## Solutions to the Exercises of Chapter 1

### 1A. The Greek Number System

**1.** $85 = \pi\varepsilon$ ; $842 = \omega\mu\beta$ ; $34{,}547 = \gamma M\overline{,\delta\phi\mu\zeta}$ ; $2{,}875{,}739 = \sigma\pi\zeta M\overline{,\varepsilon\psi\lambda\theta}$ .

### 1B. Greek Algebra

**2.** Let $x$ be the number of apples. Observe that

$$\frac{x}{5} + \frac{x}{12} + \frac{x}{8} + \frac{x}{20} + \frac{x}{4} + \frac{x}{7} + 500 = x.$$

Multiplying through by the common denominator $840 = 2^3 \cdot 3 \cdot 5 \cdot 7$, shows that

$$168x + 70x + 105x + 42x + 210x + 120x + (840)(500) = 840x,$$

and therefore, that $125x = (840)(500)$. So $x = 4(840) = 3360$.

**Note:** The mina is an early unit of weight. It was introduced by the Babylonians and used by several ancient peoples including the Greeks. There were different versions of the mina in the ancient world. One version from the Babylonian period was equivalent to about 640 grams and another to about 980 grams. The Greek mina is thought to be equivalent to about 430 grams or one pound.

**3.** Let $x, y, z$ and $w$ be the respective weights of the gold, brass, tin, and iron in the crown. Observe that

$$x + y = \frac{2}{3} \cdot 60 = 40;\ x + z = \frac{3}{4} \cdot 60 = 45;\ x + w = \frac{3}{5} \cdot 60 = 36;\ \text{and } x + y + z + w = 60.$$

This system of 4 equations in 4 unknowns can be solved as follows: Since $y = 40-x, z = 45-x$, and $w = 36 - x$, we get that $x + (40 - x) + (45 - x) + (36 - x) = 60$. So $-2x + 121 = 60$, and $x = 30\frac{1}{2}$. Therefore, $y = 9\frac{1}{2}, z = 14\frac{1}{2}$, and $w = 5\frac{1}{2}$.

**4.** Presumably the question concerns the weight $x$ of the bowl. (This seems to be the only thing we can solve for). Since the mass weighs $x+\frac{x}{3}+\frac{x}{4}+\frac{x}{12} = 1$, we find that $\frac{12x+4x+3x+x}{12} = \frac{20}{12}x = 1$. So $x = \frac{12}{20} = \frac{3}{5}$.

**5.** We will assume that the daily production of the son and son-in-law is 200 and 250 bricks respectively (even though they do not appear to have worked a full day). So the three together can make $300 + 200 + 250 = 750$ bricks in one day. Therefore, they will require $\frac{300}{750} = \frac{2}{5}$ of a day to make 300. Observe that the answer needs to be given in hours. While the wording of the problem suggests that "day" means "working day," it is not clear how many hours such a working day has. If it has eight hours, then the three brick makers will need $\frac{2}{5} \cdot 8 = \frac{16}{5} = 3\frac{1}{5}$ hours. And if it has twelve?

## 1C. The Quadratic Formula

**6.** **i.** $x^2 - 5x + 4 = 0$, exactly when $(x - \frac{5}{2})^2 = \frac{9}{4}$; in other words when $x - \frac{5}{2} = \pm\frac{3}{2}$. So $x = \frac{5}{2} \pm \frac{3}{2}$, and $x = 1$ and $x = 4$ are the two solutions.

**ii.** The smallest value that $x^2 - 5x + 4 = (x - \frac{5}{2})^2 - \frac{9}{4}$ can have is $-\frac{9}{4}$. This is so because $(x - \frac{5}{2})^2$ is greater than or equal to 0 for any $x$. This largest value is achieved for $x = \frac{5}{2}$.

**7.** Note that $3x^2 + 21x + 12 = 3(x^2 + 7x + 4)$. Completing the square gives:

$$x^2 + 7x + 4 = x^2 + 7x + (\frac{7}{2})^2 - (\frac{7}{2})^2 + 4 = (x + \frac{7}{2})^2 + 4 - \frac{49}{4} = (x + \frac{7}{2})^2 - \frac{33}{4}.$$

So $x^2 + 7x + 4$ is equal to 0 precisely when $x = -\frac{7}{2} \pm \frac{\sqrt{33}}{2}$.

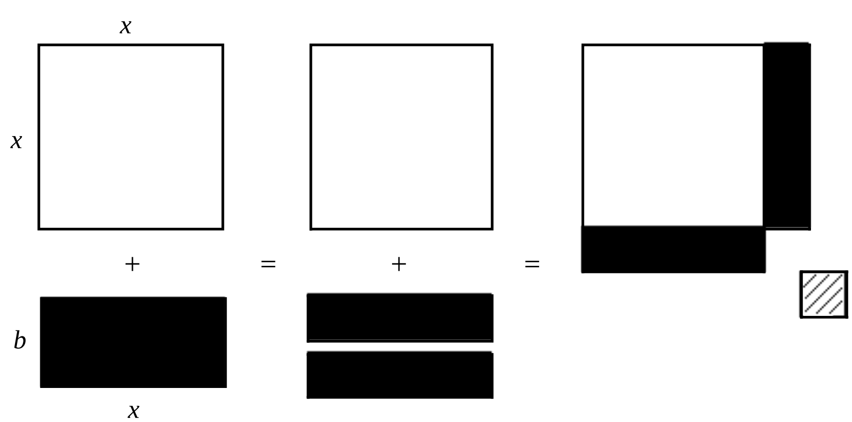
**8.** If $a = 0$, then the equation reduces to $bx + c = 0$. If $b = 0$, then any $x$ is a solution. If $b \neq 0$, then $x = -cb^{-1}$ is the only solution. Assume $a \neq 0$. Because $ax^2 + bx + c = a(x^2 + \frac{b}{a}x + \frac{c}{a})$ and $a \neq 0$, we can deal with $x^2 + \frac{b}{a}x + \frac{c}{a}$ instead. Completing the square shows that

$$x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a} = (x + \frac{b}{2a})^2 + \frac{c}{a} - (\frac{b}{2a})^2.$$

So, $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, precisely when $(x + \frac{b}{2a})^2 = (\frac{b}{2a})^2 - \frac{c}{a} = \frac{b^2-4ac}{4a^2}$. Taking square roots of both sides, gives $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}} = \pm\frac{\sqrt{b^2-4ac}}{2a}$. It is now clear that $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, as required. Incidentally, is $\sqrt{4a^2}$ always equal to $2a$? If not, why doesn't it matter here?

**Note:** The following diagram provides a "picture" of the completion of the square

$$x^2 + bx = (x + \frac{b}{2})^2 - (\frac{b}{2})^2.$$




## 1D. Rational and Real Numbers

**9.** The price of a single item must be $33\frac{1}{3}$ cents. Since the penny is the smallest coin, $\frac{1}{3}$ of a cent does not exist in the system. Put another way, one-third is not a rational number of the form $x\frac{yz}{100}$. The cashier will generally charge 34 cents for the item.

**10.** **i.** Let $r = 1.333333\ldots$ . Note that $10r = 13.33333\ldots$ , so that by subtracting,

$$9r = 10r - r = 13.33333\ldots - 1.33333\ldots = 12.00000\ldots .$$

Therefore, $r = \frac{12}{9} = \frac{4}{3}$.

**ii.** Let $r = 2.676767\ldots$ . Note that $100r = 267.676767\ldots$ , so that by subtracting,

$$99r = 100r - r = 267.676767\ldots - 2.676767\ldots = 265.000000\ldots .$$

So $r = \frac{265}{99}$.

**iii.** Let $r = 4.728728\ldots$ . Note that $1000r = 4728.728728\ldots$ , so that by subtracting,

$$999r = 1000r - r = 4728.728728\ldots - 4.728728\ldots = 4724.000000\ldots .$$

So $r = \frac{4724}{999}$.

**11.** $\frac{5}{4} = 1.250000\ldots$ and $\frac{468}{198} = 2.363636\ldots$ .

**Correction.** In the Note following Exercise 11, observe that with $r = 0.3636\ldots$ , we get $100r = 36.36\ldots$, so that $99r = 36$ and hence $r = \frac{36}{99}$. Therefore, $52\frac{468}{198}$ is incorrect. It can be replaced by any of the following: $52\frac{36}{99}$, $52\frac{72}{198}$, or $50\frac{468}{198}$.

**12.** **i.** $28 = 4 \cdot 7 = 2^2 \cdot 7$; $192 = 64 \cdot 3 = 2^6 \cdot 3$; $143 = 11 \cdot 13$. The strategy is this: see if the prime 2 divides the number. If not, go to the next prime 3. If yes, keep dividing by 2 until division by 2 is no longer possible. Repeat the procedure for 3, then 5, then 7, etc.

**ii.** Assume that $\sqrt{3}$ is rational. So $\sqrt{3} = \frac{m}{n}$, where $m$ and $n$ are positive integers. Let $m = p_1^{r_1} \cdots p_i^{r_i}$ be the factorization of $m$ into powers of primes, and let $n = q_1^{s_1} \cdots p_j^{s_j}$ be that of $n$. Note that the prime factorizations of $m^2$ and $n^2$ are $m^2 = p_1^{2r_1} \cdots p_i^{2r_i}$ and $n^2 = q_1^{2s_1} \cdots q_j^{2s_j}$ respectively. Since $m^2 = 3n^2$, it follows that $p_1^{2r_1} \cdots p_i^{2r_i} = 3q_1^{2s_1} \cdots q_j^{2s_j}$. In the factorization on the right the prime 3 must occur to an odd power, but on the left all primes occur to an even power. Therefore, the number $m^2 = 3n^2$ has two different factorizations into prime powers. Since a number has only one factorization into primes, this is not possible. Therefore, $\sqrt{3}$ cannot be a rational number.

**iii.** Suppose $n = p_1^{r_1} \cdots p_i^{r_i}$ with $r_1, \ldots, r_i$ all even. Put $r_1 = 2s_1, r_2 = 2s_2, \ldots, r_i = 2s_i$, and let $m = p_1^{s_1} \cdots p_i^{s_i}$. Notice that $m^2 = p_1^{2s_1} \cdots p_i^{2s_i} = n$. So $n$ is a square. If $n$ is a square, then $n = m^2$, for some $m$. Let $m = p_1^{s_1} \cdots p_i^{s_i}$ be the factorization into prime powers, and observe that $n = p_1^{2s_1} \cdots p_i^{2s_i}$. Therefore in the prime factorization of $n$ all the exponents are even.

**iv.** Suppose $\sqrt{n}$ is a rational number. So $\sqrt{n} = \frac{r}{s}$ for some positive integers $r$ and $s$. Let $n = p_1^{r_1} \cdots p_i^{r_i}$ be the factorization of $n$ into prime powers. Since, $ns^2 = r^2$ and all the exponents of the prime factorizations of $s^2$ and $r^2$ are both even, all the exponents of the prime factorization of $n$ must be even also. So by (iii), $n$ is a square.

## 1E. Angles and Circular Arcs

**13.** **i.** 1 radian $= \frac{180}{\pi} \approx 57.3$ degrees

**ii.** 1 degree $= \frac{\pi}{180} \approx 0.017$ radians

**iii.** $78.5° = 78.5 \times \frac{\pi}{180}$ radians $\approx 1.37$ radians

**iv.** 1.238 radians $= 1.238 \times \frac{180}{\pi}$ degrees $\approx 70.9°$

**14.** Since $\theta \approx 57.3°$, the radian measure of $\theta$ is 1 (at least approximately). Since $\theta$ in radian measure is also $\frac{\text{arc AB}}{3}$, it follows that arc $AB \approx 3$.

**15.** By a basic fact from geometry, the tangent to the circle at $A$ is perpendicular to the radius through $A$. So $AA'P'$ is a right triangle with right angle at $A$. Since $A'P'$ is the hypothenuse of this right triangle, $AP' < A'P'$. Assuming that arc $AP \leq AP' + P'P$, we get arc $AB < A'P' + P'P = A'P = t$. In the same way, arc $PB' < t$ and hence arc $AB < 2t$.

**16.** The rock is twirled in circular arc of radius 3 feet at a velocity of 4 revolutions per second. The circle has a circumference of $2\pi r = 6\pi = 18.85$ feet. So one revolution corresponds to a distance of 18.85 feet. Thus the twirled rock covers $4(18.85) = 75.4$ feet in one second. So it has a speed of 75.4 feet per second when it flies off.

**17.** Concentrate on the tip of the arrow. Since the radius of the circle is 2 feet, its circumference is $4\pi$ feet. Since the arrow does one revolution in 12 hours, its tip is moving at a rate of $\frac{4\pi}{12} = \frac{\pi}{3}$ feet per hour. The tip of the arrow travels from $A$ to $D$ in 7.5 hours, and from $B$ to $C$ in 6 hours. Therefore it takes 1.5 hours to trace out the arcs $AB$ and $CD$. From the geometry of the configuration, these two arcs are the same. So the tip of the arrow traces out the arc $CD$ in exactly $0.75 = \frac{3}{4}$ hours. Since it moves at a rate of $\frac{\pi}{3}$ feet per hour, it follows that arc $CD$ has length $\frac{3}{4} \cdot \frac{\pi}{3} = \frac{\pi}{4} = 0.785$ feet.

## 1F. Basic Trigonometry

**18.** From Figures 1.19 and 1.20 of the text:

**i.** $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

**ii.** $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

**iii.** $\cos \frac{\pi}{3} = \frac{1}{2}$

**iv.** $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$

**v.** $\tan \frac{\pi}{4} = 1$

**vi.** $\tan \frac{\pi}{3} = \sqrt{3}$

**Correction:** In the statement of part (ii) of Exercise 19, $\cos\theta$ should be replaced by $\cos\phi$, and in part (iv), $\tan\theta$ should be replaced by $\tan\phi$.

**19.** From Figure 1.21, $\cos\theta = \frac{b}{1} = b$ and $\tan\theta = \frac{a}{b}$. Pushing $\theta$ to 0 (and keeping the length of the hypotenuse fixed at 1) pushes $b$ outward to 1 and $a$ to 0. Forcing $\phi$ to $\frac{\pi}{2}$ (by pushing the hypotenuse downward) has the same effect: $a$ is pushed to 0 and $b$ to 1. It follows that

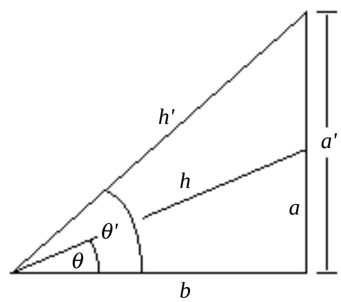
**i.** $\lim_{\theta\to 0} \cos\theta = 1$

**ii.** $\lim_{\phi\to\frac{\pi}{2}} \cos\phi = 0$

**iii.** $\lim_{\theta\to 0} \tan\theta = 0$

**iv.** $\lim_{\phi\to\frac{\pi}{2}} \tan\phi = \frac{1}{0}$ (or undefined).

**20.** Consider the diagram:




Since $\cos\theta = \frac{b}{h}$, $\cos\theta' = \frac{b}{h'}$, and $h < h'$, it follows that $\cos\theta > \cos\theta'$. Since $\tan\theta = \frac{a}{b}$ and $\tan\theta' = \frac{a'}{b}$, it follows that $\tan\theta' > \tan\theta$.

**21.** Since $\sin^2\theta + \cos^2\theta = 1$, we get $\sec^2\theta = \tan^2\theta + 1$, by dividing both sides by $\cos^2\theta$.

**22.** With 7 decimal accuracy:

**i.** $\alpha = 0.1$ radians: $\sin\alpha = 0.0998334$, $\tan\alpha = 0.1003347$.

**ii.** $\alpha = 0.01$ radians: $\sin\alpha = 0.0099998$, $\tan\alpha = 0.010003$.

**iii.** $\alpha = 0.001$ radians: $\sin\alpha = 0.0009999$, $\tan\alpha = 0.0010000$.

**Note:** Exercise 22 provides some numerical evidence for the approximations $\sin\theta \approx \theta$ and $\tan\theta \approx \theta$ for small $\theta$ in radian measure. Show that an expanded version of Figure 1.22 gives evidence for the inequality $\sin\theta < \theta < \tan\theta$ for $0 < \theta < \frac{\pi}{2}$.

## 1G. Distances and Sizes in the Universe

**23.** We will take the value $\pi = 3.1420$ and work with 4 decimal accuracy. In Figure 1.26, $3^\circ$ is replaced by $\frac{1}{6}^\circ$. This angle equals $\frac{1}{6}\frac{\pi}{180} = \frac{\pi}{1080} = 0.0029$ in radians. Since it is a small angle, $\sin 0.0029 = 0.0029$. Therefore, $\frac{D_M}{D_S} = \frac{r_M}{r_S} = 0.0029$. In Figure 1.27, $1^\circ$ is replaced by $\frac{1}{4}^\circ$. In radian measure $\frac{1}{4}^\circ$ is equal to $\frac{1}{4}\frac{\pi}{180} = \frac{\pi}{720} = 0.0044$ radians. This is a small angle. Again, $\sin 0.0044 = 0.0044$. Therefore, $\frac{r_M}{D_M} = 0.0044$. If $4r_M$ is replaced by $5r_M$, and hence $2r_M$ by $2.5r_M$, then a repetition of the analysis that led to the formula $\frac{r_E}{r_M} + \frac{r_E}{r_S} = 3$ will give

$\frac{r_E}{r_M} + \frac{r_E}{r_S} = 3.5$ instead. From $\frac{r_M}{r_S} = 0.0029$, we get $r_S = 345r_M$. Inserting $r_S = 345r_M$, shows that $3.5 = \frac{r_E}{r_M} + \frac{r_E}{345r_M} = \frac{346r_E}{345r_M}$, and therefore that $r_M = \frac{346r_E}{(3.5)(345)} = 0.2865r_E$. With $r_E = 3850$ miles, this gives $r_M = 1100$ miles. Since $r_S = 345r_M$, we find that $r_S = 380{,}000$ miles. Since $\frac{r_M}{D_M} = 0.0044$, $\frac{D_M}{r_M} = 227$. So $D_M = 250{,}000$ miles. Finally, $D_S = \frac{D_M}{0.0029} = \frac{250000}{0.0029}$, so $D_S = 86 \times 10^6$ miles. A look at Table 1.4 shows that the distances $r_M, r_S, D_M$, and $D_S$ just derived are within the "ball park."

**24.** Since the radius $r_E$ of the Earth is 3950 miles and the height of Mount Everest is 29,028 feet $= \frac{29028}{5280} = 5.5$ miles,

$$\frac{\text{Everest}}{r_E} = \frac{5.5}{3950} = 0.0014.$$

Since the radius of a basketball is $r = 4.7$ inches and the height of a pebble is 0.02 inches,

$$\frac{\text{pebble}}{r} = \frac{0.02}{4.7} = 0.0042.$$

So relatively speaking, the pebble is three times higher than Mount Everest.

**25.** Shrinking things by a factor of $\frac{1}{50{,}000{,}000}$ gives the following approximations:

$$\begin{aligned}
r_M &= (1080)\frac{1}{50{,}000{,}000} \text{ miles } = (1080)\frac{1}{50{,}000{,}000}(5280)(12) \text{ inches} \\
&= 1.37 \text{ inches. (A baseball has a radius of about 1.43 inches.)} \\
D_M &= (240{,}000)\ \frac{1}{50{,}000{,}000} = \frac{24}{5000} \text{ miles. So } D_M = \frac{24}{5000}(5280) \\
&= 25 \text{ feet.} \\
r_S &= (432{,}000)\frac{1}{50{,}000{,}000} = \frac{43}{5000} \text{ miles. So } r_S = \frac{43}{5000}(5280) \\
&= 45 \text{ feet.} \\
D_S &= (93 \times 10^6)\frac{1}{50{,}000{,}000} = (93 \times 10^6)\frac{1}{50 \times 10^6} = 1.86 \text{ miles.} \\
D_* &= (24 \times 10^{12})\frac{1}{50{,}000{,}000} = (24 \times 10^{12})\frac{1}{50 \times 10^6} \\
&= \frac{1}{2} \text{ million miles. (This is about twice the actual distance to the Moon.)}
\end{aligned}$$

**26.** From Table 1.4, the radius of the Sun is $(432{,}000)(5280) = 2.3 \times 10^9$ feet. Given that the radius of a basketball is 0.39 feet, the shrinking factor is $\frac{0.39}{2.3 \times 10^9} = 0.17 \times 10^{-9} = 1.7 \times 10^{-10}$. Examples of the corresponding approximations are,

$$\begin{aligned}
r_E &= (3950)(1.7 \times 10^{-10}) \text{ miles } = 6.7 \times 10^{-7} \text{ miles } = (6.7 \times 10^{-7})(5280)(12) \text{ inches} \\
&= 4.2 \times 10^{-2} \text{ inches} \approx \frac{1}{25} \text{ inches, and} \\
D_* &= (24 \times 10^{12})(1.7 \times 10^{-10}) \text{ miles } = 40 \times 10^2 \text{ miles } = 4000 \text{ miles.}
\end{aligned}$$

For the problem below, recall from the Corrections to Chapter 1 that the estimate $D_* < 1.6 \times 10^{12}$ miles should be replaced by $D_* < 3.6 \times 10^{12}$ miles.

**27.** We will continue to follow Archimedes's strategy of providing only very rough estimates. We already know that a sphere of 1 finger-breadth will hold about $10^9$ miles grains of sand. By increasing the diameter of such a sphere by a factor of $\frac{3}{2}$ one sees that a sphere of diameter one inch in would hold $\left(\frac{3}{2}\right)^3 \times 10^9$ or about $10^{10}$ grains of sand. Increasing this sphere to a diameter of 1 mile, or by a factor of (12)(5280), or roughly $10^5$, increases its volume by a factor of $(10^5)^3 = 10^{15}$. Such a sphere could hold $(10^{15})(10^{10})$ or about $10^{25}$ grains of sand. A sphere of radius $3.6 \times 10^{12}$ miles has a diameter of $7.2 \times 10^{12}$ miles. Such a sphere will hold about $(7.2 \times 10^{12})^3(10^{25}) = (7.2)^3 \times 10^{61}$ or very roughly $10^{63}$ grains of sand.

**28.** The relevant formula is $D_A = \frac{3}{p_{\text{sec}}}$. Taking $p = 0.55$, shows that Barnard is a distance of $\frac{3}{0.55} = 5.45\,\text{LY}$ away. The other distances are computed in the same way.