On Kulikov's Problem

Scott Nollet
Department of Mathematics
Texas Christian University
Fort Worth, TX 76129

Frederico Xavier *
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556

Abstract

Kulikov has exhibited an étale morphism $F : X \to \mathbb{C}^n$ of degree $d > 1$ which is surjective modulo codimension two with X simply connected, settling his generalized jacobian problem. His method reduces the problem to finding a hypersurface $D \subset \mathbb{C}^n$ and a subgroup $G \subset \pi_1(\mathbb{C}^n - D)$ of index d generated by geometric generators. By contrast we show that if D has simple normal crossings away from a set of codimension three and $\overline{D} \subset \mathbb{P}^n$ meets the hyperplane at infinity transversely, then necessarily $d = 1$.

The jacobian conjecture asks whether a polynomial function $F : \mathbb{C}^n \to \mathbb{C}^n$ with non-vanishing Jacobian must be invertible. Kulikov [6] considers the following generalization:

Question 1. Must every étale morphism $F : X \to \mathbb{C}^n$ which is surjective modulo codimension two with X simply connected be birational?

Since F is dominant, there is a hypersurface $D \subset \mathbb{C}^n$ such that the restriction $X - F^{-1}(D) \to \mathbb{C}^n - D$ is a covering of $d = \deg F$ sheets [8, 3.17], which is classified by a subgroup $G \subset \pi_1(\mathbb{C}^n - D)$ of index d. The group G is generated by geometric generators, which are the loops obtained by taking a path from a basepoint x_0 to a small circle about a smooth point of D, traversing the circle, then returning to x_0 via the same path [5]. Conversely, Kulikov shows that any subgroup of finite index generated by geometric

*Work partially supported by NSF grant DMS02-03637.

generators gives rise to a morphism F as in Question 1. The Lefschetz theorem implies that the fundamental group is determined by a general plane section, thus Question 1 is equivalent to

Question 2. If $G \subset \pi_1(\mathbb{C}^2 - D)$ is a subgroup of index $d < \infty$ generated by some geometric generators, must $d = 1$?

Kulikov has shown that the answer is no (Remark 5(e) below), but we expect that $d = 1$ in most cases. We work in the broader setting of local diffeomorphisms $F : X \to \mathbb{C}^n$ of simply connected manifolds such that the restriction map $X - F^{-1}(D) \to \mathbb{C}^n - D$ is a covering map for some hypersurface D. We show that if D is smooth and meets the hyperplane at infinity transversely, for example, then $d = 1$ or ∞. The precise statement is this:

Theorem 3. *Let $F : X \to \mathbb{C}^n$ be a local diffeomorphism of simply connected manifolds such that the restriction $X - F^{-1}(D) \to \mathbb{C}^n - D$ is a d-fold covering map for a hypersurface $D \subset \mathbb{C}^n$. Suppose further that*

(a) D has at worst normal crossing singularities away from a set of codimension three.

(b) The closure $\overline{D} \subset \mathbb{P}^n$ meets the hyperplane H at infinity transversely.

Then $d = 1$ or $d = \infty$.

Corollary 4. *Let $F : X \to \mathbb{C}^n$ be an étale morphism with X simply connected and $D \subset \mathbb{C}^n$ as in Theorem 3. Then F is injective.*

Remarks 5. Some comments are in order.
(a) Already for $n = 1$ both cases occur: $d = 1$ is achieved by the identity map and $d = \infty$ is achieved by the complex exponential map.
(b) The squaring map $F : \mathbb{C} \to \mathbb{C}$ given by $F(z) = z^2$ shows that the conclusion fails if F is not étale at even a single point. The restriction of F to $\mathbb{C} - \{0\}$ shows that the simply connected hypothesis on X is necessary.
(c) The result above should be compared to the main result in [9], where surgery theory and an interesting counting argument are used to draw the same conclusion if D is defined by the vanishing of a polynomial $P(z_1, \ldots, z_n)$ such that $P^{-1}(B) \cong D \times B$ for a small neighborhood of 0.
(d) Taking $X = \mathbb{C}^n$, the theorem shows that the jacobian conjecture holds generically with respect to the divisor $D \subset \mathbb{C}^n$. However the conjecture may well be false - see [12] for analytic evidence in this direction.

(e) Some condition on D is necessary in Theorem 3, as is shown by Kulikov's example [6, §3]. He starts with a curve $D \subset \mathbb{P}^2$ of degree 4 with three cusps, given by Zariski as the smallest degree curve whose complement has non-cyclic fundamental group (see [13, VIII, §2] or [1]). Here $\pi_1(\mathbb{P}^2 - D)$ is a non-abelian group of order 12 generated by geometric generators g_1, g_2. If $L \subset \mathbb{P}^2$ is a line meeting D transversely, a result of Nori [11, Prop. 3.27] yields the exact sequence

$$1 \to K \to \pi_1(\mathbb{P}^2 - D - L) \to \pi_1(\mathbb{P}^2 - D) \to 1$$

in which $K \cong \mathbb{Z}$ is central. He concludes that a pre-image $\overline{g_1}$ of g_1 generates a subgroup G of index 3 containing K.

Question 6. Does the conclusion of Theorem 3 hold if D is merely smooth or smooth and connected (but does not meet transversely at infinity) ?

To prove Theorem 3, we may suppose that $n > 1$ (see [9, 1.9] for $n = 1$). Let $\overline{D} \subset \mathbb{P}^n$ be the projective closure, H the hyperplane at infinity and $L \subset \mathbb{P}^n$ a general linear subspace of dimension two. The conditions on D imply that the curve $L \cap (\overline{D} \bigcup H)$ has at worst nodal singularities, hence $\pi_1(L - (\overline{D} \bigcup H))$ is Abelian by Zariski's conjecture [3, 4]. The restriction map $\pi_1(L - (\overline{D} \bigcup H)) \to \pi_1(\mathbb{P}^n - (\overline{D} \bigcup H))$ is surjective [11, 2.1], hence the group $\pi_1(\mathbb{C}^n - D) = \pi_1(\mathbb{P}^n - (\overline{D} \bigcup H))$ is Abelian as well. Thus the subgroup $F_*(\pi_1(X - F^{-1}(D))) \subset \pi_1(\mathbb{C}^n - D)$ is normal, the corresponding covering is regular, and Theorem 3 follows from:

Theorem 7. *Let $F : X \to \mathbb{C}^n$ be a local diffeomorphism of simply connected manifolds such that $F : X - F^{-1}(D) \to \mathbb{C}^n - D$ is a regular covering map for some algebraic hypersurface $D \subset \mathbb{C}^n$. Then $d = 1$ or $d = \infty$.*

Proof: Suppose that $1 < d < \infty$ and X is connected. Since

$$F : X - F^{-1}(D) \to \mathbb{C}^n - D$$

is a regular covering, the subgroup $F_*(\pi_1(X - F^{-1}(D), x_0)) \subset \pi_1(\mathbb{C}^n - D, p_0)$ is normal and the corresponding quotient group is isomorphic to the group of covering transformations [7, Ch. 5, 7.4]. Since $d > 1$ and $\pi_1(\mathbb{C}^n - D, p_0)$ is generated by geometric generators [5], there is a geometric generator γ about an irreducible component $D_1 \subset D$ which does not act as the identity, hence acts without fixed points [7, Ch. 5, 6.2].

Now suppose that $x \in F^{-1}(D_1)$ and $F(x) = p$ is a smooth point of D_1. Then F takes a small open neighborhood $x \in U$ homeomorphically onto its image $F(U)$. Choose $y \in U - F^{-1}(D_1)$ and set $q = f(y)$. With a deformation we can arrange that the geometric generator γ consists of a path τ from the basepoint p_0 to q, followed by a small loop σ about D_1 in $F(U)$, and finally by the return path τ^{-1} to p_0. Since $F^{-1}(\sigma)$ contains a loop $\tilde{\sigma}$ mapping isomorphically to σ (namely the one contained in U), it evident that γ has a fixed point, a contradiction. Thus the smooth points in D_1 have no pre-image. Since the map F is open, it follows that $F^{-1}(D_1) = \emptyset$.

Since $d < \infty$, $F^{-1}(\gamma)$ contains a closed loop $\tilde{\gamma}$ which covers γ (the universal cover $\mathbb{R} \to S^1$ shows that this may fail if $d = \infty$). If $\tilde{\gamma} \xrightarrow{F} \gamma$ has degree n, then $F_*(\tilde{\gamma}) = \gamma^n$. Now γ has infinite order in $\pi_1(\mathbb{C}^n - D_1)$, hence applying F_* to the map

$$X = X - F^{-1}(D_1) \to \mathbb{C}^n - D_1$$

shows that $\pi_1(X)$ is nontrivial, contradicting the simple connectivity of X.

To see that γ has infinite order, let D_1 be given by the vanishing of $P \in \mathbb{C}[z_1, \ldots, z_n]$ and consider the map

$$\mathbb{C}^n - D_1 \xrightarrow{P} \mathbb{C} - \{0\}.$$

Since P is smooth at p, the loop γ maps to a generator for the infinite cyclic group $\pi_1(\mathbb{C} - \{0\})$, hence must have infinite order itself.

Remark 8. The only properties of $\mathbb{C}^n$ used in the proof were simple connectivity and the fact that geometric generators γ for $\pi_1(\mathbb{C}^n - D)$ have infinite order. Thus Theorem 7 can be strengthened to include any local diffeomorphism $F : X \to Y$ of simply connected varieties for which there exists a hypersurface $D \subset Y$ and geometric generator $\gamma \in \pi_1(Y - D)$ of infinite order such that $F : X - F^{-1}(D) \to Y - D$ is a regular d-fold cover and $\gamma \notin F^*(\pi_1(X - F^{-1}(D)))$.

These problems fit in with our general program of trying to understand local versus global injectivity phenomenon [10, 9]. Indeed, for such maps with finite fibres, we are really asking when a local immersion is globally injective. The first named author thanks Victor Belfi for useful conversations.

References

[1] S. Abhyankar, Tame coverings and fundamental groups of algebraic varieties V, Three cuspidal plane quartics, Amer. J. Math. **82** (1960) 365–373.

[2] H. Bass, E. Connell and D. Wright, The Jacobian conjecture: reduction of degree and formal expansion of the inverse, Bull. A.M.S. **7** (1982) 287–330.

[3] P. Deligne, Le groupe fondamental du complment d'une courbe plane n'ayant que des points doubles ordinaires est ablien (d'aprs W. Fulton), Bourbaki Seminar, Vol. 1979/80, pp. 1–10, Lecture Notes in Math. **842**, Springer, Berlin-New York, 1981.

[4] W. Fulton, On the fundamental group of the complement of a node curve, Ann. of Math. (2) **111** (1980) 407–409.

[5] V. Kulikov, On the fundamental group of the complement of a hypersurface in $\mathbb{C}^n$, in *Algebraic geometry (Chicago, IL, 1989)*, Lecture Notes In Mathematics **1479** (1991) 122–130.

[6] V. Kulikov, Generalized and local Jacobian problems, Russian Acad. Sci. Izv. Math. **41** (1993), no. 2, 351–365.

[7] W. Massey, *Algebraic Topology: An Introduction*, Graduate Texts in Mathematics **56**. Springer-Verlag, New York-Heidelberg, 1977.

[8] D. Mumford, *Algebraic Geometry I: Complex Projective Varieties*, Grundlehren der mathematischen Wissenschaften **221**, Springer-Verlag, 1976.

[9] S. Nollet, L. Taylor and F. Xavier, Degrees of local diffeomorphisms which almost cover, Preprint, 2006.

[10] S. Nollet and F. Xavier, Holomorphic Injectivity and the Hopf map, Geom. and Funct. Anal. **14** (2004) 1339–1351.

[11] M. Nori, Zariski's conjecture and related problems, Ann. Sci. École Norm. Sup. (4) **16** (1983) 305–344.

[12] F. Xavier, Rigidity of the identity, Preprint 2005.

[13] O. Zariski, *Algebraic Surfaces*, Second supplemented edition with appendices by S. S. Abhyankar, J. Lipman, and D. Mumford, Ergebnisse der Mathematik und ihrer Grenzgebiete **61**, Springer-Verlag, New York-Heidelberg, 1971.