Spectral Stability of Symmetric Spaces

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1 Introduction

Let M be a complete Riemannian manifold. Associated to the metric g of M, there is a second order elliptic operator Δ. The Laplacian Δ extends to an unbounded self adjoint operator on L^2M. If M is compact, then Δ has pure point spectrum. In general, Δ has both discrete and continuous spectrum.

We say that g is spectrally stable if for any compactly supported two tensor h, there is a $\delta(h) > 0$, such that Δ_g is unitarily equivalent to $\Delta_{g+\lambda h}$, for any λ with $|\lambda| < \delta(h)$. We presume that $g + \lambda h$ is positive definite, which is assured for $\delta(h)$ sufficiently small. To illustrate the notion of spectral stability, here are some examples: (i) If M is compact, any metric on M is unstable, (ii) R^n is stable when endowed with the standard metric, (iii) If X is compact and $M = X \times R$ is a Riemannian product, then M is unstable.

The symmetric spaces of noncompact type have purely absolutely continuous spectrum with support a half line, contained in the non–negative real axis. Our main goal is to prove the following:

Theorem 1.1. *Any Riemannian symmetric space of noncompact type is spectrally stable.*

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In Section 2, a complete proof is given based upon harmonic analysis. Section 3 develops a different approach using convex functions. The second method is only successful for certain noncompact symmetric spaces.

2 Harmonic Analysis

Let (M, g_0) be a Riemannian symmetric space of noncompact type. The Laplacian of (M, g_0) is unitarily equivalent to multiplication by $\langle \rho, \rho \rangle + x^2$ on $L^2(R^+, dx, \mathcal{M})$, where $\mathcal{M}$ is a Hilbert space of countably infinite dimension. Here ρ denotes half the sum of the positive restricted roots, counted with multiplicity. If our manifold is the standard Euclidean space, then $\rho = 0$.

In [1], the spectral theory of (M, g) was studied, where g and g_0 coincide outside a compact subset of M. If $|\rho| > 0$, one creates finitely many eigenvalues in the interval $(0, |\rho|^2)$ by including a large Euclidean disc in M. The following result was proved in [1]:

Theorem 2.1. *The continuous part of the spectrum of Δ_g is unitarily equivalent to Δ_{g_0}. Moreover, Δ_g has no embedded eigenvalues lying in $(|\rho|^2, \infty)$.*

Given Theorem 2.1, to prove Theorem 1.1, we need only to show stability of the strict lower bound $|\rho|^2$ for the spectrum. One may assume $|\rho| > 0$, since $|\rho| = 0$ only for Euclidean spaces. If $M = G/K$, then we use the Cartan decomposition $G = KAK$. Recall that the weight of the volume element, induced on the Weyl chamber by g_0, is $w = \prod_{\alpha \in \Sigma^+} (\sinh\alpha(H))^{m_\alpha}$, where Σ^+ denotes the set of positive restricted roots and m_α is the multiplicity of $\alpha \in \Sigma^+$. Clearly, $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$.

One may consider ρ as a linear function on the Euclidean space A and consider the vector field $\vec{n} = (\vec{\nabla}\rho)/|\vec{\nabla}\rho|$. Note that

$$w^{-1}\vec{n}w = \sum_{\alpha \in \Sigma^+} m_\alpha \coth\alpha(H) \frac{\langle \alpha, \rho \rangle}{|\rho|} > 2|\rho|$$

because $\coth\alpha(H) > 1$ in the Weyl chamber.

Suppose that ϕ is compactly supported on G/K. Since w vanishes on the walls of the Weyl chamber, the divergence theorem applied to $\phi^2 w\vec{n}$ gives,

$$\int w\phi^2(w^{-1}\vec{n}w) = -2\int \phi(\vec{n}\phi)w$$

where integration runs over the Weyl chamber.

The Schwarz inequality leads to the estimate

$$|\rho|\int \phi^2\left(\frac{\vec{n}w}{2|\rho|w}\right)w \leq \left(\int \phi^2 w\right)^{1/2}\left(\int |\nabla\phi|^2 w\right)^{1/2}$$

We noted above that $\vec{n}w > 2|\rho|w$, so it is natural to write

$$|\rho|^2\left[\int \phi^2 w + \int w\phi^2\left(\frac{\vec{n}w}{2|\rho|w} - 1\right)\right] \leq \int |\nabla\phi|^2 w.$$

This shows that the spectrum of Δ is bounded below by $|\rho|^2$. Stability of the lower bound follows since $\vec{n}w > 2|\rho|w$ inside the Weyl chamber. The proof of Theorem 1.1 is complete.

3 Convex functions

We now develop a second approach, to the proof of Theorem 1.1, based upon the construction of suitable convex functions. At present, the method only works under additional assumptions, on the symmetric space, which will be specified shortly. However, the hope is that the convex function method may potentially be more amenable to generalization outside the class of symmetric spaces.

The starting point is a refinement of the main result in [2]:

Theorem 3.1. *Let M be a complete Riemannian manifold with metric g_0. Suppose that M supports a strictly convex function f, which has bounded gradient, and satisfies $\Delta^2 f \leq 0$. Assume further that, for all $\phi \in C_0^\infty(M)$, one has*

$$\int_M Hess f(\nabla\phi, \nabla\phi) \geq \int_M \eta|\phi|^2.$$

Here η is a positive continuous function. Then Δ_{g_0} has purely absolutely continuous spectrum. Moreover, if g is a sufficiently small compactly supported perturbation of g_0, then Δ_g also has purely absolutely continuous spectrum.

It was shown in [1] that the absolutely continuous part of the spectrum is invariant under any compactly supported perturbation of the metric. So the proof of Theorem 1.1 is reduced to constructing convex functions f which satisfy the hypotheses of Theorem 3.1.

Let $G = KAN$ be the Iwasawa decomposition, with $M = K|G$. If r is the rank of $M = AN$, then the metric $g_0 = dx_1^2 + dx_2^2 + \ldots + dx_r^2 + \sum_\alpha e^{2\alpha(x)} dw_\alpha^2$. Each x_i is convex and Δx_i is constant. By proper choice of coordinates for A, we may assume that $\mathrm{Hess} x_1$ is strictly positive upon restriction to TN.

Suppose that N contains an abelian subgroup B, with $\dim B \geq 3$. It follows from Hardy's inequality on B and Fubini's theorem on the fiber bundle $B \to N \to N/B$, that there is a positive function χ satisfying

$$\int_N \mathrm{Hess}_{g_0} x_1(d\phi, d\phi) \geq \int_N \chi \phi^2.$$

If g_0 is perturbed to g, then $\|\mathrm{Hess}_g x_1 - \mathrm{Hess}_{g_0} x_1\|$ is bounded by a small positive compactly supported function ϵ. Another application of Fubini's theorem gives

$$\int_M \mathrm{Hess}_g x_1(d\phi, d\phi) \geq \int_M \chi \phi^2 - \int_M \epsilon |d\phi|^2$$

There exists a strictly positive convex function f_1 on M, provided we assume that the Ricci curvature of M is strictly negative. Let $\{z_n\}$ be a countable dense set in the ideal boundary of the geodesic compactification of M. We define $f_1(p) = \sum_n 2^{-n} \beta(z_n, p)$, where β denotes the Busemann function. The infinite sum is a strictly convex function with constant Laplacian. Negative Ricci curvature is needed to establish the strict convexity.

Define $f = x_1 + f_1$. Then f satisfies the hypotheses of Theorem 3.1. This completes the alternative proof of Theorem 1.1 under the additional hypotheses: (i) M has negative Ricci curvature and (ii) The nilpotent group N contains an abelian subgroup of dimension at least three. These hypotheses hold for all irreducible symmetric spaces with finitely many exceptions. More exceptional cases are obtained by taking products.

References

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