Honors Analysis - Homework 1
The questions marked with $*$ are harder and might take a while to solve.

1. Suppose that $f : X \to Y$ is a map, and $\{B_\alpha\}$ is a collection of subsets of Y. Prove that

$$f^{-1}\left(\bigcap_\alpha B_\alpha\right) = \bigcap_\alpha f^{-1}(B_\alpha).$$

2.

(a) Suppose that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ are all finite non-empty sets. Prove that

$$\bigcap_{k \geqslant 1} A_k$$

is non-empty.

(b) Is the statement in (a) still true if the A_k are allowed to be infinite?

3. Let A be a set and let S be the set of all subsets of A. Prove that there does not exist a one-to-one map from A onto S.

4. Let M be an infinite set, and A be a countable set. Show that there exists a one-to-one map from M onto $M \cup A$.

5. Prove that $\mathbf{R} \times \mathbf{R}$ has the same cardinality as $\mathbf{R}$.

6. Does the set of all continuous real functions on $\mathbf{R}$ have the same cardinality as $\mathbf{R}$? Justify your answer.

7. For $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{R}^n$ and $p > 0$ write

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}.$$

Prove that

$$\lim_{p \to \infty} \|\mathbf{x}\|_p = \max_{1 \leqslant i \leqslant n} |x_i|.$$

8. In the notation of the previous question, define a "distance" function $\rho_p(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_p$ for $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$. Is $(\mathbf{R}^n, \rho_p)$ a metric space for $p \in (0, 1)$ and $n > 1$?

9.* Suppose that $A \subseteq B \subseteq C$ are three sets. Show that if A and C have the same cardinality, then B also has the same cardinality as A and C.