Honors Analysis - Homework 5

1. Let V be a Banach space, and $W \subset V$ a closed subspace. Show that the quotient space V/W is also complete, i.e. a Banach space.

2. Suppose that V is a normed linear space, and $f : V \to \mathbf{R}$ is a linear functional (which may not be continuous). Show that f is continuous if and only if the kernel $\operatorname{Ker} f$ is closed.

3. Suppose that $F : C_{[0,1]} \to \mathbf{R}$ is a linear functional, which satisfies the property that $F(g) \geqslant 0$ whenever $g \in C_{[0,1]}$ is a non-negative function (i.e. if $g(x) \geqslant 0$ for all $x \in [0,1]$). Prove that F is a continuous linear functional, with respect to the sup norm on $C_{[0,1]}$.

4. Let V be a normed linear space.

(a) Prove that every finite dimensional subspace of V is closed.

(b) For two subspaces $A, B \subset V$ we define the sum

$$A + B = \{x + y \,|\, x \in A, y \in B\}.$$

Prove that if A is a closed subspace, and B is finite dimensional, then $A + B$ is a closed subspace of V.

5. Let V be a normed linear space.

(a) Suppose that $W \subset V$ is a closed subspace. Show that there exists an element $v \in V \setminus W$ such that $\|v\| = 1$ and

$$\|v - w\| > \frac{1}{2}$$

for every $w \in W$.

(b) Prove that if the closed unit ball $\{x \in V \,|\, \|x\| \leqslant 1\}$ is compact, then V is finite dimensional.

6. Let (X, ρ) be a complete metric space, and let S be the set of all non-empty compact subsets of X. For $A, B \in S$, define the distance

$$d(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} \rho(x, y), \sup_{y \in B} \inf_{x \in A} \rho(x, y)\}.$$

In other words $d(A, B) \leq k$ means that for every point $x \in A$ there is a point $y \in B$ with $d(x, y) \leq k$ and vice versa, i.e. for every point $y \in B$ there is an $x \in A$ with $d(x, y) \leq k$.

(a) Show that (S, d) is complete.

(b) Assuming that (X, ρ) is compact, prove that (S, d) is totally bounded (so, combined with (a), this means (S, d) is compact).