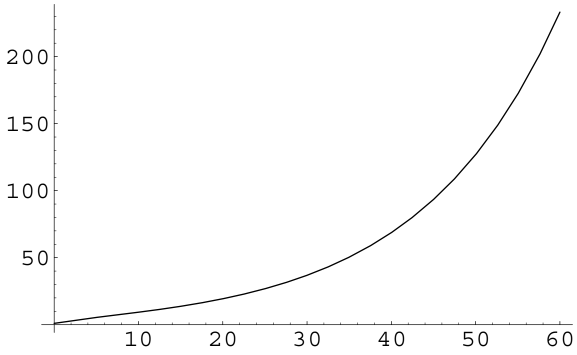
If the formula $s(t) = \frac{50}{t^2} + 1$ is valid moments after the culture is first observed at time $t = 0$, then $y_0 = y(0) = e^0 \cdot e^C = e^C$. Hence

$$y(t) = y_0\, e^{0.06t - 0.54\sqrt{5}\ \tan^{-1}\left(\frac{t}{\sqrt{5}}\right)}.$$

Let's have a look at the graph of the function $y(t)$. It is sketched below for $0 \le t \le 15$



days. Notice that it is essentially a straight line. Because the term $\sqrt{5}\tan^{-1}\left(\frac{t}{\sqrt{5}}\right)$ is bounded by $\frac{\sqrt{5}\pi}{2}$ (see Figure 10.14 in Chapter 10.5), it follows that for t large enough,



$e^{0.06t}$ will make the dominant contribution to the function $y(t)$. The graph of $y(t)$ above for $0 \le t \le 60$ reflects this.

50. **i.** Monod's equation says that $\frac{y'}{y} = \frac{\mu_{\max} s(t)}{s_{1/2\max} + s(t)}$. Letting $a = \mu_{\max}$ and $b = s_{1/2\max}$ we can rewrite this as $\frac{y'}{y} = \frac{as(t)}{b+s(t)}$. So we need to set

$$f(t) = \frac{as(t)}{b + s(t)}$$

and solve for $s(t)$. Doing so, we get $f(t)[b+s(t)] = as(t)$, hence $s(t)f(t) - as(t) = -bf(t)$, and therefore

$$s(t) = \frac{-bf(t)}{f(t) - a} = \frac{bf(t)}{a - f(t)}\,.$$

So $\frac{y'(t)}{y(t)} = f(t)$ can be achieved by taking $s(t) = \frac{bf(t)}{a - f(t)}$ in Monod's equation.

ii. By equation (11j), $\frac{y'}{y} = \mu - ky$. From the discussion in Section 11.5B we know that $k = \frac{\mu}{M}$ and $y(t) = \frac{Me^{\mu t}}{\left(\frac{M}{y_0} - 1\right) + e^{\mu t}}$. So

$$\begin{aligned}
\frac{y'}{y} &= \mu - \frac{\mu}{M}\left[\frac{Me^{\mu t}}{\left(\frac{M}{y_0} - 1\right) + e^{\mu t}}\right] = \mu - \mu\left[\frac{e^{\mu t}}{\left(\frac{M - y_0}{y_0}\right) + e^{\mu t}}\right] \\
&= \mu - \mu\left[\frac{y_0 e^{\mu t}}{(M - y_0) + y_0 e^{\mu t}}\right] = \mu\left[\frac{M - y_0 + y_0 e^{\mu t} - y_0 e^{\mu t}}{(M - y_0) + y_0 e^{\mu t}}\right] \\
&= \frac{\mu(M - y_0)}{(M - y_0) + y_0 e^{\mu t}} = \frac{\mu}{1 + \frac{y_0}{M - y_0} e^{\mu t}} \\
&= \frac{\mu e^{-\mu t}}{\frac{y_0}{M - y_0} + e^{-\mu t}}.
\end{aligned}$$

So $\frac{y'}{y} = f(t)$ with $f(t) = \frac{\mu e^{-\mu t}}{\frac{y_0}{M - y_0} + e^{-\mu t}}$. In reference to Monod's equation, note that $s(t) = e^{-\mu t}$, $\mu_{\max} = \mu$ (which is the case by equation (11j)), and $s_{1/2\max} = \frac{y_0}{M - y_0}$.