Honors Analysis - Homework 3

1. Give examples of a metric space X and a map $F : X \to X$ with no fixed points, satisfying

(a) X complete, and $\rho(F(x), F(y)) < \rho(x, y)$ for all $x, y \in X$, $x \neq y$.

(b) X not complete, and there is an $\alpha < 1$ such that $\rho(F(x), F(y)) \leq \alpha\rho(x, y)$ for all $x, y \in X$.

2. Let Z denote the space $\mathbf{R}$ with the Zariski topology. Recall that this means that the open sets in Z are the empty set, and the complements of finite sets.

(a) Is Z Hausdorff?

(b) Is Z first countable?

(c) Is Z compact?

(d) Show that polynomial functions give continuous maps $Z \to Z$.

(e) Given an example of a map $f : \mathbf{R} \to \mathbf{R}$ which is continuous with respect to the usual topology on $\mathbf{R}$, but is not continuous as a map $Z \to Z$.

3. Show that a map $f : X \to Y$ of topological spaces is continuous if and only if $f^{-1}(U)$ is an open subset of X, for all open subsets $U \subseteq Y$.

4. Give an example of a topological space X and a compact set $K \subset X$ such that K is not closed.

5. Give an example of a topological space X and a map $f : X \to \mathbf{R}$, such that for every convergent sequence $x_n \to x$ in X we have $f(x_n) \to f(x)$, but f is not continuous.

6. In this problem you will show that the set of nowhere differentiable functions is dense in $C_{[0,1]}$, where we are using the usual metric $\rho(f, g) = \sup |f - g|$.

(a) Show that if $f \in C_{[0,1]}$ is differentiable at $x_0 \in [0, 1]$, then there is a constant M, such that

$$|f(x) - f(x_0)| \leq M|x - x_0|, \qquad \text{for all } x \in [0, 1].$$

(b) Define $A_n \subseteq C_{[0,1]}$ to be the set of functions $f \in C_{[0,1]}$ for which there exists a point $x_0 \in [0, 1]$ such that

$$|f(x) - f(x_0)| \leq n|x - x_0| \qquad \text{for all } x \in [0, 1].$$

Show that A_n is closed in $C_{[0,1]}$.

(c) Show that A_n has empty interior in $C_{[0,1]}$. (*Hint: given an* $f \in A_n$ *and* $\epsilon > 0$*, you need to find a* g *such that* $\rho(f, g) < \epsilon$ *and* g *oscillates very rapidly, so that* $g \notin A_n$*.*)

(d) Conclude using Baire's theorem, that the set of nowhere differentiable functions is dense in $C_{[0,1]}$.