Honors Analysis - Homework 4

1. Suppose that X is a totally bounded metric space. Prove that X has a countable dense subset.

2. Let M be a bounded subset of the metric space $C_{[0,1]}$ with the metric $\rho(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$. Prove that the set of all functions $F \in C_{[0,1]}$ of the form

$$F : [0,1] \to \mathbf{R}$$
$$F(x) = \int_0^x f(t)\,dt,$$

for $f \in M$ is totally bounded in $C_{[0,1]}$.

3. Let (R,ρ) be a complete metric space, and $S \subset R$ a dense subset (i.e. R is the completion of S). Suppose that

$$f : S \to \mathbf{R}$$

is a uniformly continuous function on S. Prove that there is a unique continuous function $F : R \to \mathbf{R}$, such that $F(x) = f(x)$ for all $x \in S$. (You might want to try the special case where $R = [0,1]$ and $S = (0,1]$ first).

4. Suppose that $f : X \to \mathbf{R}$ is a lower semicontinuous function on a topological space X. Suppose that $\lim_{x\to\infty} x_n = x$, where $x_n, x \in X$.

(a) Prove that

$$f(x) \le \lim_{n\to\infty} \inf_{k\ge n} \{f(x_k)\}.$$

(b) Give an example to show that the limit $\lim_{n\to\infty} f(x_n)$ might not exist.

5. Let (X,ρ) be a complete metric space, and let S be the set of all non-empty compact subsets of X. For $A, B \in S$, define the distance

$$d(A,B) = \max\{\sup_{x\in A} \inf_{y\in B} \rho(x,y), \sup_{y\in B} \inf_{x\in A} \rho(x,y)\}.$$

In other words $d(A,B) \le k$ means that for every point $x \in A$ there is a point $y \in B$ with $d(x,y) \le k$ and vice versa, i.e. for every point $y \in B$ there is an $x \in A$ with $d(x,y) \le k$.

(a) Show that (S,d) is a metric space.

(b) Show that (S,d) is complete.

(c) Assuming that (X,ρ) is compact, prove that (S,d) is totally bounded (so, combined with (a), this means (S,d) is compact).