RIGIDITY OF THE IDENTITY

Frederico Xavier

University of Notre Dame

Abstract. The structure of the group $\mathrm{Aut}(\mathbb{C}^n)$ of biholomorphisms of $\mathbb{C}^n$ is largely unknown if $n > 1$. In stark contrast $\mathrm{Aut}(\mathbb{C})$ is rather small, consisting of the non-constant affine linear maps. The description of $\mathrm{Aut}(\mathbb{C})$ follows from the observation that an injective holomorphic function $f : \mathbb{C} \to \mathbb{C}$ satisfying $f(0) = 0$ and $f'(0) = 1$ must be the identity. These considerations suggest that similar characterizations of the identity might be useful in understanding the structure of $\mathrm{Aut}(\mathbb{C}^n)$. Using geometric methods we prove that an injective holomorphic map $f : \mathbb{C}^n \to \mathbb{C}^n$ is the identity I if and only if the power series at 0 of $f - I$ has no terms of order $\leq 2n+1$ and the function $|\det Df(z)|\, |z|^{2n} |f(z)|^{-2n}$ is subharmonic throughout $\mathbb{C}^n$.

§1 Introduction

The group $\mathrm{Aut}(\mathbb{C}^n)$ of biholomorphisms of $\mathbb{C}^n$ is not well understood if $n > 1$, despite significant work by many authors (see, for instance, [1], [8], [9], [10]). In stark contrast $\mathrm{Aut}(\mathbb{C})$ is rather small, consisting of the non-constant affine linear maps. The description of $\mathrm{Aut}(\mathbb{C})$ follows from an equally simple statement: an injective holomorphic function $f : \mathbb{C} \to \mathbb{C}$ satisfying $f(0) = 0$ and $f'(0) = 1$ must be the identity. This suggests the possibility that similar characterizations of the identity I might be useful in the study of $\mathrm{Aut}(\mathbb{C}^n)$. On the other hand, any expectations in this regard should be tempered by the fact that injective entire maps are not necessarily in $\mathrm{Aut}(\mathbb{C}^n)$ for $n > 1$, as shown by the Fatou-Bieberbach examples ([5], p. 45). In this paper we use (real) geometric methods to prove the following rigidity theorem for the identity I, in all dimensions:

Theorem 1. *An injective holomorphic map $f : \mathbb{C}^n \to \mathbb{C}^n$ is the identity if and only if*

i) The power series at 0 *of* $f - I$ *has no terms of order* $\leq 2n+1$.
ii) The function $|\det Df(z)|\, |z|^{2n} |f(z)|^{-2n}$ *is subharmonic on* $\mathbb{C}^n$.

Partially supported by NSF grant DMS02-03637

The automorphisms of $\mathbb{C}^2$ of the form $f(x,y) = (x+p(y), y)$ show that for $n \geq 2$ the identity cannot be characterized, among injective holomorphic maps, solely by finitely many pointwise conditions at 0. Hence some kind of global condition, such as the one in ii), is necessary. On the other hand, one would like to improve the vanishing condition in i) to $\leq 2n-1$, or even replace it with $f(0) = 0$, $Df(0) = I$. If this could be done the resulting statement would be sharp, regarding the dimension, already with $n = 1$. Indeed, when $n = 1$ the function in ii) can be realized as the absolute value squared of a holomorphic function, and therefore it is automatically subharmonic.

From a technical standpoint, in this paper we are concerned with smooth maps that are defined on a closed ball of $\mathbb{R}^n$ and coincide at the center with their linearizations, up to a high enough order. We prove that, if injective, these maps must satisfy certain integral estimates.

Although our arguments are essentially of a real-variables nature they are inspired by the so-called area theorem, an elementary result in the theory of univalent functions ([7], [17]). But as one would expect, the transition to higher dimensions requires a different perspective. In the classical setting an important role is played by the complex inversion $z \to z^{-1}$ in $\mathbb{C}$. The main idea in the higher dimensional case is to use the geometric inversion in the unit sphere of $\mathbb{R}^n$, to the extent that it is possible, as a substitute for the complex inversion. At the conceptual level, Fourier (power) series are then replaced with spherical harmonics.

Much of the impetus for studying the general mechanisms that make a local diffeomorphism injective stems from the jacobian conjecture. In one of its many formulations this well known problem states that all polynomial local biholomorphisms of $\mathbb{C}^n$ into itself must be injective (and then surjective, by general principles in algebraic geometry). Many solutions have been proposed over the years, but the consensus is that in its full generality the conjecture remains open. For more on the conjecture, as well as results on the general topic of global invertibility, see [4],[6], [13], [14],[15],[16],[18]. A more recent result is the following beautiful theorem of E. Balreira [3]: a local diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective if and only if the pre-image of every affine hyperplane is non-empty and acyclic.

In [15] surgery theory and other topological tools were used to prove that the jacobian conjecture is at least "generically" true, in a sense that will be made precise shortly. We provide a brief discussion of some aspects of [15]. Given a non-constant polynomial $P : \mathbb{C}^n \to \mathbb{C}$ there is a finite set $B \subset \mathbb{C}$ such that the restriction of f to $\mathbb{C}^n - f^{-1}(B)$ is a locally trivial differentiable fibration onto $\mathbb{C} - B$. For the purpose of stating our results succintly, we say that a complex hypersurface $\Sigma \subset \mathbb{C}^n$ is P-generic if $\Sigma = P^{-1}(a)$, where $a \notin B$. In [15] we work in the context of local diffeomorphisms $f : M \to \mathbb{C}^n \cong \mathbb{R}^{2n}$, where the differentiable manifold M is connected and its first homology group vanishes. We also assume the existence of a complex hypersurface Σ such that the restriction of f to $M - f^{-1}(\Sigma)$ is a d-sheeted cover map onto $\mathbb{C}^n - \Sigma$ (these assumptions are realized by any polynomial local biholomorphism $\mathbb{C}^n \to \mathbb{C}^n$). We prove that if $\Sigma \subset \mathbb{C}^n$ is P-generic for some P, then either $d = 1$ or $d = \infty$. In particular, if f is algebraic the degree is finite and so f has to be injective, thus realizing the conclusion in the jacobian conjecture. Other arguments in [15] show that the dichotomy $d = 1$ or $d = \infty$ must also hold if the cover in question is regular, or if Σ is the general hypersurface of a fixed degree. The starting point of our investigations leading up to [15] was the work of Kulikov [11], where algebraic examples had been constructed with $1 < d < \infty$.

The topological results in [11] and [15] suggest that the jacobian conjecture may well be false. For this reason, in §2 we collect several analytic results related to Theorem 1 that are potentially useful in the search for counterexamples to the conjecture.

§2 Integral estimates for injective maps

The estimate in Lemma 1 below is the first application of our method. The constant c_n represents the volume of the unit ball of $\mathbb{R}^n$. Its explicit value is ([2], p.239):

$$c_n = \frac{\pi^{\frac{n}{2}}}{(\frac{n}{2})!} \ (n \ \text{ even}), \quad c_n = \frac{2^{\frac{(n+1)}{2}} \pi^{\frac{(n-1)}{2}}}{(1)(3)(5)...(n)} \ (n \ \text{ odd}).$$

Let $B(R) \subset \mathbb{R}^n$ be the closed ball of radius R centered at 0, I the identity map. The next two results will be proved in §3.

Lemma 1. *Let $f : B(R) \to \mathbb{R}^n$ be a smooth map with the property that $f(0) = 0$, $Df(0) = I$ and $D^\alpha f(0) = 0$ for every multi-index α such that $2 \leq |\alpha| \leq n+1$. If f is injective, then*

$$\int_{B(R)} \left[\frac{|\det Df(x)|}{|f(x)|^{2n}} - \frac{1}{|x|^{2n}} \right] dV(x) \leq c_n R^{-n}.$$

For globally defined maps one has

Corollary 1. *Let $f, g : \mathbb{R}^n \to \mathbb{R}^n$ be diffeomorphisms satisfying $f(0) = g(0) = 0$, $Df(0) = Dg(0) = I$ and $D^\alpha f(0) = D^\alpha g(0) = 0$ for every multi-index α such that $2 \leq |\alpha| \leq n+1$. Then*

$$\int_{\mathbb{R}^n} \left[\frac{|\det Df(x)|}{|f(x)|^{2n}} - \frac{|\det Dg(x)|}{|g(x)|^{2n}} \right] dV(x) = 0.$$

As mentioned in the introduction, the topological results of [11] and [15] suggest that the jacobian conjecture may be false, perhaps already in $\mathbb{C}^2$. The corollaries below might be useful in checking that a candidate for a counterexample is not injective.

Corollary 2. *Let $f_j : \mathbb{C}^2 \to \mathbb{C}^2$, $j = 1, 2$, be polynomial maps such that $f_j(0) = 0$, $Df_j(0) = I$ and $\det Df_j = 1$. Suppose that the components of f_j contain no monomials of degrees 2, 3, 4 and 5. If the jacobian conjecture is true, then*

$$\int_{B(1)} \left[\frac{1}{|f_1(z)|^8} - \frac{1}{|z|^8}\right] dV \leq \frac{\pi^2}{2},$$

$$\int_{\mathbb{C}^2} \left[\frac{1}{|f_1(z)|^8} - \frac{1}{|f_2(z)|^8}\right] dV = 0.$$

Corollary 3. *Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map such that $f(0) = 0$, $Df(0) = I$ and $\det Df = 1$. Suppose that the components of f contain no monomials of degrees 2, 3, 4 and 5. If the jacobian conjecture is true and $f \neq I$, then*

i) $|z|^2|f(z)|^{-2}$ is not subharmonic.

ii) $|z|^{-2}|f(z)|^2$ is not plurisubharmonic.

Setting $h(z) = |z|^2|f(z)|^{-2}$ one computes $\Delta h^n = nh^{n-1}\Delta h + n(n-1)h^{n-2}|\nabla h|^2$ and i) of Corollary 3 follows from Theorem 1. Next, assume ii) fails. Since f is a biholomorphism, the function $|z|^2|f^{-1}(z)|^{-2}$ is also plurisubharmonic, hence subharmonic. But this violates i) (applied to f^{-1}) since repeated applications of the chain rule show that $D^\alpha f^{-1}(0) = 0$ if the multi-index α satisfies $2 \leq |\alpha| \leq 5$.

§3 Geometric arguments

In this section we prove Lemma 1 as well as Corollary 1. Theorem 1 will be proven in §4. Assume for now that f is an injective smooth map defined on the closed unit ball $B = B(1)$ of $\mathbb{R}^n$. Let $\sigma : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$ be the inversion in the unit sphere ∂B, $\sigma(x) = |x|^{-2}x = r^{-2}x$. Since σ is conformal and a unit radial vector at x is mapped by $D\sigma(x)$ into a vector of length r^{-2}, the image under $D\sigma(x)$ of a unit cube is another cube whose volume is $|\det D\sigma(x)| = r^{-2n}$. Letting $F : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$, $F(x) = \sigma(f(x))$, the chain rule gives

$$|\det DF(x)| = \frac{|\det Df(x)|}{|f(x)|^{2n}}. \tag{1}$$

For $0 < a < 1$, consider the following pairwise disjoint subsets of B:

$$C_a = \{x \in B,\ |x| = a\},\ \ U_a = \{x \in B,\ \ 0 < |x| < a\},\ \ V_a = \{x \in B,\ \ a < |x| < 1\}.$$

Since f (and then F) is continuous and injective, it is a homeomorphism onto its image (invariance of domain). Hence $F(C_a)$ is a compact embedded topological hypersurface. In particular, its complement in $\mathbb{R}^n$ has two connected components (Alexander duality). Let W_a be the bounded component of $\mathbb{R}^n - F(C_a)$. Observe that the sets $F(C_a)$, $F(U_a - \{0\})$ and $F(V_a)$ are pairwise disjoint as well. Since U_a contains 0, $F(U_a - \{0\})$ must be contained in the unbounded component of $\mathbb{R}^n - F(C_a)$. From this one readily sees that $F(V_a) \subset W_a$. In particular,

$$\operatorname{vol}(F(V_a)) \leq \operatorname{vol}(W_a). \tag{2}$$

We now proceed to estimate $\operatorname{vol}(W_a)$. Writing $f = I + \xi$, where $\xi(x) = O(|x|^N)$, $N \geq 2$, we have

$$F(x) = \frac{x + \xi(x)}{|x + \xi(x)|^2} = \frac{x}{|x|^2} + \frac{|x|^2\xi(x) - (|\xi(x)|^2 + 2\langle x, \xi(x)\rangle)x}{|x|^2|x + \xi(x)|^2},$$

and so

$$F(x) = \sigma(x) + O(|x|^{N-2}). \tag{3}$$

More precisely, from the above we obtain an estimate for the Hausdorff distance between the sets $F(C_a)$ and $\sigma(C_a)$: there exist η, $\delta > 0$, $0 < \delta < 1$, such that

$$d_H(F(C_a),\ \sigma(C_a)) \leq \eta a^{N-2}, \tag{4}$$

whenever $0 < a \leq \delta$. In particular, W_a is contained in a ball centered at 0 of radius $a^{-1} + 2\eta a^{N-2}$ whose volume is

$$c_n(a^{-1} + 2\eta a^{N-2})^n = c_n a^{-n}(1 + 2\eta a^{N-1})^n.$$

From the binomial expansion and (2) one obtains a constant K such that

$$\operatorname{vol}(F(V_a)) \leq c_n a^{-n} + K a^{N-1-n}, \tag{5}$$

for all sufficiently small a. From (1) and (5) one has

$$\operatorname{vol}(F(V_a)) = \int_{V_a} \frac{|\det Df(x)|}{|f(x)|^{2n}} dV(x) \leq c_n a^{-n} + K a^{N-1-n}. \tag{6}$$

On the other hand,

$$\operatorname{vol}(F(V_a)) = \int_a^1 r^{-n-1}\Big(\int_{S^{n-1}} |\det Df(r\omega)|[r^2|f(r\omega)|^{-2}]^n d\omega\Big) dr, \tag{7}$$

and from $f = I + \xi$,

$$[r^2|f(r\omega)|^{-2}]^n = \left[1 + 2\langle\omega, \frac{\xi(x)}{r}\rangle + \frac{|\xi(z)|^2}{r^2}\right]^{-n}.$$

Since $\xi = O(r^N) = O(r^2)$, this expression can be inverted for small r using a geometric series:

$$[r^2|f(r\omega)|^{-2}]^n = \left[1 + \sum_{j=1}^{\infty}(-1)^j \left[2\langle\omega, \frac{\xi(x)}{r}\rangle + \frac{|\xi(x)|^2}{r^2}\right]^j\right]^n.$$

In particular,

$$(8) \qquad |\det Df(r\omega)|[r^2|f(r\omega)|^{-2}]^n = 1 + Q(x), \;\; Q(x) = O(r^{N-1}), \; x = r\omega, |\omega| = 1.$$

From (6), (7) and (8),

$$c_n a^{-n} + K a^{N-1-n} \geq \int_a^1 \left[r^{-n-1}\int_{S^{n-1}}(1 + Q(r\omega))dw\right]dr.$$

Denoting by d_n the $(n-1)$ – volume of S^{n-1}, the previous estimate yields

$$c_n a^{-n} + K a^{N-1-n} \geq a^{-n}\frac{d_n}{n} - \frac{d_n}{n} + \int_a^1 \int_{S^{n-1}} r^{-n-1}Q(r\omega)dwdr.$$

Observing that $d_n = nc_n$ we have

$$(9) \qquad \int_a^1 \int_{S^{n-1}} r^{-n-1}Q(r\omega)dwdr \leq c_n + K a^{N-1-n}.$$

Let now f be as in the statement of Lemma 1, so that $N = n + 2$. We suppose $R = 1$ first. Letting $a \to 0$ in (9) and using $Q(x) = O(r^{N-1}) = O(r^{n+1})$, we obtain

$$(10) \qquad \int_0^1 \int_{S^{n-1}} r^{-n-1}Q(r\omega)dwdr \leq c_n,$$

which, in view of (8), can be rewritten as

$$(11) \qquad \int_B \left[\frac{|\det Df(x)|}{|f(x)|^{2n}} - \frac{1}{|x|^{2n}}\right]dV(x) \leq c_n,$$

We can now finish the proof of Lemma 1. Suppose that f is defined on $B(R)$. Applying (11) to the map $f_R : B \to \mathbb{R}^n$, $f_R(x) = R^{-1}f(Rx)$, and changing variables we finally have

$$(12) \qquad \int_{B(R)} \left[\frac{|\det Df(x)|}{|f(x)|^{2n}} - \frac{1}{|x|^{2n}}\right]dV(x) \leq c_n R^{-n}.$$

To prove Corollary 1 we proceed as follows. Suppose now that f and g are defined on $\mathbb{R}^n$. Observe that the value of the integral of $|\det Df(x)||f(x)|^{-2n}$ over a neighborhood of infinity equals the volume of a (punctured) neighborhood of 0. In particular, 0 and ∞ are integrable singularities in (12). Letting $R \to \infty$, we have

$$\int_{\mathbb{R}^n} \left[\frac{|\det Df(x)|}{|f(x)|^{2n}} - \frac{1}{|x|^{2n}} \right] dV(x) \leq 0. \tag{13}$$

Using the chain rule one can easily see that the set of diffeomorphisms $f : \mathbb{R}^n \to \mathbb{R}^n$ for which the $(n+1)$ – jet of $f - I$ vanishes at 0 forms a group under composition. Applying (13) to $g \circ f^{-1}$ and changing variables we have

$$0 \geq \int_{\mathbb{R}^n} \left[\frac{|\det D(g \circ f^{-1})(x)|}{|g(f^{-1}(x))|^{2n}} - \frac{1}{|x|^{2n}} \right] dV(x) = \int_{\mathbb{R}^n} \left[\frac{|\det Dg(y)|}{|g(y)|^{2n}} - \frac{|\det Df(y)|}{|f(y)|^{2n}} \right] dV(y).$$

The conclusion now follows by reversing the roles of f and g in the above estimate. ◇

§4 Rigidity of the Identity

In this section we prove Theorem 1. We start by assuming that f satisfies the hypotheses of Lemma 1 (with $R = 1$). For future use we estimate (7) in a way that, at least in spirit, is closer to the original line of argument in the classical area theorem.

Setting

$$\eta_r(\omega) = |\det Df(r\omega)|^{\frac{1}{2}} [r|f(r\omega)|^{-1}]^n$$

one can rewrite (7) as

$$\mathrm{vol}(F(V_a)) = \int_a^1 r^{-n-1} \Big(\int_{S^{n-1}} [\eta_r(\omega)]^2 d\omega \Big) dr. \tag{14}$$

For fixed r, we write the orthogonal decomposition of $\eta_r(\omega)$ relative to the space of constant functions in $L^2(S^{n-1})$ (of which $\{d_n^{-1/2}\}$ is an orthonormal basis where, as before, $d_n = \mathrm{vol}_{n-1}(S^{n-1})$):

$$\eta_r(\omega) = \int_{S^{n-1}} d_n^{-1} \eta_r(\omega) d\omega + \left[\eta_r(\omega) - \int_{S^{n-1}} d_n^{-1} \eta_r(\alpha) d\alpha \right].$$

(A more refined decomposition would involve the spherical harmonics in $\{d_n^{-1/2}\}^\perp$.) Hence (14) can be written as

$$\mathrm{vol}(F(V_a)) = \int_a^1 r^{-n-1} [\phi^{(1)}(r) + \phi^{(2)}(r)] dr, \tag{15}$$

where

$$\phi^{(1)}(r) = d_n^{-1}\Big(\int_{S^{n-1}} \eta_r(\omega)d\omega\Big)^2 \tag{16}$$

and

$$\phi^{(2)}(r) = \int_{S^{n-1}} \big[\eta_r(\omega) - d_n^{-1}\int_{S^{n-1}} \eta_r(\alpha)d\alpha\big]^2 d\omega. \tag{17}$$

Also,

$$\int_a^1 r^{-n-1}(\phi^{(1)}(r))dr = \int_a^1 r^{-n-1}(\phi^{(1)}(r) - d_n)dr + d_n\frac{a^{-n}}{n} - \frac{d_n}{n}. \tag{18}$$

From (5), (16) and (17) we then have

$$c_n a^{-n} + Ka^{N-1-n} \geq \int_a^1 r^{-n-1}\phi^{(2)}(r)dr + \int_a^1 r^{-n-1}(\phi^{(1)}(r) - d_n)dr + d_n\frac{a^{-n}}{n} - \frac{d_n}{n},$$

which simplifies to

$$c_n + Ka^{N-1-n} \geq \int_a^1 r^{-n-1}\phi^{(2)}(r)dr + \int_a^1 r^{-n-1}(\phi^{(1)}(r) - d_n)dr$$

We now take $N = n+2$ and let $a \to 0$ in the above expression. Since $\phi^{(2)}(r) \geq 0$ and the second integral on the right converges on account of (8), we obtain a new version of Lemma 1:

Lemma 2. *Let $f : B(1) \to \mathbb{R}^n$ be a smooth map with the property that $f(0) = 0$, $Df(0) = I$ and $D^\alpha f(0) = 0$ for every multi-index α such that $2 \leq |\alpha| \leq n+1$. If f is injective then, with the above notation,*

$$\int_0^1 r^{-n-1}[\phi^{(2)}(r) + (\phi^{(1)}(r) - d_n)]dr \leq c_n.$$

Suppose now that the map in Lemma 2 is the restriction of an injective smooth map f defined on the entire space $\mathbb{R}^n$. If f is replaced by $R^{-1}f(Rx)$, $R > 0$, $|x| \leq 1$, the corresponding function $\tilde{\eta}_r^R$ satisfies $\tilde{\eta}_r^R(\omega) = \eta_{rR}(\omega)$. From(16) and (17) one sees that, likewise, the new functions $\phi_R^{(j)}$ satisfy $\phi_R^{(j)}(r) = \phi^{(j)}(rR)$, $j = 1, 2$. Lemma 2 then gives

$$\int_0^R r^{-n-1}[\phi^{(2)}(r) + (\phi^{(1)}(r) - d_n)]dr \leq c_n R^{-n}. \tag{19}$$

Proof of Theorem 1. We may view the map $f : \mathbb{C}^n \to \mathbb{C}^n$ in the statement of the theorem as an injective smooth map defined on $\mathbb{R}^{2n}$. Since all complex derivatives of f of order $\leq 2n+1$ vanish at zero and f is holomorphic, the real derivatives of order $\leq 2n+1$ also vanish at zero. Hence we may apply (19) to the induced real map.

Since $\det D_{\mathbb{R}} f = |\det D_{\mathbb{C}} f|^2$ for any holomorphic map, hypothesis ii) in the statement of the theorem says precisely that the function $z = r\eta \to \eta_r(\omega)$ is subharmonic where, as before, $\eta_r(\omega) = |\det D_{\mathbb{R}} f|^{\frac{1}{2}}[r|f(r\omega)|^{-1}|^{2n}$. The submean value property for subharmonic functions together with (16) imply that $\phi^{(1)}(r) - d_{2n} \geq 0$ for all $r > 0$. Also, $\phi^{(2)}$ is manifestly non-negative. Letting $R \to \infty$ in (19), one then has

$$\int_0^\infty r^{-2n-1}[\phi^{(2)}(r) + (\phi^{(1)}(r) - d_{2n})]dr = 0.$$

In particular, both $\phi^{(2)}(r)$ and $\phi^{(1)}(r) - d_{2n}$ must vanish for all $r > 0$. In terms of the orthogonal decomposition of $L^2(S^{2n-1})$, vanishing of $\phi^{(2)}(r)$ means, for each r, that the function η_r must be constant on S^{2n-1}. In turn, vanishing of $\phi^{(1)}(r) - d_{2n}$ implies that this constant must be one. In other words, $\eta_r(\omega) = 1$ for all r and ω. Hence $|\det D_{\mathbb{C}} f(z)|\ |z|^{2n}|f(z)|^{-2n} = 1$. Consider now the holomorphic map $G(z) = (\det D_{\mathbb{C}} f(z))^{-\frac{1}{2n}} f(z)$, so that $|G(z)|^{2n} = |z|^{2n}$. In particular, G applies the unit ball into itself. Since $G(0) = 0$ and $DG(0) = I$, by a general result of H. Cartan ([12], p.66), $G = I$. Hence $f = hI$, with h holomorphic, $h(0) = 1$. In order to finish the proof of the theorem it remains to show that h is constant. Suppose this is not the case and let l be a complex line passing through 0 such that the restriction of h to l is non-constant. We choose coordinates $(z_1, z_2, ..., z_n)$ in $\mathbb{C}^n$ such that l is given by $z_j = 0$, $2 \leq j \leq n$. From $f = hI$ we have $f(z_1, 0, ..., 0) = (h(z_1, 0, ..., 0)z_1, 0, ..., 0)$. Since f is injective, the function $z_1 \to h(z_1, 0, ..., 0)z_1$, $z \in \mathbb{C}$, must also be injective. But an injective entire map in dimension one must be affine linear. Hence $z_1 \to h(z_1, 0, ..., 0)$ must be constant, and this contradicts our choice of l.

References

[1] Andersen, E., Lempert, L., *On the group of automorphisms of* $\mathbb{C}^n$, Invent. Math. **110** (1992), 371-388.

[2] Axler, S., Bourdon, P., Ramey, W., *Harmonic function theory*, vol. 137, Graduate Texts in Mathematics, Springer-Verlag, 2001.

[3] Balreira, E., *Detecting invertibility from the topology of pull backs of hyperplanes*, Doctoral Dissertation, University of Notre Dame (2006).

[4] Bass, H., Connell, E. Wright, D., *The Jacobian Conjecture: Reduction of the degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. **7** (1982), 287-330.

[5] Bochner, S., Martin, W., *Several Complex Variables*, Princeton University Press, 1948.

[6] den Essen, V., A., *Polynomial Automorphisms and the Jacobian Conjecture*, Progress in Mathematics 190, Birkhauser, 2000.

[7] Duren, P., *Univalent functions*, Springer-Verlag, 1983.

[8] Fornaess, J.E., Sibony, N., *Some problems in higher dimensional complex analysis and complex dynamics*, Publ. Mat. **45** (2001), 529- 647.

[9] Forstneric, F., Rosay, J.P., *Approximation of biholomorphic mappings by automorphisms of* $\mathbb{C}^k$, Invent. Math. **112** (1993), 323-349.

[10] Jung, H.W.E., *Über ganze birationale Transformationen der Ebene*, J. Reine Angew. Math. **184** (1942), 161-174.

[11] Kulikov, V., *Generalized and local Jacobian problems*, Russian Acad. Sci. Izv. Math **41**, 351-365.

[12] Narasimhan, R., *Several complex variables*, The University of Chicago Press, 1971.

[13] Nollet, S., Xavier, F., *Global Inversion via the Palais-Smale Condition*, Discrete and Continuous Dynamical Systems A **8** (2002), 17-28.

[14] Nollet, S., Xavier, F., *Holomorphic Injectivity and the Hopf map*, GAFA (Geometric and Functional Analysis) **14** (2004), 1339-1351.

[15] Nollet, S., Taylor, L., Xavier, F., *Birationality of étale maps via surgery, preprint* (2005).

[16] Pinchuk, S., *A counterexample to the strong real Jacobian conjecture*, Math. Zeitschr. **217** (1994), 1-4.

[17] Rudin, W., *Real and Complex Analysis*, McGraw-Hill Book Company, 1987.

[18] Xavier, F., *Injectivity as a transversality phenomenon in geometries of negative curvature*, Illinois Journal of Mathematics. **43** (1999), 256-263.

Frederico Xavier
Department of Mathematics
University of Notre Dame, Notre Dame IN, 46635
email xavier.1@nd.edu