Honors Analysis - Homework 2

1. Define a map $F : C_{[0,1]} \to \mathbf{R}$ by letting

$$F(f) = \int_0^1 |f(x)|^{3/2}\, dx.$$

For the following choices of metrics, determine whether F is continuous:

(a) $\rho(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$

(b) $\rho_1(f, g) = \int_0^1 |f(x) - g(x)|\, dx$

(c) $\rho_2(f, g) = \left(\int_0^1 |f(x) - g(x)|^2\, dx\right)^{1/2}$

2. Prove Hölder's inequality for integrals. Namely if $f, g \in C_{[0,1]}$, and $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_0^1 f(x)g(x)\, dx \leqslant \left(\int_0^1 |f(x)|^p\, dx\right)^{1/p} \left(\int_0^1 |g(x)|^q\, dx\right)^{1/q}.$$

3. Let (X, ρ) be a metric space.

(a) Fix a point x_0 in X and define the function $f : X \to \mathbf{R}$ by letting $f(x) = \rho(x, x_0)$ for all x. Prove that f is continuous.

(b) Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in X converging to x and y respectively. Prove that

$$\lim_{n \to \infty} \rho(x_n, y_n) = \rho(x, y).$$

4. Let $f : X \to Y$ be a map between metric spaces, and let $x \in X$ be a point. Prove that f is continuous at x, if and only if whenever $\{x_n\}$ is a sequence converging to x, the sequence $\{f(x_n)\}$ converges to $f(x)$.

5. Recall the metrics ρ_2 and ρ_∞ on $\mathbf{R}^n$ that we defined by

$$\rho_2(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^n |x_i - y_i|^2\right)^{1/2}$$
$$\rho_\infty(\mathbf{x}, \mathbf{y}) = \max_{i=1,\ldots,n} |x_i - y_i|.$$

Show that $(\mathbf{R}^2, \rho_2)$ and $(\mathbf{R}^2, \rho_\infty)$ are not isometric. (Note that it is not enough to consider the identity map.)

6. Suppose that $f_n \in C_{[0,1]}$ is a sequence converging to a function $f : [0,1] \to \mathbf{R}$ uniformly. I.e. we have

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| = 0.$$

Prove that f is continuous.

7. Recall the metric space m, whose elements are the bounded infinite sequences of real numbers, with the metric

$$\rho(\mathbf{x}, \mathbf{y}) = \sup_{i=1,2,\ldots} |x_i - y_i|.$$

Prove that m is complete.

8. Let $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ be a nested sequence of non-empty closed sets in a complete metric space (R, ρ).

(a) Give an example to show that the intersection $\bigcap_{k\geqslant 1} A_k$ may be empty.

(b) Suppose that the sets A_k are bounded, i.e. we can define the diameters

$$d(A_k) = \sup_{x,y\in A_k} \rho(x, y).$$

Prove that if $\lim_{k\to\infty} d(A_k) = 0$, then

$$\bigcap_{k\geqslant 1} A_k \neq \emptyset.$$

(c) What can we say when the sets A_k are bounded, but the diameters do not converge to zero? Is the intersection $\bigcap_{k\geqslant 1} A_k$ non-empty?