A COMPLEX-ANALYTIC PROOF OF CATALAN'S THEOREM

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Abstract. According to the Colding-Minicozzi theory, embedded minimal discs in $\mathbb{R}^3$ resemble, loosely speaking, either graphs or multigraphs. The latter gives the familiar helicoid-like pattern. Hence, away from the set where the Gaussian curvature vanishes, the product of the curvatures of the asymptotic lines – a quantity that we call the *Catalan curvature* of the surface – may be taken as a measure of how far the surface is from being ruled, and thus how much the multigraph deviates from being an actual helicoid, by the classical theorem of Catalan. As a first step in the program of understanding the structure of embedded minimal discs using classical tools, we give in this paper a complex-analytic proof of this theorem, using the idea behind the Catalan curvature. In terms of the Weierstrass-Enneper representation the problem is reduced, after careful normalizations, to the uniqueness of solutions of certain holomorphic differential equations.

1. Introduction

The past few years have witnessed great advances in the study of minimal surfaces. Thanks to the Colding-Minicozzi theory [1]-[5] the structure of embedded minimal discs is now well understood. Using this theory, as well as other tools, Meeks and Rosenberg showed in a landmark paper [6] that the helicoid and the plane are the only complete embedded simply-connected minimal surfaces in $\mathbb{R}^3$. Except for an earlier partial result [8], none of the recent developments pertaining to embedded simply-connnected minimal surfaces relied on the classical link between minimal surfaces and complex analysis, a connection that has been used with great success to tackle other fundamental problems in the theory of minimal surfaces.

One would expect that more can be revealed about the nature of embedded minimal discs if the Colding-Minicozzi theory can be understood from the standpoint of complex analysis. For instance, there is a compelling analogy between the theory of conformal harmonic embeddings of the open unit disc $D \subset \mathbb{C}$ into $\mathbb{R}^3$, and the very rich theory of (holomorphic) univalent functions on D. It is an easy matter to use the standard estimates about univalent functions on D to conclude that any

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entire univalent function must be of the form $az + b$, $b \neq 0$. By analogy, this leads one to conjecture that *the only conformal harmonic embeddings of* $\mathbb{C}$ *into* $\mathbb{R}^3$ *are given by the standard embeddings of the plane and the helicoid.* (See also [9], where similar considerations yield a rigidity theorem in the realm of several complex variables). If true, this conjecture would provide a striking complement to the Meeks-Rosenberg theorem, by shifting the focus away from the completeness condition. At the moment, there are hopeful signs indicating that large portions of the powerful Lowner theory of univalent functions can be carried over to the more general context of conformal harmonic embeddings. We hope to come back to this point in a future publication.

According to the Colding-Minicozzi theory, embedded minimal discs resemble, loosely speaking, either graphs or multigraphs. The latter gives the familiar helicoid-like pattern. Hence, away from the set where the Gaussian curvature vanishes, the product of the curvatures of the asymptotic lines – a quantity that we call the *Catalan curvature* of the surface – may be taken as a measure of how far the surface is from being ruled, and thus how much the multigraph deviates from being an actual helicoid, by the classical theorem of Catalan:

Theorem(Catalan) *If a non-flat connected minimal surface in* $\mathbb{R}^3$ *is ruled, then it is part of a helicoid.*

In keeping with the above remarks, we present in this paper a complex-analytic proof of Catalan's theorem that is couched on the fact that the Catalan curvature of a non-flat ruled minimal surface must vanish. This is to be regarded as a first step in the program of understanding the structure of embedded minimal discs using classical tools. In terms of the Weierstrass-Enneper representation the problem is reduced, after careful normalizations, to the uniqueness of solutions of certain holomorphic differential equations.

Before we begin the actual proof of Catalan's theorem, to be given in §2–§4, we recall some standard facts about the Weierstrass-Enneper representation of minimal surfaces in $\mathbb{R}^3$. Let $X(u, v)$ be a conformal parametrization of a minimal surface. Setting
$z = u + iv$ and

$$\Phi_i(z) = \frac{\partial X_i}{\partial u} - i\frac{\partial X_i}{\partial v}, \quad i = 1, 2, 3, \tag{1.1}$$

one has that the $\Phi_i's$ are holomorphic and, furthermore,

$$\langle \Phi, \overline{\Phi} \rangle = \sum_{i=1}^{3} \Phi_i^2 = 0. \tag{1.2}$$

In terms of the $\Phi_i's$, the parametrization can be recovered by integration:

$$X_i(z) = \mathrm{Re}\left\{\int_0^z \Phi_i(\zeta)d\zeta\right\} + C_i, \;\; i = 1,2,3. \tag{1.3}$$

Let f be the holomorphic function and g be the meromorphic function associated with the parametrization X, that is,

$$f = \Phi_1 - i\Phi_2\,, \qquad g = \frac{\phi_3}{\Phi_1 - i\Phi_2}\,. \tag{1.4}$$

The pair (f, g) will be called the Weierstrass-Enneper pair associated with the parametrization X. In terms of (f, g), the function $\Phi = (\Phi_1, \Phi_2, \Phi_3)$, the first fundamental form g_{ij} and the unit normal vector field N of the immersion are given, respectively, by

$$\Phi = \left(\frac{1}{2}f(1-g^2), \frac{i}{2}f(1+g^2), fg\right), \tag{1.5}$$

$$g_{ij} = \lambda^2\delta_{ij}\,, \qquad \lambda^2 = \left(\frac{|f|(1+|g|^2)}{2}\right)^2. \tag{1.6}$$

$$N = \left(\frac{2\mathrm{Re}\{g\}}{|g|^2+1}, \frac{2\mathrm{Im}\{g\}}{|g|^2+1}, \frac{|g|^2-1}{|g|^2+1}\right). \tag{1.7}$$

From (1.1) and (1.6) we obtain

$$|\Phi|^2 = \langle\Phi, \Phi\rangle = \sum_{i=1}^{3}|\Phi_i|^2 = \frac{|f|^2(1+|g|^2)^2}{2}, \tag{1.8}$$

Since the minimal surface is regular and Φ is holomorphic, it follows from (1.5) and (1.6) that the zeros of f occur only on the poles of g and the order of each zero of f is exactly twice the order of the corresponding pole of g. Up to a composition with the inverse stereographic projection, the meromorphic function g represents the Gauss map of the immersion. For more details on the connection between minimal surfaces and complex analysis, see [7].

If g is holomorphic and has a local inverse, we can consider $\tau = g(z)$ as a new parameter and, defining

$$F(\tau) = \frac{f(z)}{g'(z)}, \tag{1.9}$$

we obtain $F(\tau)d\tau = f(z)dz$. Therefore, from (1.3) and (1.5) we conclude that, in a neighborhood of a point where $g' \neq 0$, the surface can be parametrized in terms of F only:

$$X(z) = \mathrm{Re}\left\{\left(\int(1-\tau^2)F(\tau)d\tau, \int i(1+\tau^2)F(\tau)d\tau, \int 2\tau F(\tau)d\tau\right)\right\}. \tag{1.10}$$

2. Asymptotic Lines of Minimal Surfaces

In this section, we will obtain a second order differential equation that must be satisfied by a ruled minimal surface in $\mathbb{R}^3$ in a neighborhood of each point where the Gaussian curvature does not vanish. We begin with the following lemma:

Lemma 2.1. *Let $X(u,v)$ be a conformal parametrization of a minimal surface and let (f,g) be the associated Weierstrass-Enneper pair. Then, for all curves $\alpha(t) = X(\zeta(t))$ in the image of the parametrization, one has*

$$\alpha' \wedge \alpha'' = \frac{1}{4}(1+|g|^2)\mathrm{Im}\left\{|f|^2(1+|g|^2)\overline{\zeta'}\zeta'' + |\zeta'|^2(1+|g|^2)\zeta'\overline{f}f' + 2|\zeta'|^2|f|^2g'\overline{g}\zeta'\right\}N$$
$$+\mathrm{Re}\{fg'\zeta'^2\}N \wedge \alpha'. \tag{2.1}$$

Proof. We have $\Phi = 2X_z$ by (1.1), and so the velocity vector of $\alpha(t)$ is given by

$$\alpha'(t) = X_z\zeta' + X_{\bar{z}}\bar{\zeta}' = X_z\zeta' + \overline{(X_z)}\bar{\zeta}' = 2\mathrm{Re}\{X_z\zeta'\} = \mathrm{Re}\{\zeta'\Phi\}. \tag{2.2}$$

The aceleration of $\alpha(t)$ is thus

$$\alpha''(t) = \frac{d}{dt}\mathrm{Re}\{\zeta'\Phi\} = \mathrm{Re}\left\{\frac{d}{dt}(\zeta'\Phi)\right\} = \mathrm{Re}\left\{\zeta'\frac{d\Phi}{dt} + \zeta''\Phi\right\} = \mathrm{Re}\{\zeta'^2\Phi' + \zeta''\Phi\}. \tag{2.3}$$

Using the formula

$$\mathrm{Re}\{v\} \wedge \mathrm{Re}\{w\} = \frac{1}{2}\mathrm{Re}\{v \wedge w + \overline{v} \wedge w\}, \tag{2.4}$$

valid for all $v, w \in \mathbb{C}^3$, and equations (2.2) and (2.3), we get

$$\alpha' \wedge \alpha'' = \frac{1}{2}\mathrm{Re}\{(\zeta'\Phi) \wedge (\zeta'^2\Phi' + \zeta''\Phi) + (\overline{\zeta'\Phi}) \wedge (\zeta'^2\Phi' + \zeta''\Phi)\}$$
$$= \frac{1}{2}\mathrm{Re}\{\zeta'^3\Phi \wedge \Phi' + |\zeta'|^2\zeta'\overline{\Phi} \wedge \Phi' + \overline{\zeta'}\zeta''\overline{\Phi} \wedge \Phi\}. \tag{2.5}$$

From (1.5) we have

$$\Phi' = \left(\frac{1}{2}f'(1-g^2) - fgg', \frac{i}{2}f'(1+g^2) + ifgg', f'g + fg'\right), \tag{2.6}$$

and

$$\overline{\Phi} = \left(\frac{1}{2}\overline{f}(1-\overline{g}^2), -\frac{i}{2}\overline{f}(1+\overline{g}^2), \overline{fg}\right). \tag{2.7}$$

After a long but straightforward calculation, we obtain

$$\Phi \wedge \Phi' = ifg'\left(\frac{1}{2}f(1-g^2), \frac{i}{2}f(1+g^2), fg\right) = ifg'\Phi, \tag{2.8}$$

$$\overline{\Phi} \wedge \Phi = -\frac{i}{2}|f|^2(1+|g|^2)^2N, \tag{2.9}$$

$$\overline{\Phi}\wedge\Phi' = -\frac{i}{2}\overline{f}f'(1+|g|^2)^2N - i|f|^2g'\left(\frac{1}{2}+\frac{\overline{g}^2}{2}+|g|^2, -\frac{i}{2}+\frac{i}{2}\overline{g}^2 - i|g|^2, |g|^2\overline{g}\right). \tag{2.10}$$

We will now determine each term on the right hand side of (2.5). A simple calculation gives us

$$N\wedge\Phi = i\Phi. \tag{2.11}$$

Using the equality above and equations (2.2) and (2.4), we obtain

$$N\wedge\alpha' = \mathrm{Re}\{N\wedge(\zeta'\Phi)\} = \mathrm{Re}\{\zeta' N\wedge\Phi)\} = \mathrm{Re}\{i\zeta'\Phi\} = -\mathrm{Im}\{\zeta'\Phi\}. \tag{2.12}$$

The first term on the right hand side of (2.5) can now be determined with the aid of (2.8) and (2.12):

$$\begin{aligned}\mathrm{Re}\{\zeta'^3\Phi\wedge\Phi'\} &= \mathrm{Re}\{\zeta'^3 ifg'\Phi\} = \mathrm{Re}\{ifg'\zeta'^2(\zeta'\Phi)\}\\ &= \mathrm{Re}\{ifg'\zeta'^2(\mathrm{Re}\{\zeta'\Phi\}+i\mathrm{Im}\{\zeta'\Phi\})\}\\ &= \mathrm{Re}\{ifg'\zeta'^2\alpha' + fg'\zeta'^2 N\wedge\alpha'\}\\ &= \mathrm{Re}\{ifg'\zeta'^2\}\alpha' + \mathrm{Re}\{fg'\zeta'^2\}N\wedge\alpha'.\end{aligned} \tag{2.13}$$

The third term follows easily from (2.9):

$$\begin{aligned}\mathrm{Re}\{\overline{\zeta'}\zeta''\overline{\Phi}\wedge\Phi\} &= \mathrm{Re}\left\{\overline{\zeta'}\zeta''\left(-\frac{i}{2}|f|^2(1+|g|^2)^2N\right)\right\}\\ &= |f|^2(1+|g|^2)^2\mathrm{Re}\left\{-\frac{i}{2}\overline{\zeta'}\zeta''\right\}N.\end{aligned} \tag{2.14}$$

From (2.10) we can write the second term on the right hand side of (2.5) as

$$\begin{aligned}\mathrm{Re}\{|\zeta'|^2\zeta'\overline{\Phi}\wedge\Phi'\} &= |\zeta'|^2(1+|g|^2)^2\mathrm{Re}\left\{-\frac{i}{2}\zeta'\overline{f}f'\right\}N\\ &-|f|^2|\zeta'|^2\mathrm{Re}\left\{ig'\zeta'\left(\frac{1}{2}+\frac{\overline{g}^2}{2}+|g|^2, -\frac{i}{2}+\frac{i}{2}\overline{g}^2 - i|g|^2, |g|^2\overline{g}\right)\right\}.\end{aligned} \tag{2.15}$$

Setting

$$v = \left(\frac{1}{2}+\frac{\overline{g}^2}{2}+|g|^2, -\frac{i}{2}+\frac{i}{2}\overline{g}^2 - i|g|^2, |g|^2\overline{g}\right),$$

we need now to determine the components of $\mathrm{Re}\{ig'\zeta'v\}$ in the basis $\left\{\frac{\alpha'}{|\alpha'|}, N, N\wedge\frac{\alpha'}{|\alpha'|}\right\}$. Using the formula

$$\langle\mathrm{Re}\{v\},\mathrm{Re}\{w\}\rangle = \frac{1}{2}\mathrm{Re}\{\langle v,w\rangle + \langle v,\overline{w}\rangle\}, \tag{2.16}$$

valid for all $v, w\in\mathbb{C}^3$, we have

$$\left\langle\mathrm{Re}\{ig'\zeta'v\}, \frac{\alpha'}{|\alpha'|}\right\rangle = \frac{1}{|\alpha'|}\langle\mathrm{Re}\{ig'\zeta'v\},\mathrm{Re}\{\zeta'\Phi\}\rangle$$

$$= \frac{1}{2|\alpha'|}\mathrm{Re}\{\langle ig'\zeta' v, \zeta'\Phi\rangle + \langle ig'\zeta' v, \overline{\zeta'\Phi}\rangle\}$$
$$= \frac{1}{2|\alpha'|}\mathrm{Re}\{ig'|\zeta'|^2\langle v, \Phi\rangle + ig'\zeta'^2\langle v, \overline{\Phi}\rangle\}.$$

But a simple calculation gives

$$\langle v, \Phi\rangle = 0\,, \qquad \langle v, \overline{\Phi}\rangle = \frac{1}{2}f(1+|g|^2)^2, \tag{2.17}$$

and so

$$\left\langle \mathrm{Re}\{ig'\zeta' v\}, \frac{\alpha'}{|\alpha'|}\right\rangle \frac{\alpha'}{|\alpha'|} = \frac{1}{4|\alpha'|^2}(1+|g|^2)^2\mathrm{Re}\{ifg'\zeta'^2\}\alpha'. \tag{2.18}$$

On the other hand, by (1.2), (1.8) and (2.16) we have

$$|\alpha'|^2 = \langle \mathrm{Re}\{\zeta'\Phi\}, \mathrm{Re}\{\zeta'\Phi\}\rangle = \frac{1}{2}\mathrm{Re}\{\langle \zeta'\Phi, \zeta'\Phi\rangle + \langle \zeta'\Phi, \overline{\zeta'\Phi}\rangle\}$$
$$= \frac{1}{2}\mathrm{Re}\{|\zeta'|^2\langle \Phi, \Phi\rangle + \zeta'^2\langle \Phi, \overline{\Phi}\rangle\}$$
$$= \frac{|f|^2(1+|g|^2)^2}{4}|\zeta'|^2. \tag{2.19}$$

Thus

$$\left\langle \mathrm{Re}\{ig'\zeta' v\}, \frac{\alpha'}{|\alpha'|}\right\rangle \frac{\alpha'}{|\alpha'|} = \frac{\mathrm{Re}\{ifg'\zeta'^2\}}{|f|^2|\zeta'|^2}\alpha'. \tag{2.20}$$

Another simple calculation shows that

$$\langle v, N\rangle = \overline{g}(1+|g|^2),$$

which together with (2.16) implies

$$\langle \mathrm{Re}\{ig'\zeta' v\}, N\rangle N = \mathrm{Re}\{\langle ig'\zeta' v, N\rangle\}N = \mathrm{Re}\{ig'\zeta'\overline{g}(1+|g|^2)\}N$$
$$= (1+|g|^2)\mathrm{Re}\{ig'\overline{g}\zeta'\}N. \tag{2.21}$$

It remains to compute the component of $\mathrm{Re}\{ig'\zeta' v\}$ in the direction of the vector $N \wedge \frac{\alpha'}{|\alpha'|}$. To do this first observe that from (2.12), (2.16) and (2.17) we have

$$\langle \mathrm{Re}\{ig'\zeta' v\}, N \times \alpha'\rangle = \langle \mathrm{Re}\{ig'\zeta' v\}, \mathrm{Re}\{i\zeta'\Phi\}\rangle$$
$$= \frac{1}{2}\mathrm{Re}\{\langle ig'\zeta' v, i\zeta'\Phi\rangle + \langle ig'\zeta' v, -i\overline{\zeta'\Phi}\rangle\}$$
$$= \frac{1}{2}\mathrm{Re}\{|\zeta'|^2 g'\langle v, \Phi\rangle - \zeta'^2 g'\langle v, \overline{\Phi}\rangle\}$$
$$= -\frac{1}{4}(1+|g|^2)^2\mathrm{Re}\{fg'\zeta'^2\}. \tag{2.22}$$

From (2.19) and (2.22) we conclude that

$$\left\langle \mathrm{Re}\{ig'\zeta' v\}, N\wedge\frac{\alpha'}{|\alpha'|}\right\rangle N\wedge\frac{\alpha'}{|\alpha'|} = -\frac{1}{|\zeta'|^2|f|^2}\mathrm{Re}\{fg'\zeta'^2\}N\wedge\alpha'. \tag{2.23}$$

Combining (2.15), (2.20), (2.21) and (2.23) we get

$$\mathrm{Re}\{|\zeta'|^2\zeta'\overline{\Phi}\wedge\Phi'\} = -\mathrm{Re}\{ifg'\zeta'^2\}\alpha' + \mathrm{Re}\{fg'\zeta'^2\}N\wedge\alpha'$$
$$-\left(|\zeta'|^2(1+|g|^2)^2\mathrm{Re}\left\{\frac{i}{2}\zeta'\overline{f}f'\right\} + |\zeta'|^2|f|^2(1+|g|^2)\mathrm{Re}\{ig'\overline{g}\zeta'\}\right)N. \tag{2.24}$$

Formula (2.1) now follows from (2.5), (2.13), (2.14) and (2.24), after simplifications. □

Corollary 2.2. *Let $X : D \to S$ be a conformal parametrization of a minimal surface S and (f, g) be the associated Weierstrass-Enneper pair. Then, for any regular curve $\alpha(t) = X(\zeta(t))$ contained in $X(D)$, the normal curvature k_n and the geodesic curvature k_g of $\alpha(t)$ are given, respectively, by*

$$k_n = \frac{4\mathrm{Re}\{-fg'\zeta'^2\}}{|\zeta'|^2|f|^2(1+|g|^2)^2}, \tag{2.25}$$

$$k_g = \frac{2}{|\zeta'|^3|f|^3(1+|g|^2)^2}\mathrm{Im}\{|f|^2(1+|g|^2)\overline{\zeta'}\zeta'' + |\zeta'|^2(1+|g|^2)\zeta'\overline{f}f' + 2|\zeta'|^2|f|^2g'\overline{g}\zeta'\}. \tag{2.26}$$

Proof. The Corollary follows from (2.1) and the well known formula

$$\alpha'\wedge\alpha'' = -|\alpha'|^2k_nN\wedge\alpha' + |\alpha'|^3k_gN, \tag{2.27}$$

for the cross product of α' and α''. □

Lemma 2.3. *Let $X : D \to S$ be an isothermic parametrization of a ruled minimal surface S and let $\{f, g\}$ be the corresponding Weierstrass-Enneper pair. Then at each point of D where g is holomorphic and $g' \neq 0$, it holds that*

$$(1+|g|^2)\left(\frac{f'}{f} - \frac{g''}{g'}\right) + 4g'\overline{g} = \pm i\frac{fg'}{|fg'|}\left[(1+|g|^2)\left(\frac{\overline{f'}}{\overline{f}} - \frac{\overline{g''}}{\overline{g'}}\right) + 4\overline{g'}g\right]. \tag{2.28}$$

Proof. Let $z_o \in D$ be a point where g is holomorphic and $g' \neq 0$. We have $f(z_o)g'(z_o) \neq 0$ and thus $fg' \neq 0$ in a neighborhood of z_o. Let $\zeta(t)$ be a regular curve in D with $\zeta(0) = z_o$ such that $\alpha(t) = X(\zeta(t))$ is a parametrization of a straight line contained in S. Such a curve exists because the surface is ruled. Recalling that the curvature of α em $\mathbb{R}^3$ is given by

$$k(t) = \frac{|\alpha'\times\alpha''|}{|\alpha'|^3},$$

we obtain from (2.1) that

$$\mathrm{Re}\{fg'\zeta'^2\} = 0, \tag{2.29}$$

and

$$\mathrm{Im}\left\{|f|^2(1+|g|^2)\overline{\zeta'}\zeta'' + |\zeta'|^2(1+|g|^2)\zeta'\overline{f}f' + 2|\zeta'|^2|f|^2g'\overline{g}\zeta'\right\} = 0. \tag{2.30}$$

From (2.29) we get $fg'\zeta'^2 = ib(t), b(t) \in \mathbb{R} - \{0\}$. Reparametrizing the curve, if necessary, we may assume

$$\zeta'^2 = \pm i\overline{fg'}. \tag{2.31}$$

Differentiating the above equation with respect to t, we obtain

$$2\zeta'\zeta'' = \pm i\frac{d}{dt}[\overline{fg'}] = \pm i[(\overline{fg'})_z\zeta' + (\overline{fg'})_{\overline{z}}\overline{\zeta'}] = \pm i\overline{\zeta'}(\overline{f'g'} + \overline{fg''}). \tag{2.32}$$

Using (2.31) and (2.32) to compute each term on the right hand side of (2.30), we obtain, after some work,

$$\mathrm{Im}\left\{\zeta'\left[(1+|g|^2)\left(\frac{f'}{f} - \frac{g''}{g'}\right) + 4g'\overline{g}\right]\right\} = 0. \tag{2.33}$$

Using again (2.31), we conclude that (2.28) holds along $\zeta(t)$. □

Using Lemma 2.3, we will derive in the next lemma a second order holomorphic differential equation that, roughly speaking, must be satisfied by a ruled minimal surface in a neighborhood of each point where the Gaussian curvature does not vanish. This differential equation will play an important role in our proof of Catalan's Theorem.

Lemma 2.4. *Let (f, g) be the Weierstrass-Enneper pair associated with a conformal parametrization $X : D \to S$ of a ruled minimal surface S. Assume that $g(z_o) = 0$ and $g'(z_o) \neq 0$, for some $z_o \in D$. Then, considering $\tau = g(z)$ as a new local parameter and defining $F(\tau)$ by (1.9), we have*

$$\left(\frac{F_\tau}{F}\right)^2 - 4\left(\frac{F_\tau}{F}\right)_\tau = cF, \tag{2.34}$$

where

$$c = -\frac{(F_\tau(0))^2}{F(0)^3} + \frac{16i}{|F(0)|}. \tag{2.35}$$

Proof. Since

$$\frac{f'}{f} - \frac{g''}{g'} = \frac{F'}{F},$$

the previous lemma implies

$$(1+|g|^2)\frac{F'}{F} + 4g'\overline{g} = \pm i\frac{fg'}{|fg'|}\left[(1+|g|^2)\frac{\overline{F'}}{\overline{F}} + 4\overline{g'}g\right],$$

where ' stands for differentiation with respect to z. By the above equation and the chain rule, we obtain

$$(1+|\tau|^2)\frac{F_\tau}{F} + 4\overline{\tau} = \pm i\frac{f\overline{g'}}{|fg'|}\left[(1+|\tau|^2)\frac{\overline{F_\tau}}{\overline{F}} + 4\tau\right]$$

$$= \pm i \frac{f|g'|}{|f|g'} \left[(1+|\tau|^2) \frac{\overline{F_\tau}}{\overline{F}} + 4\tau \right]$$

$$= \pm i \frac{F}{|F|} \left[(1+|\tau|^2) \frac{\overline{F_\tau}}{\overline{F}} + 4\tau \right]. \tag{2.36}$$

The function

$$G(\tau) = (1+|\tau|^2) \frac{F_\tau}{F} + 4\overline{\tau} \tag{2.37}$$

cannot vanish on an open set since, otherwise, we woud have

$$\frac{F_\tau}{F} = -\frac{4\overline{\tau}}{1+|\tau|^2},$$

and so $-\frac{4\overline{\tau}}{1+|\tau|^2}$ would be holomorphic in this open set, a contradiction. Since each side of (2.34) is a holomorphic function, it suffices to establish (2.34) on an open subset U where $G \neq 0$. From (2.36) we obtain

$$\pm iF = \frac{G|F|}{\overline{G}}. \tag{2.38}$$

Taking the logarithm on both sides of (2.38) we have

$$\log(\pm iF) = \log G + \log|F| - \log\overline{G}.$$

Differentiating the last expression with respect to $\overline{\tau}$ gives

$$\frac{G_{\overline{\tau}}}{G} - \overline{\left(\frac{G_\tau}{G}\right)} + \frac{1}{2}\overline{\left(\frac{F_\tau}{F}\right)} = 0.$$

Taking the conjugate, it follows that

$$\frac{1}{2}\frac{F_\tau}{F} = \frac{G_\tau}{G} - \overline{\left(\frac{G_{\overline{\tau}}}{G}\right)}. \tag{2.39}$$

Differentiating again with respect to $\overline{\tau}$ yields

$$\frac{GG_{\tau\overline{\tau}} - G_\tau G_{\overline{\tau}}}{G^2} - \overline{\left(\frac{GG_{\tau\overline{\tau}} - G_\tau G_{\overline{\tau}}}{G^2}\right)} = 0,$$

which implies that

$$h := \frac{GG_{\tau\overline{\tau}} - G_\tau G_{\overline{\tau}}}{G^2}, \tag{2.40}$$

is a real function. From (2.37) one easily obtains

$$G_{\overline{\tau}} = \tau \frac{F_\tau}{F} + 4\,, \qquad G_\tau = \overline{\tau}\frac{F_\tau}{F} + (1+|\tau|^2)\left(\frac{F_\tau}{F}\right)_\tau, \tag{2.41}$$

$$G_{\tau\overline{\tau}} = \frac{F_\tau}{F} + \tau\left(\frac{F_\tau}{F}\right)_\tau, \tag{2.42}$$

and thus

$$GG_{\tau\overline{\tau}} - G_\tau G_{\overline{\tau}} = \left(\frac{F_\tau}{F}\right)^2 - 4\left(\frac{F_\tau}{F}\right)_\tau. \tag{2.43}$$

From (2.38), (2.40) and (2.43) we obtain

$$\frac{\left(\frac{F_\tau}{F}\right)^2 - 4\left(\frac{F_\tau}{F}\right)_\tau}{F} = \pm i\frac{|G|^2}{|F|}h. \tag{2.44}$$

Since the left hand side of the above equation is holomorphic and the right hand side assumes only purely imaginary values, we conclude that each side of (2.44) is a purely imaginary constant, thus proving (2.34). It remains to identify the constant c.

To see that c in (2.34) is given by (2.35), set $\alpha = \frac{F_\tau}{F}$ and $\beta = \left(\frac{F_\tau}{F}\right)_\tau$. At every point where $G \neq 0$, we have by (2.37), (2.39) and (2.41),

$$\frac{\alpha}{2} = \frac{\overline{\tau}\alpha + (1+|\tau|^2)\beta}{(1+|\tau|^2)\alpha + 4\overline{\tau}} - \frac{\overline{\tau\alpha} + 4}{(1+|\tau|^2)\overline{\alpha} + 4\tau},$$

from which we have

$$\beta = \frac{\frac{\alpha}{2}[(1+|\tau|^2)\alpha + 4\overline{\tau}] - i(\overline{\tau}\alpha + 4)\frac{F}{|F|} - \overline{\tau}\alpha}{1+|\tau|^2}.$$

Since 0 is an accumulation point of the set where $G \neq 0$, we can take the limit in the above equality as $\tau \to 0$ to obtain

$$\left(\frac{F_\tau}{F}\right)_\tau(0) = \beta(0) = \frac{1}{2}\left(\frac{F_\tau}{F}(0)\right)^2 - 4i\frac{F(0)}{|F(0)|}. \tag{2.45}$$

Substituting (2.45) in (2.34) we arrive at (2.35). □

3. A Family of Helicoids

In this section, we will describe a specific family of helicoids that will furnish, up to translations, homotheties and orthogonal transformations, all examples of nonflat ruled minimal surfaces in $\mathbb{R}^3$. In order to obtain this family, consider, for each $0 < b \leq 1$, the helicoid S_b parametrized by

$$X_b(u,v) = (b\,\mathrm{sinh}v\,\mathrm{cos}u, bu, b\,\mathrm{sinh}v\,\mathrm{sin}u). \tag{3.1}$$

The unit normal vector field and the Gaussian curvature of S_b at $X_b(u,v)$ are given respectively by

$$N(u,v) = \left(\frac{\mathrm{sin}u}{\mathrm{cosh}v}, \mathrm{tanh}v, -\frac{\mathrm{cos}u}{\mathrm{cosh}v}\right), \tag{3.2}$$

$$K(u,v) = -\frac{1}{b^2\mathrm{cosh}^4 v}. \tag{3.3}$$

For $0 < b \leq 1$, let $v_b \in [0, \infty)$ be the real number such that

$$\cosh v_b = \frac{1}{\sqrt{b}}. \tag{3.4}$$

It follows from (3.3) that $K(0, v_b) = -1$. Let θ_b, $0 \leq \theta_b \leq \pi$, be the angle between the vectors $(0, 0, -1)$ and $N(0, v_b)$, so that

$$\cos\theta_b = \frac{1}{\cosh v_b}. \tag{3.5}$$

For each $0 < b \leq 1$, consider the surface $\tilde{S}_b$ obtained by rotating the helicoid S_b by the angle $-\theta_b$ around the x-axis. The surface $\tilde{S}_b$, which was chosen so that its unit normal vector and its Gaussian curvature at $\tilde{X}_b(0, v_b)$ are, respectively, $(0, 0, -1)$ and -1, can be parametrized by

$$\tilde{X}_b(u, v) = (b \sinh v \cos u, bu\cos\theta_b + b\sin\theta_b \sinh v \sin u, -bu\sin\theta_b + b\cos\theta_b \sinh v \sin u). \tag{3.6}$$

It is exactly the family $\{\tilde{S}_b, 0 < b \leq 1\}$ that we referred to in the beginning of this section. Setting

$$\Phi_b(z) = ((\Phi_b)_1(z), (\Phi_b)_2(z), (\Phi_b)_3(z)) = \frac{\partial \tilde{X}_b}{\partial u} - i\frac{\partial \tilde{X}_b}{\partial v}, \tag{3.7}$$

we have

$$(\Phi_b)_1(z) = -ib\cosh(iz), \tag{3.8}$$

$$(\Phi_b)_2(z) = b\cos\theta_b - b\sin\theta_b \sinh(iz), \tag{3.9}$$

$$(\Phi_b)_3(z) = -b\sin\theta_b - b\cos\theta_b \sinh(iz). \tag{3.10}$$

It follows from (1.4) that the Weierstrass-Enneper pair $\{f_b, g_b\}$ associated to the parametrization $\tilde{X}_b$ of $\tilde{S}_b$ is given by

$$f_b(z) = (\Phi_b)_1(z) - i(\Phi_b)_2(z) = -ib[\cosh(iz) + \cos\theta_b - \sin\theta_b \sinh(iz)], \tag{3.11}$$

$$g_b(z) = \frac{(\Phi_b)_3(z)}{f_b(z)} = \frac{-b\sin\theta_b - b\cos\theta_b \sinh(iz)}{f_b(z)}. \tag{3.12}$$

At the point $z_b = iv_b$ we have

$$g_b(z_b) = 0. \tag{3.13}$$

Moreover, $f_b(z_b) = -2i\sqrt{b}$ and $g_b'(z_b) = \frac{\sqrt{b}}{2}$, which implies

$$F_b(z_b) = -4i, \tag{3.14}$$

where $F_b = f_b/g_b'$. After some computations, we obtain

$$F_b'(z_b) = -8\sqrt{1-b}. \tag{3.15}$$

Since $g_b'(z_b) \neq 0$, we can, as before, parametrize a neighborhood of $\tilde{X}_b(z_b)$ using $\tau = g_b(z)$ as the parameter. In so doing, the initial conditions (3.13), (3.14) and (3.15) become

$$F_b(0) = -4i\,, \quad (F_b)_\tau(0) = F_b'(z_b)\frac{dz}{d\tau}(0) = \frac{F_b'(z_b)}{g_b'(z_b)} = -16\sqrt{\frac{1-b}{b}}. \tag{3.16}$$

4. The Proof

The idea of the proof is to compare a nonflat ruled minimal surface S with a suitable member $\tilde{S}_b$ of the family considered in Section 3. Since, roughly speaking, we already know that both surfaces satisfy the holomorphic differential equation (2.34), all we have to do is to ensure that the two surfaces share the same initial conditions. This is achieved by performing certain orthogonal transformations in the surface S. In order to do this, we have to know the effect of these transformations on the Weierstrass-Enneper representation of the surface.

Lemma 4.1. *Let (f, g) be a Weierstrass-Enneper representation of a minimal surface S.* **(i)** *The functions f and g remain unchanged under translations in $\mathcal{R}^3$.*

(ii) *The function f remains the same and g changes sign under the reflexion in the plane $\{z = 0\}$.*

(iii) *Assume that $g(z_0) = 0$ and $g'(z_o) \neq 0$, for some z_o. Consider $\tau = g(z)$ as a local parameter and define F by (1.9). Then both functions f and g, as well as the initial condition $F_\tau(0)$, change sign under the reflexion in the z-axis.*

Proof. The statements (i) and (ii) and the first assertion of (iii) follow easily from (1.1) and (1.4). To prove the second assertion of (iii), observe first that F remains the same, by the first assertion. Since, by the chain rule,

$$F_\tau(0) = F'(z_o)\frac{dz}{d\tau}(0) = \frac{F'(z_o)}{g'(z_o)},$$

and $g'(z_o)$ changes sign while $F'(z_o)$ remains the same, the conclusion follows. □

Proof of Catalan's Theorem. Let S be a ruled minimal surface. If the Gaussian curvature of S vanishes everywhere, the surface is totally geodesic and so is part of a plane. If not, take a point $p \in S$ where the Gaussian curvature is not zero. Since homotheties in $\mathbb{R}^3$ takes helicoids into helicoids, we can suppose that the Gaussian curvature of S at p is $K(p) = -1$. Applying a translation and orthogonal transformations, if necessary, we can suppose further that $p = 0$, the unit normal vector of S at p is $N(p) = (0, 0, -1)$ and that a portion of the x-axis containing p lies in the surface. Consider a conformal parametrization $X : D \to S$ of a neighborhood of p such that $X(0) = p$, and let (f, g) be the Weierstrass-Enneper pair associated to X. From $N(p) = (0, 0, -1)$ and (1.7) we have

$$g(0) = 0. \tag{4.1}$$

From (2.25) it is easy to see that the Gaussian curvature is given by

$$K = -\left[\frac{4|g'|}{|f|(1 + |g|^2)^2}\right]^2. \tag{4.2}$$

Since $K(p) = -1$, it follows that $g'(0) \neq 0$. Restricting D, if necessary, we have that g has an inverse function g^{-1} in D which is holomorphic as well. Defining, as before, $F = f/g'$, we have, by (4.1), (4.2) and the fact that $K(p) = -1$,

$$|F(0)| = \left|\frac{f}{g'}(0)\right| = 4. \tag{4.3}$$

We claim that

$$F(0) = \pm 4i. \tag{4.4}$$

To prove the claim, let $\alpha(t) = X(\zeta(t))$ be a regular parametrization of the x-axis. As in the proof of Lemma 2.3, we can reparametrize ζ so that $\zeta'^2 = \pm i\overline{fg'}$. Suppose first $\zeta'^2 = i\overline{fg'}$. Since $g(0) = 0$ and $\alpha'(0) = (\lambda, 0, 0)$, for some real number λ, we have from (1.5) and (2.2),

$$(\lambda, 0, 0) = \zeta'(0)\left(\frac{f(0)}{4}, i\frac{f(0)}{4}, 0\right) + \overline{\zeta'(0)}\left(\frac{\overline{f(0)}}{4}, -i\frac{\overline{f(0)}}{4}, 0\right).$$

By the equality in the second coordinates we obtain $\mathrm{Re}\{i\zeta'(0)f(0)\} = 0$, that is, $\mathrm{Im}\{\zeta'(0)f(0)\} = 0$. Then there is a real number μ such that $\zeta'(0)f(0) = \mu$. Since $(\zeta'(0))^2 = i\overline{f(0)g'(0)}$, it follows that $|\zeta'(0)|^2\zeta'(0) = i\overline{\zeta'(0)f(0)g'(0)} = i\mu\overline{g'(0)}$ and consequentely

$$\zeta'(0) = \frac{i\mu\overline{g'(0)}}{|\zeta'(0)|^2} = \frac{i\mu\overline{g'(0)}}{|f(0)g'(0)|}.$$

Multiplying both sides of the above equation by $f(0)$ we get

$$\mu = i\mu\frac{f(0)}{|f(0)|}\frac{\overline{g'(0)}}{|g'(0)|} = i\mu\frac{f(0)}{|f(0)|}\frac{|g'(0)|}{g'(0)}$$

which implies

$$F(0) = \frac{f}{g'}(0) = -i\left|\frac{f}{g'}(0)\right| = -i|F(0)|.$$

Using (4.3), we finally conclude that $F(0) = -4i$. In the case $(\zeta'(0))^2 = -i\overline{f(0)g'(0)}$, a completely analogous argument shows that $F(0) = +4i$, and the claim is proved. By Lemma 4.1 (ii) and (4.4), we can suppose, performing the reflexion in the plane $\{z = 0\}$ if necessary,

$$F(0) = -4i. \tag{4.5}$$

Again by (4.4) and the proof of Lemma 2.3, equation (2.36) holds with the minus sign. Using (4.5), this implies that $F_\tau(0)$ is a real number. Using Lemma 4.1 (iii), we can suppose further, performing the reflexion in the z-axis if necessary, that $F_\tau(0) \leq 0$. Choosing $0 < b \leq 1$ so that

$$F_\tau(0) = -16\sqrt{\frac{1-b}{b}}, \tag{4.6}$$

we have by (3.16), (4.5) and (4.6) that F and F_b are both solutions of the holomorphic differential equation (2.34) satisfying the same inicial conditions. It follows that F and F_b coincide in a domain where both are defined. From (1.10) one sees that a neighborhood of p is contained in $\tilde{S}_b$, where $\tilde{S}_b$ is a member of the family of helicoids considered in Section 3. Since $\tilde{S}_b$ and S are both minimal, by the unique continuation principle we conclude that S is parte of a helicoid. This concludes the proof of Catalan's theorem. □

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