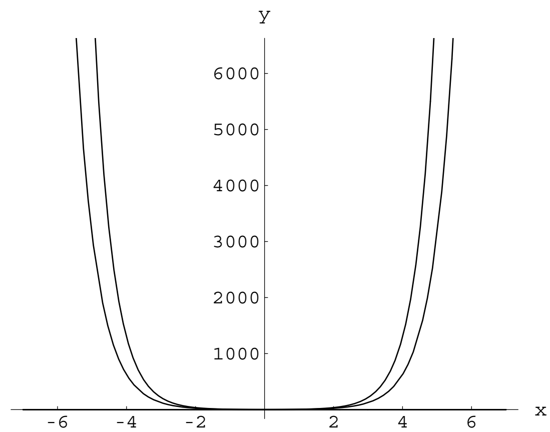
Solutions to the Exercises of Chapter 10

10A. Exponential Functions

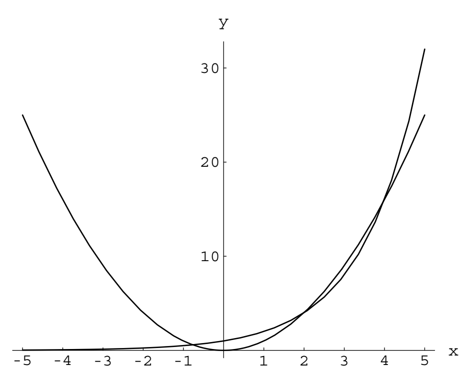
1. Because $\left(\frac{1}{5}\right)^x = (5^{-1})^x = 5^{-x}$, it follows that the graph of $y = \left(\frac{1}{5}\right)^x$ is obtained by rotating the graph of $y = 5^x$ about the y-axis. Similarly, the graph of $y = \left(\frac{1}{6}\right)^x$ is obtained by reflecting the graph $y = 6^x$ across the y-axis.



2. The table below compares the values of the functions:

x	0	1	2	3	4	5	10	15	20
x^2	0	1	4	9	16	25	100	225	400
2^x	1	2	4	8	16	32	1024	32,768	1,048,576

The graphs of $f(x) = x^2$ and $g(x) = 2^x$ follow below. Which is which?



3. **i.** Because $f(x) = e^{x^{\frac{1}{2}}}$, we get $f'(x) = e^{x^{\frac{1}{2}}} \cdot \dfrac{d}{dx}x^{\frac{1}{2}} = e^{x^{\frac{1}{2}}} \cdot \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}$.

ii. $g'(x) = e^{-5x}(-5)\cos 3x + e^{-5x}(-\sin 3x) \cdot 3$

$$= -5e^{-5x}\cos 3x - 3e^{-5x}\sin 3x.$$

iii. $\dfrac{dy}{dx} = e^{x+e^x}\dfrac{d}{dx}(x+e^x) = e^{x+e^x}(1+e^x).$

iv. $f'(x) = 2xe^x + x^2e^x = (2x+x^2)e^x.$

v. $\dfrac{dy}{dx} = e^{x^2} + xe^{x^2}\cdot 2x = (1+2x^2)e^{x^2}.$

vi. $\dfrac{dy}{dx} = e^{\frac{1}{1-x^2}}\cdot\dfrac{d}{dx}(1-x^2)^{-1} = e^{\frac{1}{1-x^2}}(-1)(1-x^2)^{-2}\dfrac{d}{dx}(1-x^2) = -e^{\frac{1}{1-x^2}}(1-x^2)^{-2}(-2x)$

$$= \frac{2xe^{\frac{1}{1-x^2}}}{(1-x^2)^2}.$$

vii. $\dfrac{dy}{dx} = \sec^2(e^{3x-2})\cdot e^{3x-2}\cdot 3 = 3e^{3x-2}\sec^2(e^{3x-2}).$

viii. $\dfrac{dy}{dx} = \dfrac{(e^x-e^{-x})(e^x-e^{-x}) - (e^x+e^{-x})(e^x+e^{-x})}{(e^x-e^{-x})^2}$

$$= \frac{e^{2x}+e^{-2x}-2-[e^{2x}+e^{-2x}+2]}{(e^x-e^{-x})^2} = \frac{-4}{(e^x-e^{-x})^2}.$$

4. The slope of the tangent is $\frac{dy}{dx}$ evaluated at $x=1$. Because $\frac{dy}{dx} = 2xe^{-x} + x^2(-e^{-x}) = (2x-x^2)e^{-x}$, this value is $e^{-1} = \frac{1}{e}$. By the point-slope form of the equation of a line, we get that the tangent has equation $\left(y - \frac{1}{e}\right) = \frac{1}{e}(x-1)$ or $y = \frac{1}{e}x$.

5. Observe that $y' = 2e^{2x} - 3e^{-3x}$ and $y'' = 4e^{2x} + 9e^{-3x}$. Therefore by substituting, we get $y'' + y' - 6y = 4e^{2x} + 9e^{-3x} + 2e^{2x} - 3e^{-3x} - 6e^{2x} - 6e^{-3x} = 0$.

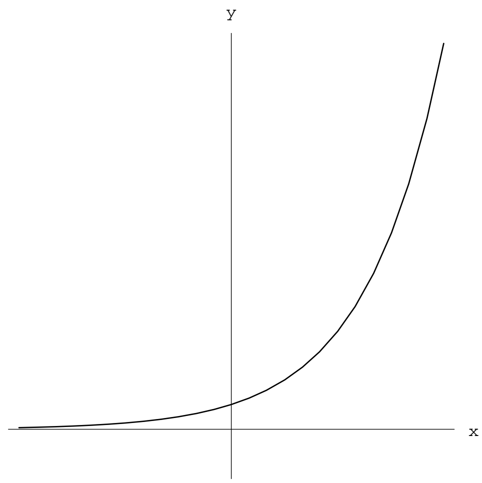
6.

$$\begin{aligned}
f'(x) &= e^{-x} + x(-e^{-x}) = -(x-1)e^{-x} \\
f''(x) &= -e^{-x} + (x-1)e^{-x} = (x-2)e^{-x} \\
f'''(x) &= e^{-x} - (x-2)e^{-x} = -(x-3)e^{-x} \\
f^{(4)}(x) &= -e^{-x} + (x-3)e^{-x} = (x-4)e^{-x} \\
f^{(5)}(x) &= e^{-x} - (x-4)e^{-x} = -(x-5)e^{-x} \\
f^{(6)}(x) &= -e^{-x} + (x-5)e^{-x} = (x-6)e^{-x}
\end{aligned}$$

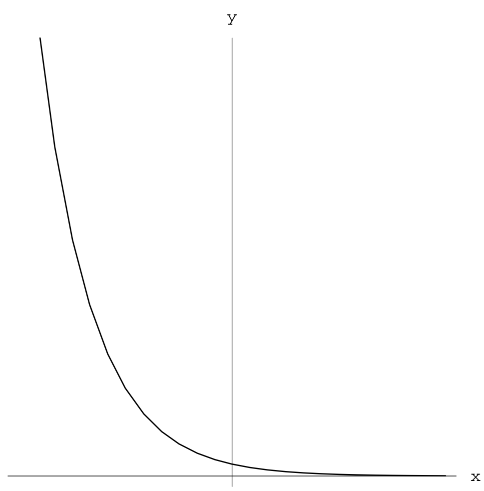
Observe the pattern that has emerged and conclude that the one hundredth derivative of $f(x)$ is $(x-100)e^{-x}$.

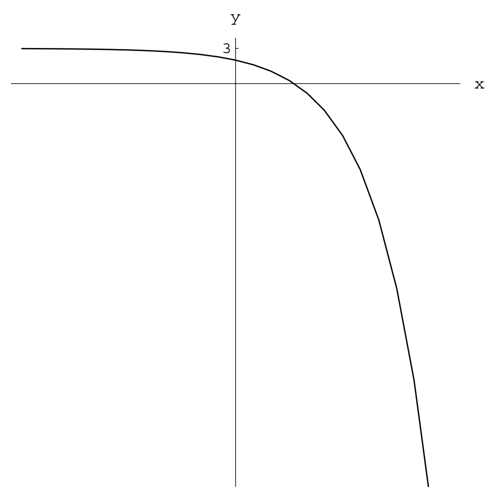
7. Let $f(x) = e^x + x$. Because $f(0) = 1$ and $f(-1) = \frac{1}{e} - 1 = \frac{1-e}{e} < 0$, it follows by the Intermediate Value Theorem and the continuity of f that $f(x) = 0$ for some x with $-1 < x < 0$.

8. Starting with the graph of e^x,



we get the graphs of e^{-x} and $3 - e^x$, see below, after thinking a little.





9. $g'(x) = \frac{e^x x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$. Observe that $g'(x) = 0$ precisely when $x = 1$. Notice that $g'(x) < 0$ when $0 < x < 1$ and that $g'(x) > 0$ when $x > 1$. So $g(x) = \frac{e^x}{x}$ is decreasing to the left of $x = 1$ and increasing to the right of $x = 1$. So g has its absolute minimum at $x = 1$. The absolute minimum value is $g(1) = e$.

10B. Inverse Functions

10. Solving $y = 4x + 7$ for x we get $4x = y - 7$ and hence $x = \frac{1}{4}(y-7)$. So $f^{-1}(y) = \frac{1}{4}(y-7)$ or, letting x be the variable, $f^{-1}(x) = \frac{1}{4}(x-7)$. Doing the same for $f(x) = \frac{x-2}{x+2}$, we solve $y = \frac{x-2}{x+2}$ for x to get $y(x+2) = x - 2$, hence $yx - x = -2 - 2y$, hence $x(y-1) = -2 - 2y$ and therefore

$$x = \frac{-2-2y}{y-1} = \frac{2+2y}{1-y}.$$

So $f^{-1}(x) = \frac{2+2x}{1-x} = \frac{2(1+x)}{1-x}$.

Comment: For Exercises 11–13 we will use the formula $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$ or, because $g(x) = f^{-1}(x)$, $g'(x) = \frac{1}{f'(g(x))}$.

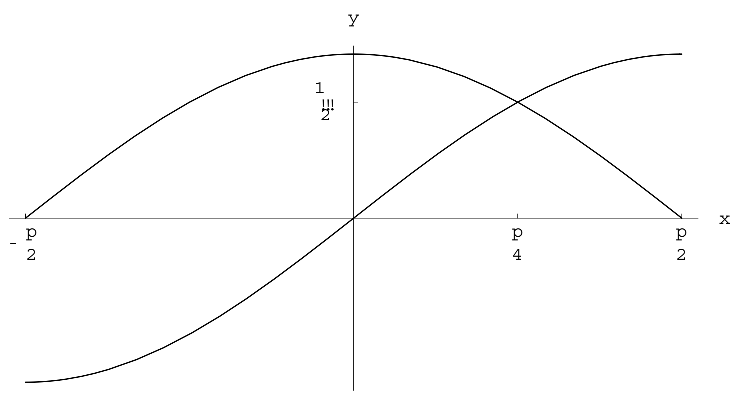
11. Notice that $f(0) = 1$. So $g(1) = f^{-1}(1) = 0$. By the formula already pointed out,

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)}.$$

Because $f'(x) = 3x^2 + 1$, we see that $f'(0) = 1$. So $g'(1) = 1$. Because $f'(x) = 3x^2 + 1 > 0$ for all x, observe that f is always increasing. Hence it has an inverse.

12. Notice that $f(1) = 1 - 1 + 2 = 2$. So $g(2) = f^{-1}(2) = 1$. Because $g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f'(1)}$ and $f'(x) = 5x^4 - 3x^2 + 2$, we get $g'(2) = \frac{1}{4}$. By the quadratic formula, $5x^4 - 3x^2 + 2 = 0$ when $x^2 = \frac{3\pm\sqrt{9-40}}{10} = \frac{3\pm\sqrt{-31}}{10}$, so never. So $f'(x) > 0$ for all x and hence f has an inverse.

13. Recall from Section 1.4 that $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. So $\sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{2}{\sqrt{2}} = \frac{\sqrt{2}\cdot\sqrt{2}}{\sqrt{2}} = \sqrt{2}$. Therefore, $g(\sqrt{2}) = f^{-1}(\sqrt{2}) = \frac{\pi}{4}$. Since $g'(\sqrt{2}) = \frac{1}{f'(g(\sqrt{2}))} = \frac{1}{f'(\frac{\pi}{4})}$, it remains to compute $f'\left(\frac{\pi}{4}\right)$. Because $f'(x) = \cos x - \sin x$, we get $f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$ Therefore $g'(\sqrt{2})$ is not defined. A look at the graphs of $\sin x$ and $\cos x$ tells us that $f'(x) > 0$ for $-\frac{\pi}{2} \le x < \frac{\pi}{4}$ and



$f'(x) < 0$ for $\frac{\pi}{4} < x \le \frac{\pi}{2}$. So the graph of f is increasing to the left of $\frac{\pi}{4}$ and decreasing to the right of $\frac{\pi}{4}$. Hence f does not have an inverse on any interval around $\frac{\pi}{4}$.

10C. Logarithms

14. **i.** $\log_2 64 = \log_2 2^6 = 6$.

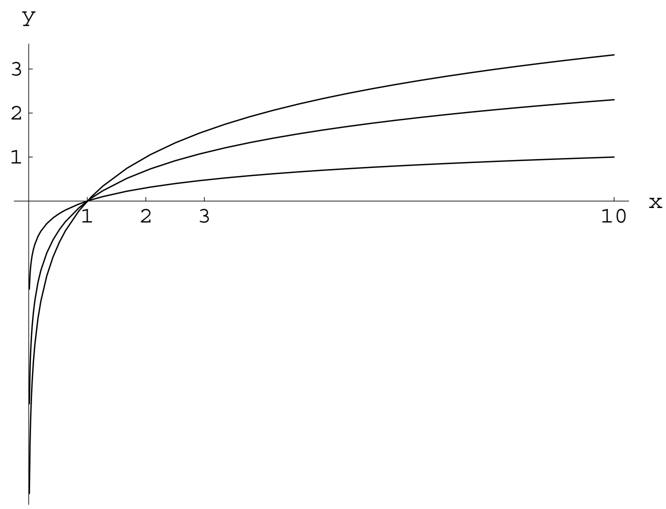
ii. $\log_3 3^{\sqrt{5}} = \sqrt{5}$.

iii. $\log_3 108 - \log_3 4 = \log_3 \frac{108}{4} = \log_3 27 = \log_3 3^3 = 3$.

iv. $\log_5 10 + \log_5 20 - 3\log_5 2 = \log_5 200 - \log_5 2^3 = \log_5 \frac{200}{8} = \log_5 25 = 2$.

15. Notice that $\log_{10} 100 = \log_{10} 10^2 = 2$ and $\log_9 81 = \log_9 9^2 = 2$. Because $\log_a x$ is an increasing function for any $a > 1$, we get $\log_9 82 > \log_9 81 = \log_{10} 100 > \log_{10} 99$.

16. With (**i**) at the top, (**ii**) in the middle and (**iii**) at the bottom, these graphs are



17. **i.** $\log_2 x + 3\log_2(x+1) + \frac{1}{4}\log_2(x-1) = \log_2 x + \log_2(x+1)^3 + \log_2(x-1)^{\frac{1}{4}}$
$= \log_2 x(x+1)^3(x-1)^{\frac{1}{4}}$.

ii. $\frac{1}{3}\ln x - 4\ln(2x+3) = \ln x^{\frac{1}{3}} - \ln(2x+3)^4 = \ln \frac{x^{\frac{1}{3}}}{(2x+3)^4}$.

18. **i.** $2^{\log_2 x} = 2^3$, so $x = 2^3 = 8$.

ii. $\ln 2^{x^2-5} = \ln 3$, so $(x^2-5)\ln 2 = \ln 3$. Hence $x^2 - 5 = \frac{\ln 3}{\ln 2}$. Therefore, $x = \pm\sqrt{5 + \frac{\ln 3}{\ln 2}}$.

iii. $\ln 5^{x^2-1} = \ln 2$, so $x^2 - 1 = \frac{\ln 2}{\ln 5}$, and $x = \pm\sqrt{1 + \frac{\ln 2}{\ln 5}}$.

iv. $\ln 4^{x^2+1} = \ln 3$, so $x^2 + 1 = \frac{\ln 3}{\ln 4}$. Because $\ln x$ is an increasing function, $\ln 4 > \ln 3$. Hence $\frac{\ln 3}{\ln 4} < 1$. But $x^2 + 1 \geq 1$. So the original equation can have no solution.

v. Given $\log_9(4x^2 - 11) = 7$, we get $9^{\log_9(4x^2-11)} = 9^7$ and hence $4x^2 - 11 = 9^7$. So $4x^2 = 4782969 + 11 = 4782980$, and therefore $x^2 = 1195745$. So $x \approx \pm\ 1093.5$.

vi. Since $\log_5(\log_5 x) = 6$, we get $5^{\log_5(\log_5 x)} = 5^6$ and hence $\log_5 x = 5^6$. Therefore, $5^{\log_5 x} = (5)^{5^6}$ and hence $x = (5)^{5^6} = 5^{15625}$.

vii. By basic properties of $\ln x$, we get $\ln[(x+6)(x-3)] = \ln[5\cdot 7]$. Applying the exponential function e^x to both sides, we get $(x+6)(x-3) = 35$. So $x^2 + 3x - 18 = 35$ and hence $x^2 + 3x - 53 = 0$. Thus, by the quadratic formula, $x = \frac{-3\pm\sqrt{221}}{2}$.

viii. From the given equation $\ln\frac{x-2}{x+1} - \ln\frac{x-3}{x+1} = 1$. So $\ln\left(\frac{x-2}{x+1}\Big/\frac{x-3}{x+1}\right) = 1$ and hence $\ln\left[\frac{x-2}{x+1}\cdot\frac{x+1}{x-3}\right] = 1$. It follows that $\ln\frac{x-2}{x-3} = 1$. Therefore, $\frac{x-2}{x-3} = e^{\ln\frac{x-2}{x-3}} = e$ and hence $x - 2 = e(x-3)$. So $(e-1)x = 3e - 2$ and $x = \frac{3e-2}{e-1}$.

19. Because e^x is an increasing function, $3x - 2 = e^{\ln(3x-2)} \leq e^0 = 1$. Therefore, $3x \leq 3$ and hence $x \leq 1$. On the other hand, $3x - 2 > 0$ since $\ln(3x-2)$ is defined. So $x > \frac{2}{3}$. Therefore, $\frac{2}{3} < x \leq 1$.

20. From $4^x - 2^{x+3} + 12 = 0$ we get $(2^2)^x - 2^x \cdot 2^3 + 12 = 0$, so $(2^x)^2 - 8\cdot 2^x + 12 = 0$. Let $y = 2^x$.

So $y^2 - 8y + 12 = 0$ and hence $(y-2)(y-6) = 0$. Therefore $y = 2$ or 6. So $2^x = 2$ or 6. Hence $x = \log_2 2^x = \log_2 2$ or $\log_2 6$. So $x = 1$ or $\log_2 6$.

21. Since the distance to the San Francisco quake is 100 km, the amplitude x on the seismograph satisfies $6.9 = \log_{10} \frac{x}{a}$ with $a = 1$ micron. So $x = 10^{6.9} \approx 7{,}943{,}282$ microns and hence $7{,}943{,}282 \times 10^{-6} = 7.943$ meters.

22. $M = \log_{10} \frac{25x}{1}$ where x is 7,943,282 microns. So $M = \log_{10}(25x) = 8.3$.

Comment: The unit of *micron* under discussion in the paragraph preceding Exercise 20 is also known as the *micrometer.*

Correction: The conclusion of Exercise 21 tells us that a standard seismograph placed 100 km from the 1989 quake recorded it in terms of a graph with an amplitude (amplitude is the maximal height of the graph) of over 7.9 meters or about 26 feet. This implies in turn that the 1906 quake produced a graph of amplitude about $25 \times 26 = 650$ feet. Unfortunately, this conclusion does not survive a reality check. Why? Simply because the drum of a standard seismograph that records the graph is less than a foot in length. As a consequence a seismograph placed at 100 km of the epicenter of a large quake (such as those of the exercises) would "go off the charts" and not provide much useful information. In Richter's original approach

Charles F. Richter, *Elementary Seismology,* W.H. Freeman and Company, San Francisco, 1958,

the formula for the magnitude M of a quake was given as follows:

$$M = \log_{10} x - \log_{10} x_0,$$

where x is the amplitude of the graph generated by the seismograph and $\log_{10} x_0$ depends on the distance of the instrument from the epicenter of the quake and is determined empirically. The

distance	$-\log_{10} x_0$
0	-1.6
10	-1.5
20	-1.3
30	-0.9
50	-0.4
100	0
150	0.3
200	0.5
300	0.9
400	1.3
500	1.7
600	1.9

table above supplies some representative values with distance given in kilometers and amplitude in microns. Notice that in the situation where the seismograph is 100 kilometers from the epicenter of the quake, this coincides with the earlier formulation. The solution of the equation

$$6.9 = \log_{10} x + 1.9$$

is $x = 10^5$ =10,000 and this tells us that the 1989 San Francisco quake registered 10^5 microns, or 10 centimeters, on a standard seismograph located 600 kilometers from the quake. Check that a quake of Richter Magnitude of 8 or more is off the charts on a standard seismograph placed at a distance of 600 kilometers from the quake.

The modern versions of the Richter Magnitude also makes use of the logarithm, but takes into account the different types of waves that an earthquake generates and also the lengths of time that the tremors last. See the readable account

Bruce Holt, *Earthquakes,* W.H. Freeman and Company, New York, 1993

for the details and for much additional information.

23. **i.** We need $1 - x > 0$. So $1 > x$.

ii. We need both $t \geq 0$ and $t^2 - 1 > 0$. So $t \geq 0$ and $t^2 > 1$. Hence $t > 1$.

24. **i.** Solving $y = \ln(x + 3)$ for x, we get $e^y = e^{\ln(x+3)} = x + 3$, so $x = e^y - 3$. So the inverse function sends y to $e^y - 3$. Hence it is the function $g(x) = e^x - 3$.

ii. Solving $y = (\ln x)^2$ for x we get $y^{\frac{1}{2}} = \ln x$ and hence $e^{y^{\frac{1}{2}}} = e^{\ln x} = x$. So the inverse is the function that sends y to $e^{y^{\frac{1}{2}}}$, or $g(x) = e^{x^{\frac{1}{2}}}$.

iii. We need to solve $y = \frac{1+e^x}{1-e^x}$ for x. Let's first solve for e^x. Because $y(1-e^x) = 1+e^x$, we get $y-ye^x = 1+e^x$, so $ye^x+e^x = y-1$ and hence $e^x = \frac{y-1}{y+1}$. Therefore, $x = \ln e^x = \ln \frac{y-1}{y+1}$. So the inverse sends y to $\ln \frac{y-1}{y+1}$. In terms of the variable x, it is the function $g(x) = \ln \frac{x-1}{x+1}$.

25. **i.** For $\ln x$ to make sense we need $x > 0$. Because $\cos\theta$ makes sense for any θ, the domain of $f(x)$ is $\{x \,|\, x > 0\}$. Note that $f'(x) = -\sin(\ln x) \cdot \frac{1}{x}$. The considerations above tell us that the domain of $f'(x)$ is also $\{x \,|\, x > 0\}$.

ii. For $f(x)$ to make sense, we need $2 - x - x^2 > 0$ or $x^2 + x - 2 < 0$. Because $x^2 + x - 2 = (x + 2)(x - 1)$, we see that $x^2 + x - 2 = 0$ when $x = -2$ or 1. Consider the x axis and take $-3, 0$ and 2 as test points to see that $x^2 + x - 2 < 0$ for $-2 < x < 1$. So the domain of $f(x)$ is $\{x \,|\, -2 < x < 1\}$. Check that $f'(x) = \frac{-1-2x}{2-x-x^2} = \frac{2x+1}{x^2+x-2}$. It must now be pointed out that the domain of $f'(x)$ can be no larger than the domain of $f(x)$. This follows from the definition $\lim\limits_{\Delta\to x} 0\frac{f(x+\Delta x)-f(x)}{\Delta x}$ of $f'(x)$. It follows that the domain of $f'(x)$ is also $\{x \,|\, -2 < x < 1\}$.

iii. For $f(x)$ to make sense we need both $x \geq 0$ and $x - 1 \geq 0$, so we need $x \geq 1$. For $\ln(\sqrt{x} - \sqrt{x-1})$ to make sense, we must have $\sqrt{x} > \sqrt{x-1}$. When $x \geq 1$, $x > x - 1$ and hence $\sqrt{x} > \sqrt{x-1}$. So the domain is $\{x \,|\, x \geq 1\}$. Because $f(x) = \ln(x^{\frac{1}{2}} - (x-1)^{\frac{1}{2}})$, we see that

$$f'(x) = \frac{\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}(x-1)^{-\frac{1}{2}}}{x^{\frac{1}{2}} - (x-1)^{\frac{1}{2}}} = \frac{\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x-1}}}{\sqrt{x} - \sqrt{x-1}}.$$

The fact that the domain of $f(x)$ is $\{x \,|\, x \geq 1\}$ and the discussion above tells us that the domain of $f'(x)$, is $\{x \,|\, x > 1\}$.

iv. We need $x^4 + 3x^2 > 0$ for $f(x)$ to make sense. For $x^4 + 3x^2 = x^2(x^2+3) > 0$ all we need to have is $x \neq 0$. So $\{x \,|\, x \neq 0\}$ is the domain of $f(x)$. Since $f(x) = \log_{11}(x^4 + 3x^2) = \frac{\ln(x^4+3x^2)}{\ln 11}$, we get by taking the derivative

$$f'(x) = \frac{1}{\ln 11}\left(\frac{4x^3 + 6x}{x^4 + 3x^2}\right).$$

So $\{x \,|\, x \neq 0\}$ is also the domain of $f'(x)$.

v. For $f(x)$ to be defined we need $x + 3x^2 > 0$. Because $x + 3x^2 = x(1+3x)$, this is so for $x > 0$. If $x < 0$, we need to have $1 + 3x < 0$, to get $x(1+3x) > 0$. But $1 + 3x < 0$ means that $3x < -1$ and hence $x < -\frac{1}{3}$. So the domain of $f(x)$ is

$$\left\{x \,|\, x > 0 \text{ or } x < -\frac{1}{3}\right\}$$

Because $f(x) = \ln(x+3x^2)^{\frac{1}{2}}$, we get $f'(x) = \frac{\frac{1}{2}(x+3x^2)^{-\frac{1}{2}}(1+6x)}{(x+3x^2)^{\frac{1}{2}}}$. Because $x + 3x^2 > 0$ for all x in the domain of $f(x)$, the domain of $f'(x)$ is the same as that of $f(x)$.

26. **i.** Check that $y' = \ln x + x \cdot \frac{1}{x} = \ln x + 1$. So $y'' = \frac{1}{x}$.

ii. Because $\frac{d}{dx}\log_a x = \frac{1}{\ln a} \cdot \frac{1}{x}$ for any base a, $y' = \frac{1}{\ln 10} \cdot \frac{1}{x} = \frac{1}{\ln 10} \cdot x^{-1}$ and $y'' = \frac{-1}{\ln 10}x^{-2} = \frac{-1}{(\ln 10)x^2}$.

iii. Check that $y' = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \sec x = (\cos x)^{-1}$. So $y'' = -(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} = (\sec x)(\tan x)$.

27. **i.** Because $g(x) = (\ln x)^{\frac{1}{2}}$, we get $g'(x) = \frac{1}{2}(\ln x)^{-\frac{1}{2}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$.

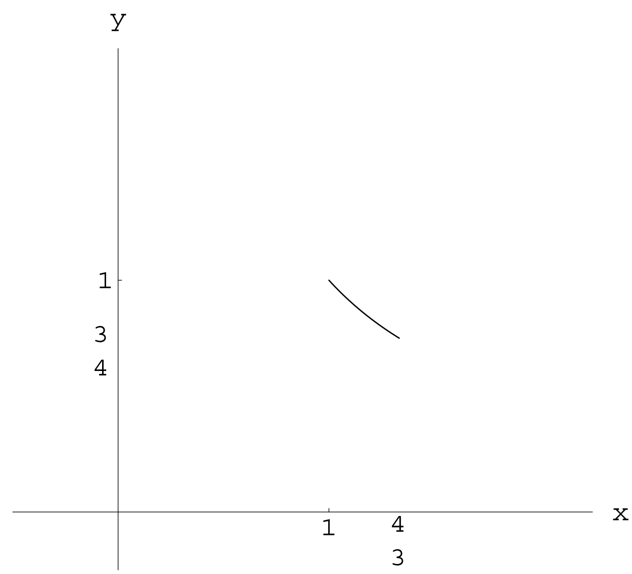
ii. $f'(t) = \frac{1}{\ln 7} \cdot \frac{1}{t^4 - t^2 + 1}(4t^3 - 2t) = \frac{4t^3 - 2t}{(\ln 7)(t^4 - t^2 + 1)}$.

iii. $f'(x) = e^x \cdot \ln x + e^x \cdot \frac{1}{x}$.

iv. $h'(t) = 3t^2 - (\ln 3)3^t$.

28. $f'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1$. Notice that $f'(x) > 0$ when $\ln x > -1$, i.e., when $x = e^{\ln x} > e^{-1} = \frac{1}{e}$, i.e., when $x > \frac{1}{e}$. Similarly, $f'(x) < 0$ when $x < \frac{1}{e}$. So $f(x)$ is decreasing to the left of $x = \frac{1}{e}$ and increasing to the right. It follows that $f(x)$ has its absolute minimum value when $x = \frac{1}{e}$. This value is $f\left(\frac{1}{e}\right) = \frac{1}{e} \cdot \ln \frac{1}{e} = \frac{1}{e}\ln(e^{-1}) = -\frac{1}{e}$.

29. Because $\ln x$ is an antiderivative of $\frac{1}{x}$, we get $\int_1^r \frac{1}{x}\, dx = \ln x\big|_1^r = \ln r - \ln 1 = \ln r$. The fact that $\int_1^{\frac{4}{3}} \frac{1}{x}\,dx = \ln\frac{4}{3}$ is the area under the graph of $\frac{1}{x}$ between 1 and $\frac{4}{3}$ and the diagram below (it shows the graph of $\frac{1}{x}$ over $\left[1, \frac{4}{3}\right]$) provides the required inequality.



30. To evaluate $\int_0^{\frac{\pi}{4}} \tan x\,dx$ we need to find an antiderivative of $\tan x = \frac{\sin x}{\cos x}$. Recall the formula

$$\frac{d}{dx}\ln g(x) = \frac{g'(x)}{g(x)}$$

from Section 10.3. Let $g(x) = \cos x$ and note that $g'(x) = -\sin x$. So $-\frac{\sin x}{\cos x} = \frac{g'(x)}{g(x)} = \frac{d}{dx}[\ln\cos x]$. It follows that $-\ln\cos x$ is an antiderivative of $\tan x = \frac{\sin x}{\cos x}$. Therefore,

$$\int_0^{\frac{\pi}{4}} \tan x\,dx = -\ln\cos x\,\Big|_0^{\frac{\pi}{4}} = -\ln\frac{\sqrt{2}}{2} - (-\ln 1) = \ln\left(\frac{\sqrt{2}}{2}\right)^{-1} - 0$$

$$= \ln\frac{2}{\sqrt{2}} = \ln\sqrt{2} \approx 0.347.$$

31. By Leibniz's formula and Figure 10.10 the length of this arc is

$$\int_6^{10} \sqrt{1 + (T'(x))^2}\,dx.$$

By Section 10.4, $a = 10$ and $T'(x) = -\frac{(10^2 - x^2)^{\frac{1}{2}}}{x}$. So

$$(T'(x))^2 + 1 = \frac{10^2 - x^2}{x^2} + 1 = \frac{10^2 - x^2 + x^2}{x^2} = \frac{10^2}{x^2}.$$

Hence

$$\int_6^{10} \sqrt{1 + (T'(x))^2}\,dx = \int_6^{10} \frac{10}{x}\,dx = 10\ln x\,\Big|_6^{10} = 10(\ln 10 - \ln 6) = 10\ln\frac{5}{3} \approx 5.11.$$

10D. Inverse Trigonometric Functions

32. **i.** What angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ has sine equal to $\frac{\sqrt{3}}{2}$? By section 1.4 the answer is $\frac{\pi}{3}$.

ii. What angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ has tangent equal to -1? By Section 1.4, $\tan\frac{\pi}{4} = 1$. So by Example 4.12 of Section 4.4, $\tan -\frac{\pi}{4} = -1$. So $-\frac{\pi}{4}$ is the answer.

iii. Because $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\sin(-x) = -\sin x$, we see that $\left(\sin -\frac{\pi}{6}\right) = -\frac{1}{2}$. So $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

iv. By Section 1.4, $\tan\frac{\pi}{3} = \sqrt{3}$. So $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$.

Correction: Exercise 33 should read "Show that $\tan(\sin^{-1} x) = \dfrac{x}{\sqrt{1-x^2}}$."

33. $\tan(\sin^{-1} x) = \frac{\sin(\sin^{-1} x)}{\cos(\sin^{-1} x)} = \frac{x}{\cos(\sin^{-1} x)}$. Because $\sin^2\theta + \cos^2\theta = 1$, we see that $\cos\theta = \sqrt{1-\sin^2\theta}$. So $\frac{x}{\cos(\sin^{-1} x)} = \frac{x}{\sqrt{1-\sin^2(\sin^{-1} x)}} = \frac{x}{\sqrt{1-x^2}}$

34. For example, $\tan^{-1} 0 = 0$ but $\frac{1}{\tan 0} = \frac{1}{0}$ is not defined. Also $\tan^{-1}(1) = \frac{\pi}{4} \approx 0.79$ but $\frac{1}{\tan 1} \approx 0.642$.

35. Because $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}\tan^{-1} x = \frac{1}{x^2+1}$, we get

i. $\frac{d}{dx}\sin^{-1} e^{2x} = \frac{1}{\sqrt{1-(e^{2x})^2}} \cdot 2e^{2x} = \frac{2e^{2x}}{\sqrt{1-e^{4x}}}$.

ii. $\frac{d}{dx}\tan^{-1}(5x-7) = \frac{1}{(5x-7)^2+1} \cdot 5 = \frac{5}{(5x-7)^2+1}$.

iii.
$$\frac{d}{dx}\left(\tfrac{1}{x} - \sin^{-1}\tfrac{1}{x}\right)^3 = 3(x^{-1} - \sin^{-1} x^{-1})^2 \cdot \Big(-x^{-2} - \tfrac{1}{\sqrt{1-x^{-2}}} \cdot (-x^{-2})\Big)$$
$$= 3\left(\tfrac{1}{x} - \sin^{-1}\tfrac{1}{x}\right)^2 \tfrac{1}{x^2}\left(\tfrac{1}{\sqrt{1-x^{-2}}} - 1\right).$$

iv. $\frac{d}{dx}\left(\frac{\tan^{-1} x}{x^2+1}\right) = \frac{\left(\frac{1}{x^2+1}\right)(x^2+1) - (\tan^{-1} x)\cdot 2x}{(x^2+1)^2} = \frac{1-2x\tan^{-1} x}{(x^2+1)^2}$.

v.
$$\frac{d}{dx}x\sin^{-1}\sqrt{4x+1} = \sin^{-1}(4x+1)^{\frac{1}{2}} + x\frac{1}{\sqrt{1-(4x+1)}} \cdot \tfrac{1}{2}(4x+1)^{-\frac{1}{2}} \cdot 4$$
$$= \sin^{-1}(4x+1)^{\frac{1}{2}} + \frac{2x}{\sqrt{-4x}\sqrt{4x+1}}.$$

vi. $\frac{d}{dx}\frac{1}{\sin^{-1} x} = \frac{d}{dx}(\sin^{-1} x)^{-1} = -(\sin^{-1} x)^{-2}\frac{1}{\sqrt{1-x^2}} = \frac{-1}{(\sin^{-1} x)^2\sqrt{1-x^2}}$.

36. **i.** $\int_{-1000}^{1000} \frac{1}{x^2+1}dx = \tan^{-1} 1000 - \tan^{-1}(-1000) = 1.570 - (-1.570) = 3.140$.

ii. $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} 1 - \sin^{-1}(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$.

10E. Concavity and Asymptotes

37. Since $h(x) = x^4 + 6x^2$, we get $h'(x) = 4x^3 + 12x$ and $h''(x) = 12x^2 + 12$. So $h''(x) > 0$ for all x and hence the graph of $h(x)$ is concave up throughout. To apply the second derivative test, we need to solve $h'(x) = 4x^3 + 12x = 0$ for x. Since $4x^3 + 12x = 4x(x^2+3)$, we see that $x = 0$. So $h(x)$ has a local minimum value at $x = 0$. There is no local maximum because 0 is the only critical point. Because $h(x) = x^4 + 6x^2$, we see that $h(0) = 0$ is the absolute minimum value and that there is no absolute maximum value.

38. $y' = e^x + 2e^{-x}$ and $y'' = e^x - 2e^{-x}$. So $y'' = \frac{e^{2x}-2}{e^x}$. Note that $y'' = 0$ when $e^{2x} = 2$, so when $2x = \ln e^{2x} = \ln 2$, and $x = \frac{1}{2}\ln 2$. When $x < \frac{1}{2}\ln 2$, we see that $2x < \ln 2$, hence $e^{2x} < e^{\ln 2} = 2$, so that $y'' < 0$. Similarly, when $x > \frac{1}{2}\ln 2$, we get $y'' > 0$. So the graph of $y = e^x - 2e^{-x}$ is concave down for $x < \frac{1}{2}\ln 2$ and concave up for $x > \frac{1}{2}\ln 2$.

39. Notice that the domain of $f(x)$ is $\{x \,|\, x > 0\}$. Now

$$\begin{aligned} f'(x) &= \frac{\frac{1}{x}\cdot\sqrt{x} - (\ln x)\frac{1}{2}x^{-\frac{1}{2}}}{x} = \frac{\frac{1}{\sqrt{x}\sqrt{x}}\cdot\sqrt{x} - \frac{\ln x}{2\sqrt{x}}}{x} = \frac{\frac{1}{\sqrt{x}}(1 - \frac{\ln x}{2})}{x} \\ &= \frac{1}{x\sqrt{x}}\left(\frac{2-\ln x}{2}\right) = \frac{2-\ln x}{2x\sqrt{x}} = \frac{2-\ln x}{2x^{\frac{3}{2}}}. \end{aligned}$$

So $f''(x) = \frac{(-\frac{1}{x})2x^{\frac{3}{2}} - (2-\ln x)3x^{\frac{1}{2}}}{4x^3} = \frac{3x^{\frac{1}{2}}\ln x - 6x^{\frac{1}{2}} - 2x^{\frac{1}{2}}}{4x^3} = \frac{x^{\frac{1}{2}}(3\ln x - 8)}{4x^3} = \frac{3\ln x - 8}{4x^{\frac{5}{2}}}$. It follows that $f''(x) = 0$ when $\ln x = \frac{8}{3}$ or $x = e^{\frac{8}{3}}$. When $0 < x < e^{\frac{8}{3}}$, we get $\ln x < \frac{8}{3}$, so $3\ln x < 8$, and hence $f''(x) < 0$. Similarly, when $e^{\frac{8}{3}} < x$, then $f''(x) > 0$. So $f(x)$ is concave down for $0 < x < e^{\frac{8}{3}}$ and concave up for $x < e^{\frac{8}{3}}$, and has an inflection point at $x = e^{\frac{8}{3}}$.

40.

i. Pushing $x \longrightarrow 5^+$, you push $x-5$ to zero from the positive side, so $\ln(x-5)$ is pushed to $-\infty$. So $x = 5$ is a vertical asymptote. Because $\ln(x-5)$ is not defined when $x - 5 \leq 0$, or $x \leq 5$, there is no graph to the left of the line $x = 5$.

ii. Pushing $x \to 0^+$, you push $4x$ to zero from the right. In the process $\log_{10} 4x$ goes to $-\infty$. So the line $x = 0$ (the y-axis) is a vertical asymptote . There is no graph to the left of the y-axis.

iii. Note that $x^2 - x = x(x-1)$; we see that $x^2 - x$ goes to $+\infty$ when $x \to \pm\infty$. So $\ln(x^2 - x)$ goes to $+\infty$. It follows that there is no horizontal asymptote. The graph of $\ln(x^2 - x) = \ln x + \ln(x-1)$ grows slowly as $x \to \infty$ or $x \to -\infty$.

iv. When $x \to \dfrac{\pi}{2}^-$ note that $\cos x$ goes to 0 through positive values. So $\ln\cos x$ goes to $-\infty$. So $x = \dfrac{\pi}{2}$ is a vertical asymptote of $\log_{10}\cos x$. The function is not defined for $\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$ because $\cos x < 0$ for such x.

41.

i. Let x be a large positive number. What can be said about $\tan^{-1} x$? The translation of the question is this: For what angles θ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ is $\tan\theta$ large and positive? Because $\tan\theta = \frac{\sin\theta}{\cos\theta}$ we see that for $\theta < \frac{\pi}{2}$, but close to $\frac{\pi}{2}$,

$$\tan\theta = \frac{\text{a number} < 1 \text{ but close to } 1}{\text{a number} > 0 \text{ but close to } 0}$$

So $\tan\theta$ will be large and positive. The closer $\theta < \frac{\pi}{2}$ is to $\frac{\pi}{2}$, the larger this positive number. It follows that when x is pushed to ∞, $\tan^{-1} x$ closes in on $\frac{\pi}{2}$ from below $\frac{\pi}{2}$.

This means that the line $y = \frac{\pi}{2}$ is a horizontal asymptote for the right side of the graph of $\tan^{-1} x$.

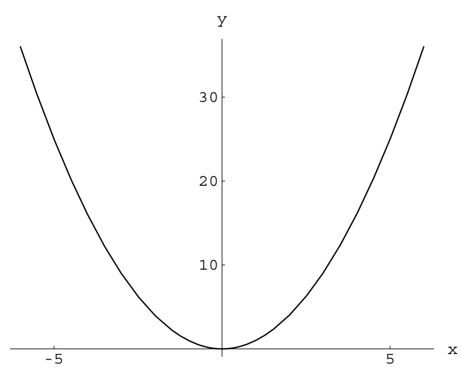
ii. A similar argument shows that $\lim\limits_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$ and hence that $y = -\frac{\pi}{2}$ is a horizontal asymptote for the left side of the graph.

iii. Study $\tan x = \frac{\sin x}{\cos x}$ for $x > \frac{\pi}{2}$ but close to $\frac{\pi}{2}$. For such $x, \cos x$ is negative and close to 0, while $\sin x$ is close to 1. So $\frac{\sin x}{\cos x}$ is large and negative . The closer $x > \frac{\pi}{2}$ is to $\frac{\pi}{2}$, the larger this negative number will be. So $x = \frac{\pi}{2}$ is a vertical asymptote for the part of the graph of $y = \tan x$, where $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

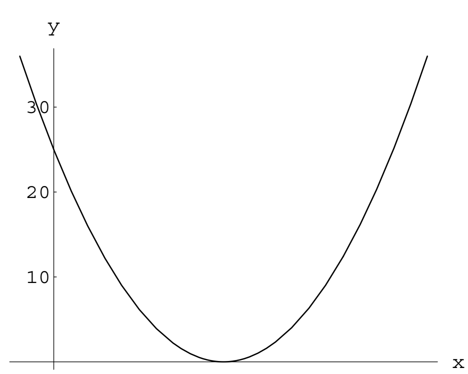
iv. A study of $\frac{\sin x}{\cos x}$ for $x < -\frac{\pi}{2}$ but close to $-\frac{\pi}{2}$, shows that $\lim\limits_{x \to -\frac{\pi}{2}^-} \tan x = +\infty$ and hence that the line $x = -\frac{\pi}{2}$ is a vertical asymptote for the graph of $\tan x$ with $-\frac{3\pi}{2} < x < \frac{-\pi}{2}$.

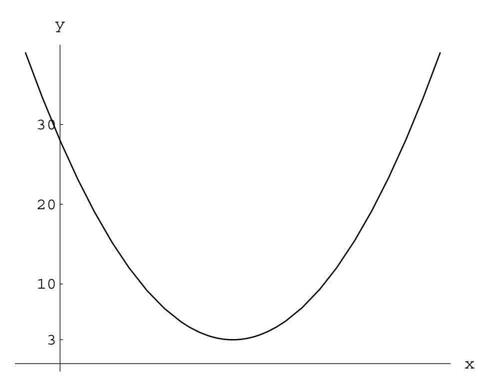
10F. Sketching Graphs

42. Correction: The concern is the graph of $y = (x-5)^2 + 3$, not $y + 3 = (x-5)^2 + 3$. Start with $y = x^2$. Replacing x by $x - 5$ shifts the graph

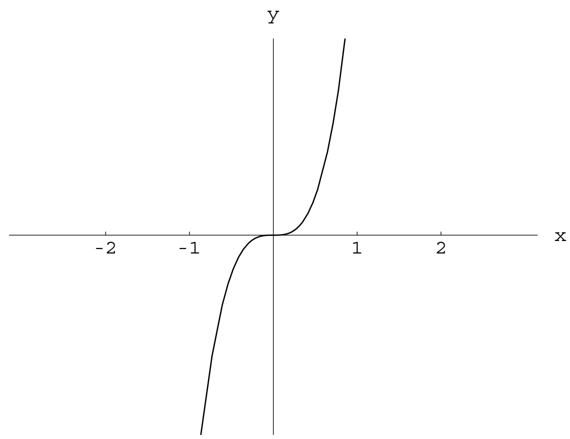


5 units to the right. The graph of $y = (x-5)^2 + 3$ is then gotten by adding 3 to every y-coordinate, in other words, by pushing up by 3 units. The graphs of $y = (x-5)^2$ and $y = (x-5)^2 + 3$ follow below.

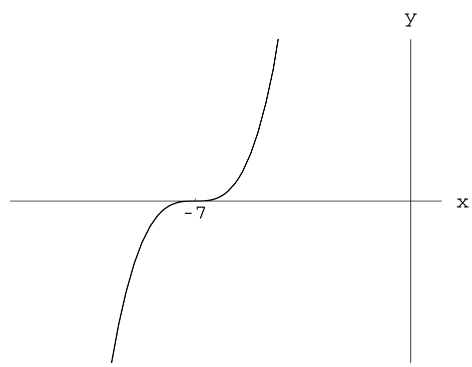




To get the graph of $y = (x+7)^3 + 4$, we start with the graph



of $y = x^3$. The graph of $y = (x+7)^3$ is gotten



by shifting it 7 units to the left. That of $y = (x+7)^3 + 4$ is obtained by raising this last graph by 4 units.

43. $y = x^2 - 3x - 5$

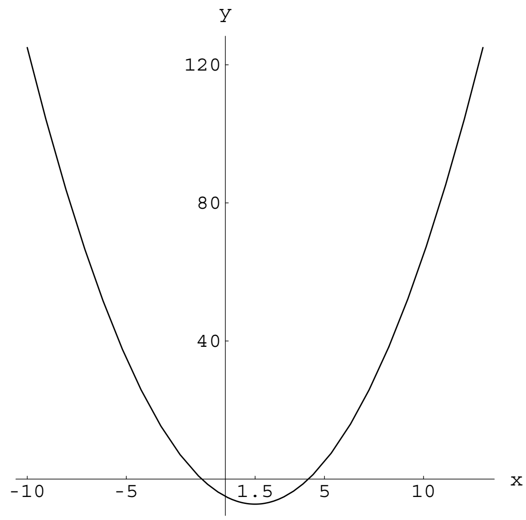
A. The function $f(x) = x^2 - 3x - 5$ makes sense for all x. So the domain is the set of all real numbers.

B. By completing the square, we get $y = x^2 - 3x + (\frac{3}{2})^2 - (\frac{3}{2})^2 - 5$. So $y = (x - \frac{3}{2})^2 - 7\frac{1}{4}$. So the graph is obtained by shifting the graph of $y = x^2$ by $\frac{3}{2}$ units to the right and then down by $7\frac{1}{4}$ units.

C. For x very large positive or negative, x^2 will dominate, so that $y \approx x^2$. So there are no horizontal asymptotes. Because $y = x^2 - 3x - 5$ is defined for all x, there are no vertical asymptotes.

D. $y' = 2x - 3 = 0$ for $x = \frac{3}{2}$. When $x < \frac{3}{2}$, we get $2x < 3$, so $y' < 0$. When $x > \frac{3}{2}, 2x > 3$, so $y' > 0$. Hence the graph decreases to the left of $x = \frac{3}{2}$ and increases to the right of $x = \frac{3}{2}$. So the graph has an absolute minimum when $x = \frac{3}{2}$. There is no local maximum.

E. Because $y'' = 2$, the graph is concave up throughout. There are no points of inflection. By the quadratic formula $x^2 - 3x - 5 = 0$ when $x = \frac{3 \pm \sqrt{9+20}}{2} = \frac{3}{2} \pm \frac{\sqrt{29}}{2} \approx 4.19$ or -1.19.



44. $y = x^3 - 3x^2 + 2$

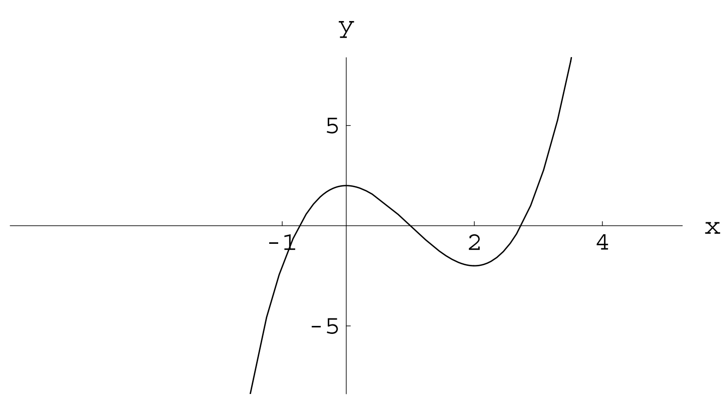
A. Domain: all real numbers.

B. No apparent symmetry.

C. When x is very large positive or negative, x^3 dominates. So $y \approx x^3$. There are no horizontal asymptotes. Because $y = x^3 - 3x^2 + 2$ is defined for all x, there are no vertical asymptotes.

D. $y' = 3x^2 - 6x = 3x(x-2)$. So $y' = 0$ when $x = 0$ or 2. Take the test points $-1, 1$, and 3. Because $y' > 0$ when $x = -1; y' < 0$ when $x = 1$; and $y' > 0$ when $x = 3$, we discover that $y = x^3 - 3x^2 + 2$ is increasing for $x < 0 :$ decreasing for $0 < x < 2$; and increasing for $2 < x$. So there is a local maximum at $x = 0$ and a local minimum at $x = 2$.

E. $y'' = 6x - 6 = 6(x-1)$. So $y'' < 0$ when $x < 1; y'' = 0$ when $x = 1$; and $y'' > 0$ when $x > 1$. It follows that $y = x^3 - 3x^2 + 2$ is concave down when $x < 1$ and concave up when $x > 1$, and has a point of inflection at the point (1,0).



45. $y = 1 - 3x + 5x^2 - x^3$

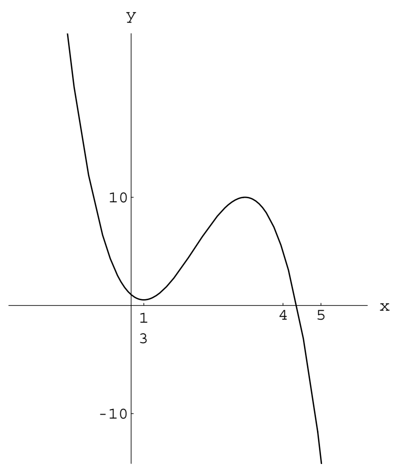
A. Domain: all real numbers.

B. No symmetry.

C. For x large and positive or negative, $-x^3$ dominates $y = 1 - 3x + 5x^2 - x^3$. So $y \approx -x^3$ and there are no horizontal asymptotes. Because polynomial functions are continuous for all real numbers, there are no vertical asymptotes.

D. $y' = -3x^2 + 10x - 3$. By the quadratic formula $y' = 0$ for $x = \frac{-10 \pm \sqrt{100-36}}{-6} = \frac{-10 \pm 8}{-6} = \frac{1}{3}$ or 3. Take 0,1, and 4 as testpoints. Because $y' < 0$ at 0; $y' > 0$ at 1; and $y' < 0$ at 4, we discover that $y = 1 - 3x + 5x^2 - x^3$ is decreasing when $x < \frac{1}{3}$; increasing when $\frac{1}{3} < x < 3$; and decreasing when $x > 3$. So there is a local minimum at $x = \frac{1}{3}$ and a local maximum at $x = 3$.

E. $y'' = -6x + 10$. So $y'' = 0$ for $x = \frac{10}{6} = \frac{5}{3}$. When $x < \frac{5}{3}$, $6x < 10$ and hence $y'' < 0$. When $x > \frac{5}{3}, 6x > 10$ and hence $y'' < 0$. So the graph is concave up when $x < \frac{5}{3}$ and concave down when $x > \frac{5}{3}$, and has a point of inflection when $x = \frac{5}{3}$.



46. $y = \frac{x}{(2x-3)^2}$

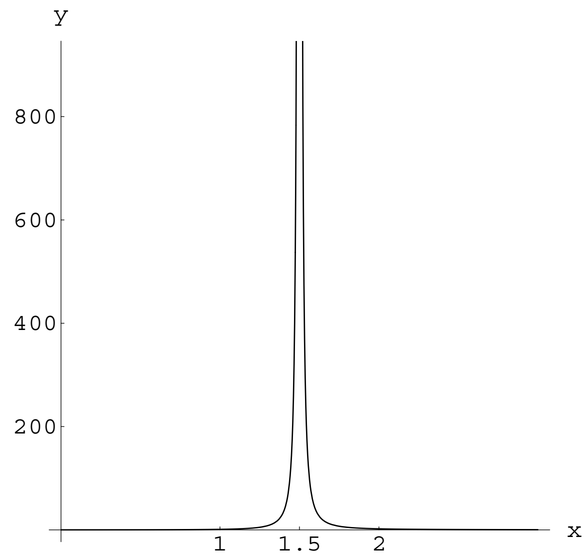
A. Domain: all x, except for $x = \frac{3}{2}$.

B. No symmetry.

C. Because $y = \frac{x}{4x^2-12x+9} = \frac{\frac{1}{x}}{4 - \frac{12}{x} + \frac{9}{x^2}}$, we see that $\lim_{x \to \pm\infty} \frac{x}{4x^2-12x+9} = \frac{0}{4-0+0} = 0$. So $y = 0$ is the horizontal asymptote. Because $y = \frac{x}{(2x-3)^2}$ is not defined for $x = \frac{3}{2}$, it is not continuous there. So the vertical line $x = \frac{3}{2}$ is a possible vertical asymptote. Check that $\lim_{x \to \frac{3}{2}^-} \frac{x}{(2x-3)^2} = +\infty$ and $\lim_{x \to \frac{3}{2}^+} \frac{x}{(2x-3)^2} = +\infty$. So indeed, $x = \frac{3}{2}$ is a vertical asymptote.

D. $y' = \frac{1 \cdot (2x-3)^2 - x \cdot 2(2x-3) \cdot 2}{(2x-3)^4} = \frac{(2x-3)^2 - 4x(2x-3)}{(2x-3)^4} = \frac{2x-3-4x}{(2x-3)^3} = -\frac{2x+3}{(2x-3)^3}$. So $y' = 0$ when $x = -\frac{3}{2}$ and y' is not defined when $x = \frac{3}{2}$. Take the test points $-2, 0$, and 2. Because $y' < 0$ at $x = -2$; $y' > 0$ at $x = 0$; and $y' < 0$ when $x = 2$, we see that $y = \frac{x}{(2x-3)^2}$ is decreasing when $x < -\frac{3}{2}$; increasing when $-\frac{3}{2} < x < \frac{3}{2}$; and decreasing when $x > \frac{3}{2}$. So there is a local minimum at $x = -\frac{3}{2}$. (Why is there not a local maximum at $x = \frac{3}{2}$?)

E. $y'' = -\left[\frac{2(2x-3)^3-(2x+3)3(2x-3)^2 2}{(2x-3)^6}\right] = -\left[\frac{2(2x-3)-6(2x+3)}{(2x-3)^4}\right] = -\frac{4x-6-12x-18}{(2x-3)^4} = \frac{8x+24}{(2x-3)^4} = \frac{8(x+3)}{(2x-3)^4}$. So $y''=0$ when $x=-3$ and y'' is not defined for $x=\frac{3}{2}$. Take the test points $-4, 0,$ and 2. Because $y''<0$ when $x=-4; y''>0$ when $x=0$, and $y''>0$ when $x=2$, we see that the graph is concave down for $x<-3$; concave up for $-3<x<\frac{3}{2}$; and concave up for $x>\frac{3}{2}$. There is a point of inflection at $\left(-3,-\frac{1}{27}\right)$.



47. $y = \frac{1}{(x-1)(x+2)}$

A. Domain: all x except $x=1, x=-2$.

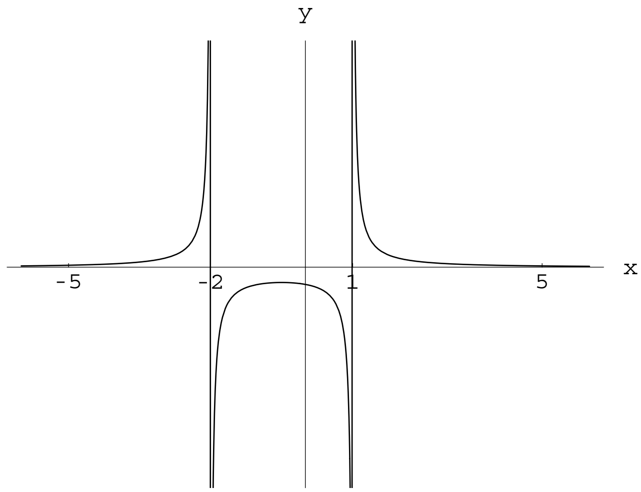
B. No symmetry.

C. Because $y = \frac{1}{x^2+x-2} = \frac{\frac{1}{x^2}}{1+\frac{1}{x}-\frac{2}{x^2}}$, we see that $\lim_{x\to\pm\infty}\frac{1}{x^2+x-2}=0$. So $y=0$ is horizontal asymptote. Because $\lim_{x\to 1^-}\frac{1}{(x-1)(x+2)}=-\infty$ and $\lim_{x\to 1^+}\frac{1}{(x-1)(x+2)}=+\infty$, we discover that $x=1$ is a vertical asymptote. Because $\lim_{x\to -2^-}\frac{1}{(x-1)(x+2)}=+\infty$ and $\lim_{x\to -2^+}\frac{1}{(x-1)(x+2)}=-\infty$, we see that $x=-2$ is also a vertical asymptote.

D. Because $y=\frac{1}{x^2+x-2}=(x^2+x-2)^{-1}$, we get $y'=-(x^2+x-2)^{-2}(2x+1)=-\frac{2x+1}{(x^2+x-2)^2}=-\frac{2x+1}{(x-1)^2(x+2)^2}$. Note that $y'=0$ when $x=-\frac{1}{2}$, and that y' is not defined when $x=1$ or -2. Take the test points $-3,-1,0,$ and 2. Because $y'>0$ at $x=-3$; $y'>0$ at $x=-1$; $y'<0$ at $x=0$; and $y'<0$ at $x=2$; we see that $y=\frac{1}{(x-1)(x+2)}$ is increasing for $x<-2$; increasing for $-2<x<-\frac{1}{2}$; decreasing for $-\frac{1}{2}<x<1$; and decreasing for $x>1$. There is a local maximum at $x=-\frac{1}{2}$.

E. $y''=-\left[\frac{2(x^2+x-2)^2-(2x+1)2(x^2+x-2)(2x+1)}{(x^2+x-2)^4}\right]=-\frac{2(x^2+x-2)-2(2x+1)(2x+1)}{(x^2+x-2)^3}=-\frac{2x^2+2x-4-2(4x^2+4x+1)}{(x^2+x-2)^3}$
$=-\frac{-6x^2-6x-6}{(x-1)^3(x+2)^3}=\frac{6(x^2+x+1)}{(x-1)^3(x+2)^3}$.
By the quadratic formula, $y''=0$ for $x=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{-3}}{2}$. Since $\sqrt{-3}$ is not a real number, it follows that y'' is never 0. Note that y'' is not defined when $x=-2$ or $x=1$.

Take the test points $-3, 0$, and 2, and check that $y'' > 0$ at $x = -3$; $y'' < 0$ at $x = 0$; and $y'' > 0$ at $x = 2$. So the graph is concave up for $x < -2$; concave down for $-2 < x < 1$; and concave up for $x > 1$. Why are there no inflection points?



48. $y = \frac{1+x^2}{1-x^2} = \frac{1+x^2}{(1-x)(1+x)}$.

A. Domain: all x, except for $x = \pm 1$.

B. The graph is symmetric about the y-axis because $f(-x) = \frac{1+x^2}{1-x^2} = f(x)$.

C. Because $y = \frac{x^2+1}{-x^2+1} = \frac{1+\frac{1}{x^2}}{-1+\frac{1}{x^2}}$, we see that $\lim_{x\to\pm\infty} \frac{x^2+1}{-x^2+1} = \frac{1+0}{-1+0} = -1$. So $y = -1$ is a horizontal asymptote. Because

$$\lim_{x\to 1^+} \frac{1+x^2}{(1-x)(1+x)} = -\infty \text{ and } \lim_{x\to 1^-} \frac{1+x^2}{(1-x)(1+x)} = +\infty,$$

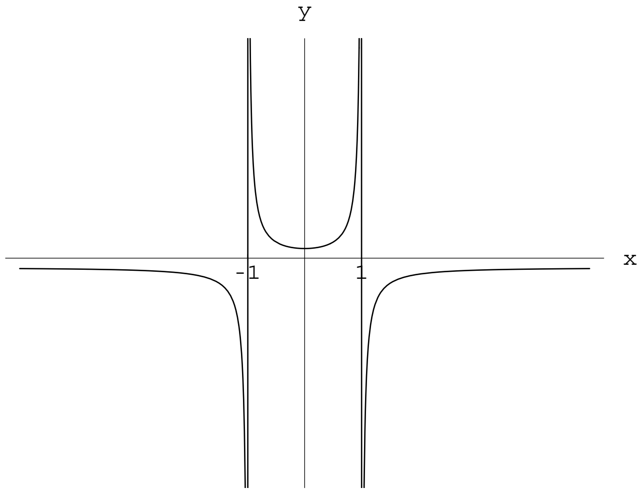
we see that $x = 1$ is a vertical asymptote. Because

$$\lim_{x\to -1^+} \frac{1+x^2}{(1-x)(1+x)} = +\infty \text{ and } \lim_{x\to -1^-} \frac{1+x^2}{(1-x)(1+x)} = -\infty,$$

we see that $x = -1$ is a vertical asymptote.

D. $y' = \frac{2x(1-x^2)-(1+x^2)(-2x)}{(1-x^2)^2} = \frac{-2x^3+2x+2x^3+2x}{(1-x^2)^2} = \frac{4x}{(1-x^2)^2}$. So $y' = 0$ when $x = 0$; and y' is not defined when $x = -1$ or $x = 1$. Take the test points $-2, -\frac{1}{2}, \frac{1}{2}$, and 2 to check that $y' < 0$ for $x < -1$; $y' < 0$ for $-1 < x < 0$; $y' > 0$ for $0 < x < 1$; and $y' > 0$ for $x > 1$. So the graph is decreasing for $x < -1$; decreasing for $-1 < x < 0$; increasing for $0 < x < 1$; and increasing for $1 < x$. There is a local minimum at (0,1).

E. $y'' = \frac{4(1-x^2)^2-4x\cdot 2(1-x^2)(-2x)}{(1-x^2)^4} = \frac{4(1-x^2)+16x^2}{(1-x^2)^3} = \frac{12x^2+4}{(1-x^2)^3} = \frac{12x^2+4}{(1-x)^3(1+x)^3}$. Notice that y'' is never zero, and that y'' is not defined when $x = -1$ or 1. Take the test points $-2, 0$, and 2, and check that $y'' < 0$ for $x < -1$; $y'' > 0$ for $-1 < x < 1$ and $y'' < 0$ for $1 < x$. So the graph is concave down for $x < -1$ and $x > 1$ and concave up for $-1 < x < 1$.



49. $y = \frac{x^3-1}{x^3+1}$

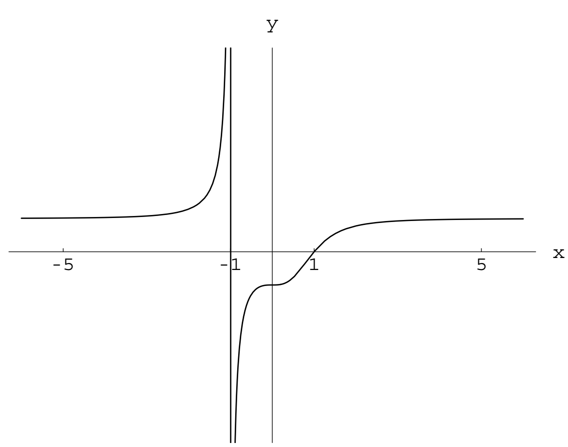
A. Domain: all x, except for $x = -1$.

B. No symmetry.

C. Because $y = \frac{x^3-1}{x^3+1} = \frac{1-\frac{1}{x^3}}{1+\frac{1}{x^3}}$, we see that $\lim_{x\to\pm\infty} \frac{x^3-1}{x^3+1} = \frac{1-0}{1+0} = 1$. So $y = 1$ is a horizontal asymptote. Note that $y = \frac{x^3-1}{x^3+1}$ is not defined at $x = -1$. So $x = -1$ is a possible vertical asymptote. Because $\lim_{x\to-1^-} \frac{x^3-1}{x^3+1} = +\infty$ and $\lim_{x\to-1^+} \frac{x^3-1}{x^3+1} = -\infty$, this is indeed the case.

D. $y' = \frac{3x^2(x^3+1)-(x^3-1)(3x^2)}{(x^3+1)^2} = \frac{3x^5+3x^2-3x^5+3x^2}{(x^3+1)^2} = \frac{6x^2}{(x^3+1)^2}$. Notice that $y' = 0$ when $x = 0$ and that y' is not defined when $x = -1$. Notice that $y' > 0$ except at $x = 0$ and $x = -1$. So the graph is increasing for $x < -1$, for $-1 < x < 0$, and for $x > 0$.

E. $y'' = \frac{12x(x^3+1)^2-6x^2\cdot 2(x^3+1)(3x^2)}{(x^3+1)^4} = \frac{12x(x^3+1)-36x^4}{(x^3+1)^3}$
$= \frac{12x^4+12x-36x^4}{(x^3+1)^3} = \frac{-24x^4+12x}{(x^3+1)^3} = \frac{-24x(x^3-\frac{1}{2})}{(x^3+1)^3}$.
Observe that $y'' = 0$ when $x = 0$ or $x = \frac{1}{\sqrt[3]{2}}$ and that y'' is not defined when $x = -1$. So $-1, 0$, and $\frac{1}{\sqrt[3]{2}} \approx 0.79$ are the critical points. Take $-2, -\frac{1}{2}, \frac{1}{2}$, and 1 as test points to check that $y'' > 0$ for $x < -1$; $y'' < 0$ for $-1 < x < 0$; $y'' > 0$ for $0 < x < \frac{1}{\sqrt[3]{2}}$; and



$y'' < 0$ for $x > \frac{1}{\sqrt[3]{2}}$. The concavity information and the points of inflection are captured on the graph.

50. $y = \frac{1}{x^3-x} = \frac{1}{x(x^2-1)} = \frac{1}{x(x-1)(x+1)}$

A. Domain: all x, except for $x = -1, 0, 1$.

B. The graph is symmetric about the origin because $f(-x) = \frac{1}{(-x)^3+x} = \frac{1}{-x^3+x} = -f(x)$.

C. Because $y = \frac{1}{x^3-x} = \frac{\frac{1}{x^3}}{1-\frac{1}{x^2}}$, we see that $\lim_{x\to\pm\infty} \frac{1}{x^3-x} = \frac{0}{1-0} = 0$. So $y = 0$ is a horizontal asymptote. The lines $x = -1, x = 0$, and $x = 1$ are all vertical asymptotes because

$$\lim_{x\to-1^-} \frac{1}{x(x-1)(x+1)} = -\infty \quad \text{and} \quad \lim_{x\to-1^+} \frac{1}{x(x-1)(x+1)} = +\infty$$
$$\lim_{x\to0^-} \frac{1}{x(x-1)(x+1)} = +\infty \quad \text{and} \quad \lim_{x\to0^+} \frac{1}{x(x-1)(x+1)} = -\infty,$$
$$\lim_{x\to1^-} \frac{1}{x(x-1)(x+1)} = -\infty \quad \text{and} \quad \lim_{x\to1^+} \frac{1}{x(x-1)(x+1)} = +\infty.$$

D. Because $y = (x^3-x)^{-1}$, we get

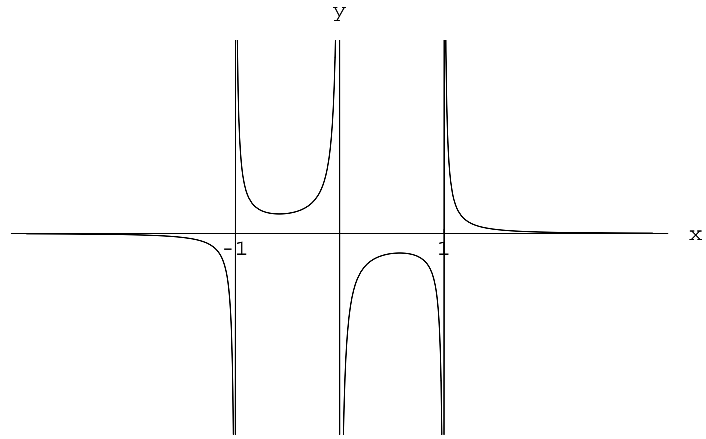
$$y' = -(x^3-x)^{-2}(3x^2-1) = \frac{-3(x^2-\frac{1}{3})}{(x^3-x)^2}.$$

So $y' = 0$ when $x = \pm\frac{1}{\sqrt{3}}$ and y' is not defined when $x = -1, 0$, or 1. So the critical points are $-1, -\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}$, 1. Because $\frac{1}{\sqrt{3}} \approx 0.58$, take the test points $-2, -\frac{2}{3}, -\frac{1}{2}, \frac{1}{2}, \frac{2}{3}$, and 2, to check that: $y' < 0$ when $x < -1$; $y' < 0$ when $-1 < x < -\frac{1}{\sqrt{3}}$; $y' > 0$ when $-\frac{1}{\sqrt{3}} < x < 0$; $y' > 0$ when $0 < x < \frac{1}{\sqrt{3}}$; $y' < 0$ when $\frac{1}{\sqrt{3}} < x < 1$: and $y' < 0$ when $x > 1$. So the graph is decreasing over the intervals $x < -1, -1 < x < -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} < x < 1$, and $x > 1$; and increasing over $-\frac{1}{\sqrt{3}} < x < 0$ and $0 < x < \frac{1}{\sqrt{3}}$. There is a local minimum at $x = -\frac{1}{\sqrt{3}}$ and a local maximum at $x = \frac{1}{\sqrt{3}}$.

E.

$$\begin{aligned} y'' &= \frac{-6x(x^3-x)^2 + 3(x^2-\frac{1}{3})2(x^3-x)(3x^2-1)}{(x^3-x)^4} \\ &= \frac{-6x(x^3-x) + 2(3x^2-1)(3x^2-1)}{(x^3-x)^3} \\ &= \frac{-6x^4+6x^2+18x^4-12x^2+2}{(x^3-x)^3} = \frac{12x^4-6x^2+2}{(x^3-x)^3} \\ &= \frac{2(6x^4-3x^2+1)}{(x^3-x)^3}. \end{aligned}$$

Because $\sqrt{9-24} = \sqrt{-15}$ is not a real number, $2(6x^4-3x^2+1) > 0$ for all x. So only the points $-1, 0$, and 1 (where y'' is not defined) play a role. Take $-2, -\frac{1}{2}, \frac{1}{2}$, and 2 as test points to check that the graph captures the concavity information.



51. $y = e^{\sqrt{x^2+1}} = e^{(x^2+1)^{\frac{1}{2}}}$.

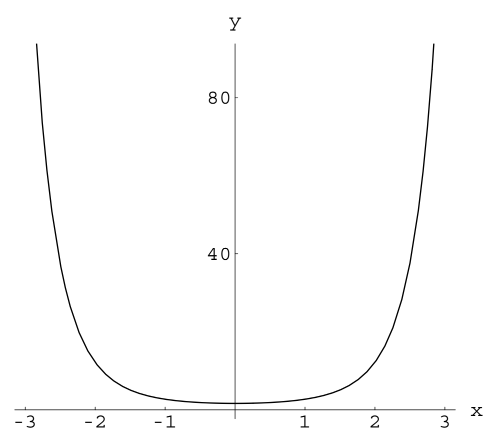
A. Domain: Note that $x^2 + 1 > 0$ for all x. So the domain is all real numbers.

B. Since $e^{\sqrt{x^2+1}} = e^{\sqrt{(-x)^2+1}}$, the graph is symmetric about the $y-$axis.

C. Because the composite of continuous function is continuous, $y = e^{\sqrt{x^2+1}}$ is continuous or all x and hence the graph has no vertical asymptotes. Because $\lim_{x\to\pm\infty} e^{\sqrt{x^2+1}} = +\infty$, there are no horizontal asymptotes.

D. $y' = e^{(x^2+1)^{\frac{1}{2}}} \cdot \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x = \frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}} = \frac{xe^{(x^2+1)^{\frac{1}{2}}}}{(x^2+1)^{\frac{1}{2}}}$. So $y' < 0$ when $x < 0$ and $y' > 0$ when $x > 0$. So the graph is decreasing for $x < 0$ and increasing for $x > 0$. When $x = 0$, there is an absolute minimum. This smallest value of y is e.

E.
$$y'' = \frac{\frac{d}{dx}(xe^{(x^2+1)^{\frac{1}{2}}})(x^2+1)^{\frac{1}{2}} - xe^{(x^2+1)^{\frac{1}{2}}}\frac{1}{2}(x^2+1)^{-\frac{1}{2}}2x}{x^2+1}$$
$$= \frac{[e^{(x^2+1)^{\frac{1}{2}}} + xe^{(x^2+1)^{\frac{1}{2}}} \cdot \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)](x^2+1)^{\frac{1}{2}} - x^2e^{(x^2+1)^{\frac{1}{2}}}(x^2+1)^{-\frac{1}{2}}}{x^2+1}$$
$$= \frac{e^{(x^2+1)^{\frac{1}{2}}}}{x^2+1}\left[(1 + x^2(x^2+1)^{-\frac{1}{2}})(x^2+1)^{\frac{1}{2}} - x^2(x^2+1)^{-\frac{1}{2}}\right]$$
$$= \frac{e^{(x^2+1)^{\frac{1}{2}}}}{x^2+1}\left[(x^2+1)^{\frac{1}{2}} + x^2 - x^2(x^2+1)^{-\frac{1}{2}}\right]$$
$$= \frac{e^{(x^2+1)^{\frac{1}{2}}}}{x^2+1}\left[\frac{x^2+1+x^2(x^2+1)^{\frac{1}{2}} - x^2}{(x^2+1)^{\frac{1}{2}}}\right] = \frac{e^{(x^2+1)^{\frac{1}{2}}}}{x^2+1}\left[\frac{1+x^2(x^2+1)^{\frac{1}{2}}}{(x^2+1)^{\frac{1}{2}}}\right].$$



Notice that $y'' > 0$ for all x. So the graph is concave up throughout.

52. $y = x^2 \cdot e^{-x^2} = \frac{x^2}{e^{x^2}}$.

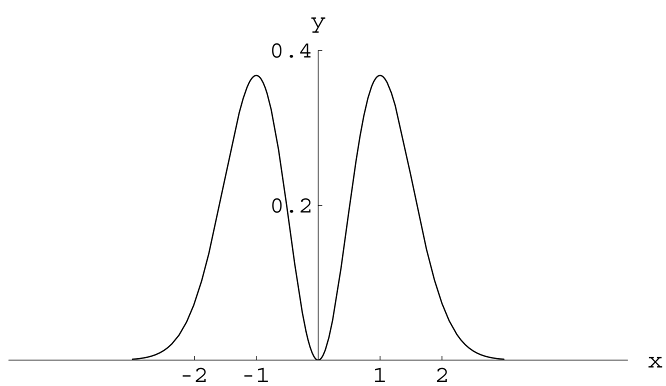
A. Domain: all real numbers.

B. The graph is symmetric about the y-axis because $(-x)^2 e^{-(-x)^2} = x^2 e^{-x^2}$.

C. Because e^{x^2} grows much faster than x^2 for increasing x, it follows that $\lim_{x\to\pm\infty} x^2 e^{-x^2} = \lim_{x\to\pm\infty} \frac{x^2}{e^{x^2}} = 0$. So $y = 0$ is a horizontal asymptote. Because composites and products of continuous functions are continuous, $y = x^2 e^{-x^2}$ is continuous for all x. So there can be no vertical asymptotes.

D. $y' = 2x \cdot e^{-x^2} + x^2 e^{-x^2}(-2x) = \frac{2x - 2x^3}{e^{x^2}} = \frac{2x(1-x^2)}{e^{x^2}} = \frac{2x(1-x)(1+x)}{e^{x^2}}$. So $y' = 0$ when $x = -1, 0,$ or 1. There are no other critical numbers. Checking at $-2, -\frac{1}{2}, \frac{1}{2}$, and 2, tells us that $y' > 0$ when $x < -1$; $y' < 0$ when $-1 < x < 0$, $y' > 0$ when $0 < x < 1$; and $y' < 0$ when $x > 1$. It follows that the graph has local maxima at $x = -1$ and $x = 1$ and a local minimum at $x = 0$.

E. $y'' = \frac{(2-6x^2)e^{x^2} - (2x-2x^3)e^{x^2}2x}{(e^{x^2})^2} = \frac{2-6x^2-4x^2+4x^4}{e^{x^2}} = \frac{4x^4-10x^2+2}{e^{x^2}}$.
By the quadratic formula, $4x^4 - 10x^2 + 2 = 0$ when $x^2 = \frac{10\pm\sqrt{100-32}}{8} = \frac{10\pm\sqrt{68}}{8} = \frac{5\pm\sqrt{17}}{4}$. So $x = \frac{\pm\sqrt{5\pm\sqrt{17}}}{2} \approx \pm 1.51$ or ± 0.47. Because $e^{x^2} \neq 0$ for all x, there are no other points to consider. Taking $-2, -1, 0, 1$ and 2 as test points, we discover that $y'' > 0$ when $x < \frac{-\sqrt{5+\sqrt{17}}}{2}$; $y'' < 0$ when $\frac{-\sqrt{5+\sqrt{17}}}{2} < x < \frac{-\sqrt{5-\sqrt{17}}}{2}$; $y'' > 0$ when $\frac{-\sqrt{5-\sqrt{17}}}{2} < x < \frac{\sqrt{5-\sqrt{17}}}{2}$; $y'' < 0$ when $\frac{\sqrt{5-\sqrt{17}}}{2} < x < \frac{\sqrt{5+\sqrt{17}}}{2}$; and $y'' > 0$ when $x > \frac{\sqrt{5+\sqrt{17}}}{2}$. The graph sketched below reflects the information just obtained.



53. $y = \ln(x+3)$

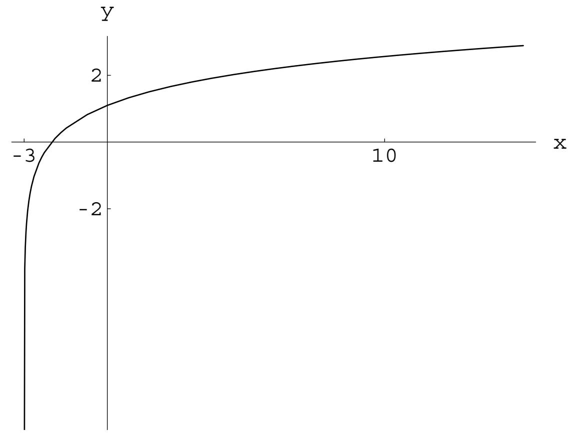
A. For this function to be defined, we need $x + 3 > 0$ or $x > -3$.

B. No symmetry. The graph is obtained by shifting the graph of $y = \ln x$ three units to the left.

C. Because $\lim_{x\to\infty} \ln(x+3) = +\infty$, there is no horizontal asymptote. The line $x = -3$ is a possible vertical asymptote. Because $\lim_{x\to -3^+} \ln(x+3) = -\infty$, this is indeed so.

D. $y' = \frac{1}{x+3}$. Because $x > -3, y' > 0$ throughout. So the graph is always increasing.

E. $y'' = -(x+3)^{-2} = \frac{-1}{(x+3)^2}$. So the graph is concave down throughout.



54. $y = x^2 + \ln x$

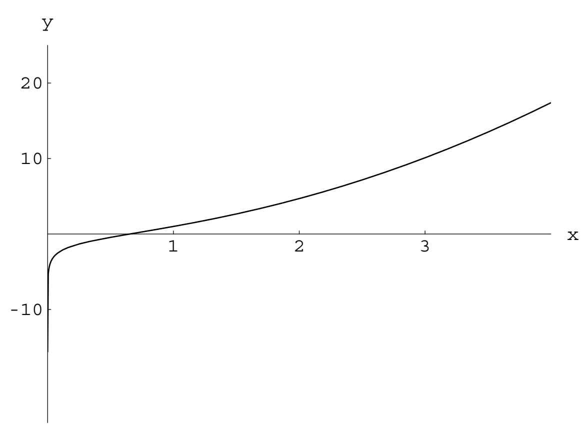
A. Domain: We need $x > 0$ for $\ln x$ to make sense.

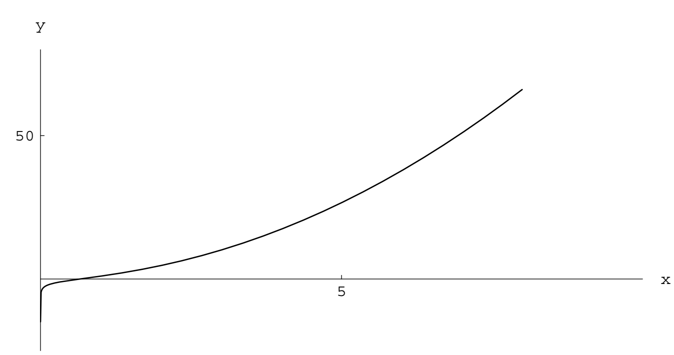
B. No symmetry.

C. For x large (and positive) x^2 dominates and hence $y \approx x^2$. For x small (and positive), $\ln x$ dominates, so $y \approx \ln x$. It follows that there is no horizontal asymptote, but that $x = 0$ is a vertical asymptote.

D. $y' = 2x + \frac{1}{x}$. Because $x > 0,\ y' > 0$ and the graph is increasing throughout.

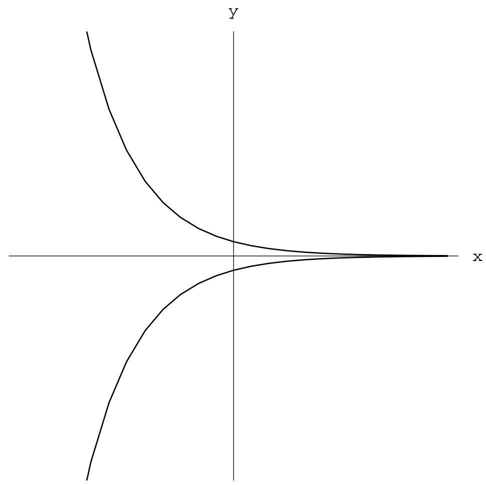
E. $y'' = 2 - x^{-2} = \frac{2x^2-1}{x^2}$. Note that $y'' = 0$ when $x = \sqrt{\frac{1}{2}}$. For $0 < x < \sqrt{\frac{1}{2}}$, notice that $y'' > 0$ and for $x > \sqrt{\frac{1}{2}},\ y'' < 0$. So the graph is concave up when $0 < x < \sqrt{\frac{1}{2}}$, concave down when $x > \sqrt{\frac{1}{2}}$, and it has a point of inflection at $(\sqrt{\frac{1}{2}}, \frac{1}{2} + \ln\sqrt{\frac{1}{2}})$. Where exactly





does the graph cross the x-axis? Use a calculator to verify that the $x-$coordinate of this point satisfies $0.652 \le x \le 0.653$.

55. Refer to Exercise 8 to see how to obtain the graphs of e^{-x} and $-e^{-x}$. Because $-1 \le \sin x \le 1$ it follows that $-e^{-x} \le e^{-x}\sin x \le e^{-x}$. So the graph of $g(x) = e^{-x}\sin x$ has to lie between the graphs of e^{-x} and $-e^{-x}$. These (refer to Exercise 8) are sketched below. Because $\sin x = 0$



for $x = \cdots - 3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$ and $\sin x = 1$ for $x = \cdots - \frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5x}{2}, \ldots$ and finally, $\sin x = -1$ for $x = \cdots - \frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$, we find that the following points are on the graph of $e^{-x}\sin x$:

$\cdots(-\frac{7\pi}{2}, e^{\frac{7\pi}{2}}),\ (-3\pi, 0), (-\frac{5\pi}{2}, -e^{\frac{5\pi}{2}}),\ (-2\pi, 0), (-\frac{3\pi}{2}, e^{\frac{3\pi}{2}}),\ (-\pi, 0),\ (0,0), (\frac{\pi}{2}, e^{-\frac{\pi}{2}}), (\pi, 0)$, $(\frac{3\pi}{2}, e^{-\frac{3\pi}{2}}),\ (2\pi, 0), \ldots$. Plot these points to get some idea of the "flow" of the graph.

A. Domain: Both e^{-x} and $\sin x$ make sense for all x. So the domain of $g(x)$ is the set of all real numbers x.

B. No symmetry. Shifting does not illuminate things either.

C. By assertions already made, the line $y = 0$ is a horizontal asymptote, and there cannot be a vertical asymptote.

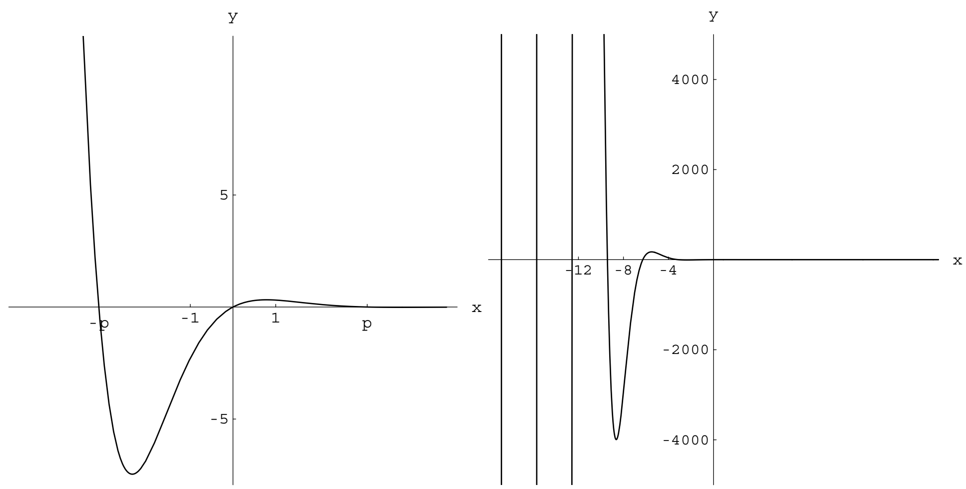
D. $g'(x) = -e^{-x}\sin x + e^{-x}\cos x = \frac{\cos x - \sin x}{e^x}$. Recall from Section 1.4 that $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. All other x for which $\cos x = \sin x$ are obtained by using Example 4.12. $\cos\frac{5\pi}{4} = \cos(\frac{\pi}{4} + \pi) = -\cos\frac{\pi}{4} = -\sin\frac{\pi}{4} = \sin(\frac{\pi}{4} + \pi) = \sin\frac{5\pi}{4}$. Also $\cos(\frac{\pi}{4} + \pi - 2\pi) = \sin(\frac{\pi}{4} + \pi - 2\pi)$, so $\cos(-\frac{3\pi}{4}) = \sin(-\frac{3\pi}{4})$. It follows that $g'(x) = 0$ for $x = \cdots \frac{-11\pi}{4}, \frac{-7\pi}{4}, \frac{-3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots$.

Let's illustrate what happens over the interval $[-\pi, \pi]$. A comparison of the graphs of $\sin x$ and $\cos x$ (see Figures 10.28 and10.29) shows that for $[-\pi, \frac{-3\pi}{4}], \sin x > \cos x$; for $[\frac{-3\pi}{4}, \frac{\pi}{4}], \cos x > \sin x$; and for $[\frac{\pi}{4}, \pi], \sin x > \cos x$. So $g(x) = e^{-x}\sin x$ is decreasing for $[-\pi, \frac{-3\pi}{4}]$; increasing for $[-\frac{3\pi}{4}, \frac{\pi}{4}]$; and decreasing for $[\frac{\pi}{4}, \pi]$.

E. $g''(x) = \frac{(-\sin x - \cos x)e^x - (\cos x - \sin x)e^x}{(e^x)^2} = \frac{-2\cos x}{e^x}$. So the sign of the cosine determines the concavity of the graph of $g(x)$. In particular, over $[-\pi, \pi]$, $g(x)$ is concave up over

$[-\pi, \frac{-\pi}{2}]$; concave down over $[\frac{-\pi}{2}, \frac{\pi}{2}]$; and concave up over $[\frac{\pi}{2}, \pi]$.

Two versions of the graph follow below. The one on the left concentrates on what happens near the origin. The one on the right shows what it looks like "from a distance". Because $g(x) = e^{-x} \sin x$ is a product of two continuous functions, $g(x)$ is also continuous. Observe



therefore that the graph on the left is not complete: the lines come together very high above (respectively below) the x axis.

56. $f(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$.

A. Because $1 + x^2$ is never zero, the domain of f is the set of all real numbers.

B. The graph is symmetric about the y-axis because $f(x) = f(-x)$.

C. Because $\frac{1}{1+x^2} = \frac{\frac{1}{x^2}}{\frac{1}{x^2}+1}$, $\lim_{x \to \pm\infty} f(x) = \frac{0}{0+1} = 0$. So $y = 0$ is horizontal asymptote. There is no vertical asymptote because f is continuous for all x. (It is the quotient of two functions that are continuous for all x.)

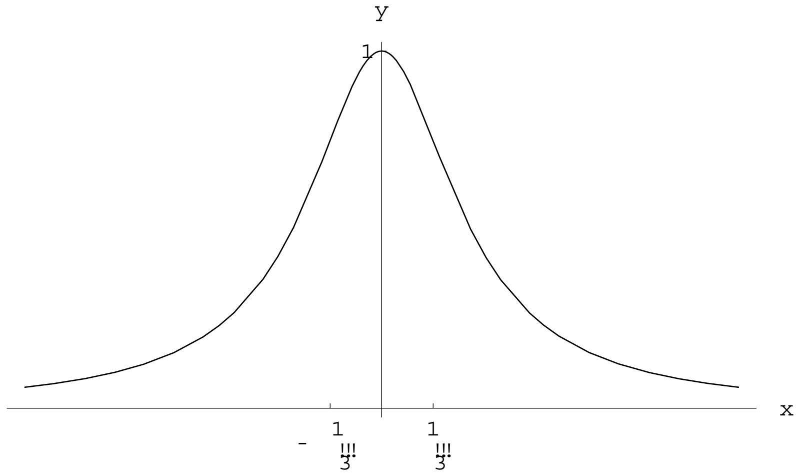
D. $f'(x) = -1(1+x^2)^{-2}(2x) = \frac{-2x}{(1+x^2)^2}$. Observe that $f'(x) > 0$ when $x < 0$; that $f'(x) < 0$ when $x > 0$; and that $f'(x) = 0$ when $x = 0$. So $f(x)$ is increasing for $x < 0$, decreasing for $x > 0$, and has an absolute maximum value at $x = 0$. This value is $f(0) = 1$.

E.

$$
\begin{aligned}
f''(x) &= \frac{-2(1+x^2)^2 + 2x \cdot 2(1+x^2)2x}{(1+x^2)^4} \\
&= \frac{-2(1+x^2) + 8x^2}{(1+x^2)^3} \\
&= \frac{6x^2 - 2}{(1+x^2)^3}.
\end{aligned}
$$

Observe that $f''(x) = 0$ when $6x^2 = 2$, i.e., when $x^2 = \frac{1}{3}$, or $x = \pm\frac{1}{\sqrt{3}} \approx \pm 0.58$. Taking $-1, 0$, and 1 as test points, we see that $f''(x) > 0$ for $x < -\frac{1}{\sqrt{3}}$; that $f''(x) < 0$ for

$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$; and that $f''(x) > 0$ for $x > \frac{1}{\sqrt{3}}$. So the graph of f exhibits the concavity properties shown and has points of inflection when $x = \pm\frac{1}{\sqrt{3}}$.



57. **A.** Observe that there are no limitations on x and hence that the domain of f is the set of all real numbers.

B. Replacing x by $-x$ has no effect on $f(x)$. Hence the graph is symmetric about the y-axis.

C. Pushing x to a larger and larger number pushes $\frac{x^2}{2\sigma^2}$ and hence also $e^{\frac{x^2}{2\sigma^2}}$ to a larger and larger number. It follows that

$$\lim_{x\to\pm\infty} e^{-\frac{x^2}{2\sigma^2}} = \lim_{x\to\infty} \frac{1}{e^{\frac{x^2}{2\sigma^2}}} = 0.$$

So the x-axis is a horizontal asymptote.

D. Observe that by the chain rule,

$$f'(x) = e^{-\frac{x^2}{2\sigma^2}}\left(-\frac{x}{\sigma^2}\right) = \frac{-x}{\sigma^2 e^{\frac{x^2}{2\sigma^2}}}.$$

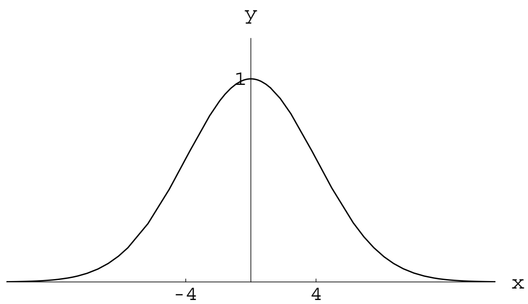
Because $\sigma^2 e^{\frac{x^2}{2\sigma^2}} > 0$, it follows that 0 is the only critical number. If $x < 0$, then $f'(x) > 0$, and if $x > 0$, $f'(x) < 0$. So $f(x)$ is increasing when $x < 0$ and decreasing when $x > 0$. It has an (absolute) maximum at $x = 0$.

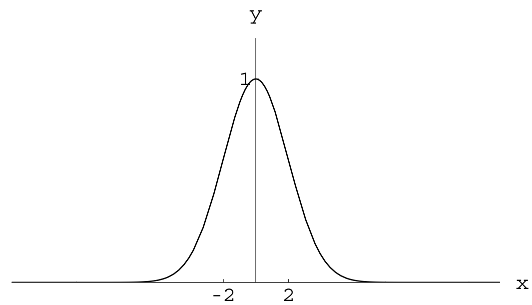
E.

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(-\frac{x}{\sigma^2}\cdot e^{-\frac{x^2}{2\sigma^2}}\right) = -\frac{1}{\sigma^2}\cdot e^{-\frac{x^2}{2\sigma^2}} + \left(-\frac{x}{\sigma^2}\right)\cdot e^{-\frac{x^2}{2\sigma^2}}\left(-\frac{x}{\sigma^2}\right) \\ &= -\frac{1}{\sigma^2}\cdot e^{-\frac{x^2}{2\sigma^2}} + \frac{x}{\sigma^2}\cdot e^{-\frac{x^2}{2\sigma^2}}\cdot\frac{x}{\sigma^2} = \frac{1}{\sigma^2}\cdot e^{-\frac{x^2}{2\sigma^2}}\left(\frac{x^2}{\sigma^2} - 1\right) \\ &= \frac{1}{\sigma^2}\cdot e^{-\frac{x^2}{2\sigma^2}}\left(\frac{x}{\sigma}+1\right)\left(\frac{x}{\sigma}-1\right). \end{aligned}$$

Since $\frac{1}{\sigma^2}\cdot e^{-\frac{x^2}{2\sigma^2}} > 0$, it is clear that $f''(x) = 0$ for $x = -\sigma$ and $x = \sigma$. Notice that $f''(x) > 0$ for x in $(-\infty, -\sigma)$ and (σ, ∞) and $f''(x) < 0$ for x in $(-\sigma, \sigma)$. So $f(x)$

is concave up over $(-\infty, -\sigma)$ and (σ, ∞), and concave down over $(-\sigma, \sigma)$. The points $(-\sigma, f(-\sigma)) = (-\sigma, e^{-\frac{1}{2}})$ and $(\sigma, f(\sigma)) = (\sigma, e^{-\frac{1}{2}})$ are the two points of inflection.

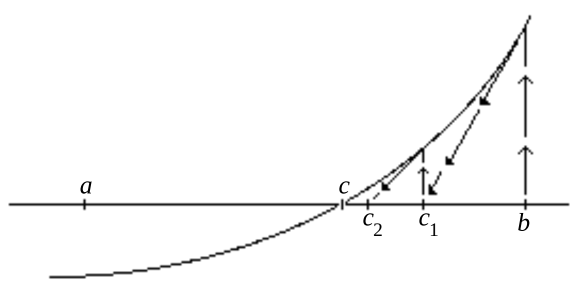




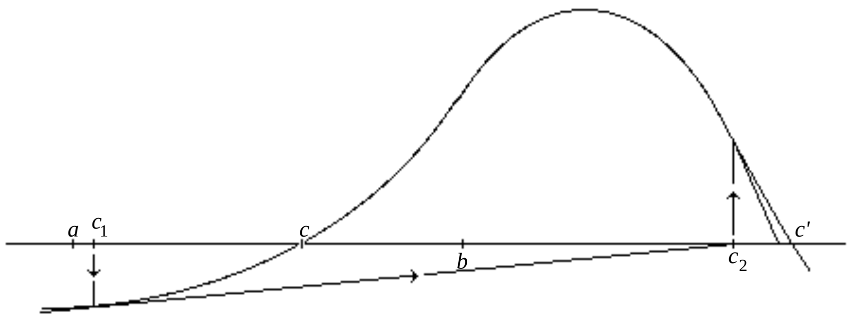
Comment: The specifics of the graph depend on σ. The cases $\sigma = 2$ and $\sigma = 4$ are shown above. The parameter σ is called the *standard deviation*. Refer to **(E)** above and observe that σ is the distance from 0 to the x-coordinate of the point of inflection. It is a measure of the amount of "spread" in the curve. The total area $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}}\,dx$ under the curve turns out to be finite! It can be shown to equal $\sigma\sqrt{2\pi}$. It follows that $\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}\,dx = 1$. The "normalized" functions $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$ provide the *bell shaped* curves that arise in the statistical and probabilistic analyses of data.

10G. Concavity and Newton's Method

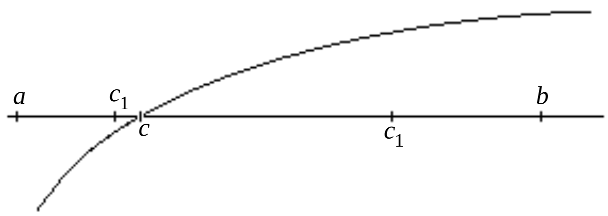
58. Consider a typical graph that satisfies the requirements: Take any guess c_1 with $c < c_1 < b$,



and observe from the figure how Newton's method closes in on c. Suppose next that you have started with a guess c_1 with $a < c_1 < c$. the indicated shape. Notice that the approximation c_2 falls to the right of c on the x-axis. The course of Newton's method now depends on where c_2 falls. If it falls into an interval I containing b such that the graph of f is concave up on I, then the flow depicted in the figure above is still valid and Newton's method will close in on c. However, if the tangent line to the graph at $(c_1, f(c_1))$ intersects the x-axis beyond $x = b$, then all bets are off. In particular, in the example sketched below, Newton's method will converge to a zero c' of $y = f(x)$ that falls outside the interval $[a, b]$.



59. The graph sketched below shows that the situation of Exercise 58 has been

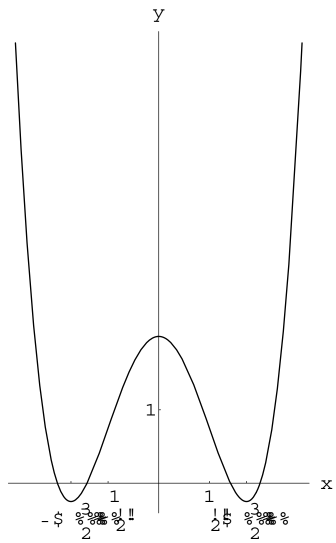


reversed: Newton's Method will converge to c for a guess c_1 with $a < c_1 < c$, and it may or may not converge to c for a guess that satisfies $c < c_1 < b$.

60. Only the increasing/decreasing and the concavity information is relevant, and this is all we will pursue. Because

$$\begin{aligned} f'(x) &= 4x^3 - 6x = 4x\left(x^2 - \frac{3}{2}\right) \\ &= 4x\left(x - \sqrt{\frac{3}{2}}\right)\left(x + \sqrt{\frac{3}{2}}\right) \end{aligned}$$

we find that the graph is decreasing on $\left(-\infty, -\sqrt{\frac{3}{2}}\right)$ and $\left(0, \sqrt{\frac{3}{2}}\right)$ and increasing on $\left(-\sqrt{\frac{3}{2}}, 0\right)$ and $\left(\sqrt{\frac{3}{2}}, \infty\right)$. Because $f''(x) = 12x^2 - 6 = 12(x^2 - \frac{1}{2}) = 12\left(x - \sqrt{\frac{1}{2}}\right)\left(x + \sqrt{\frac{1}{2}}\right)$ we see that the graph is concave up over $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$ and concave down over $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$. We already know that $f(x) = 0$ when $x = -\sqrt{2}, -1, 1$, and $\sqrt{2}$. The graph of f can now be sketched as follows:



61. Start by considering any guess c_1 with $c_1 > 0$. If c_1 falls into the interval $\left[\sqrt{2}, \infty\right)$ then Newton's method will converge to the root $\sqrt{2}$. This follows from Exercises 58 and 60. If c_1 falls into the interval $\left(\frac{1}{\sqrt{2}}, 1\right]$, then Newton's method will converge to the root 1. This follows from Exercises 59 and 60. Let's consider $c_1 = 0.1$ next. Going from $c_1 = 0.1$ up to the graph and back down along the tangent to the $x-$axis locates c_2 in the interval $\left(\sqrt{2}, \infty\right)$. It follows that with $c_1 = 0.1$, Newton's method converges to $\sqrt{2}$. For the starting point $c_1 = 1.2$, what happens is less clear, but it turns out that $c_2 \approx 0.344$ and $c_3 \approx 1.214$. The next step would show that c_4 is to the left of c_1, and ultimately, that Newton's method converges to -1. The following table shows how Newton's method converges for selected values of c_1:

c_1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4
c	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	1	1	1	1	1	1	1	1	-1	$\sqrt{2}$	$\sqrt{2}$

.

The area around 1.2 is very volatile: Though 1.2 converges to -1, the values 1.1999 and 1.2001 converge to $-\sqrt{2}$; 1.1998 and 1.2002 converge to $\sqrt{2}$; and 1.1997 converges to 1!