Honors Analysis - Homework 6
The questions marked with $*$ are harder.

1. Consider the space $C_{[0,1]}$ with the norm

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p},$$

for some $p > 1$. Let $\phi \in C_{[0,1]}$, and define the functional $F : C_{[0,1]} \to \mathbf{R}$ given by

$$F(g) = \int_0^1 g(x)\phi(x) \, dx.$$

Prove that F is a bounded functional and find the norm $\|F\|$.

2. Let

$$l_1 = \{(x_1, x_2, \dots) \mid \sum_{i=1}^{\infty} |x_i| < \infty\}$$

with norm $\|\mathbf{x}\|_1 = \sum_{i=1}^{\infty} |x_i|$, and let

$$m = \{(x_1, x_2, \dots) \mid \sup_{i>0} |x_i| < \infty\}$$

with norm $\|\mathbf{x}\|_\infty = \sup_{i>0} |x_i|$. Prove that $l_1^* = m$.

3.* Suppose that we have a sequence $\mathbf{x}_k \in l_1$, converging weakly $\mathbf{x}_k \rightharpoonup 0$. Show that then $\mathbf{x}_k \to 0$ in l_1, i.e. $\lim_{k\to\infty} \|\mathbf{x}_k\|_1 = 0$.

4. Let $S \subseteq V$ be a subset of a normed linear space V. Suppose that S is weakly bounded, i.e. for every linear functional $f \in V^*$ there is a constant C (depending on f) such that $|f(x)| < C$ for all $x \in S$. Show that S is bounded in norm, i.e. there is a constant D such that $\|x\| < D$ for all $x \in S$.

5. Show that weak limits are unique. I.e. if $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$ in a normed linear space V, then $x = y$.

6.* A function $f : [0,1] \to \mathbf{R}$ is called Hölder continuous, if there exists an $\alpha \in (0,1]$ and $C > 0$ such that

$$|f(x) - f(y)| \leqslant C|x-y|^\alpha$$

for all $x, y \in [0,1]$.

Suppose that $E \subset C_{[0,1]}$ is a closed subspace, using the sup norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$. Assume that every element of E is Hölder continuous.

(a) Show that there exists a $\beta \in (0,1]$ and $K > 0$ such that

$$|f(x) - f(y)| \leqslant K\|f\| \, |x-y|^\beta,$$

for all $f \in E$ and $x, y \in [0,1]$. (Note that in the definition of Hölder continuity the α and C can depend on f.)

(b) Show that E is finite dimensional.