SCHRÖDINGER OPERATORS AND TOPOLOGICAL PRESSURE ON MANIFOLDS OF NEGATIVE CURVATURE

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Abstract. The bilaplacians of the Busemann functions on the universal cover of a compact manifold M of negative curvature induce a function $\mathcal{B}^{(2)}$ on the unit tangent bundle of M. It is conjectured that $\mathcal{B}^{(2)}$ is cohomologous to a non-positive continuous function if and only if M is locally symmetric. It is shown here that $\mathcal{B}^{(2)}$ is not cohomologous to a negative continuous function. In particular, its topological pressure is non-negative. The proof is based on the construction of certain Schrödinger operators.

§1. Introduction

Let N be a complete simply-connected manifold of negative curvature, Γ a co-compact lattice of the group of isometries of N and $\pi : UN \to N$ the natural projection of the unit tangent bundle of N. To every $v \in UN$ one can associate the Busemann function $B_v : N \to \mathbb{R}$ given by $B_v(x) = \lim_{t\to\infty}(d(\gamma_v(t), x) - t)$, where γ_v is the geodesic of N satisfying $\gamma_v(0) = \pi(v), \gamma'(0) = v$. If g is an isometry of N one has $B_{gv} \circ g = B_v$. In particular, $\Delta^j B_v = \Delta^j B_{gv} \circ g, j \geq 1$, where Δ is the Laplace-Beltrami operator of N. Hence $UN \ni v \to (\Delta^j B_v)(\pi v)$ is Γ-invariant and as such it descends to a (Hölder) function $\mathcal{B}^{(j)}$ on the unit tangent bundle UM of $M := N/\Gamma$. The function $\mathcal{B}^{(1)}$, which assigns to v the mean curvature at πv of the v-horosphere, is especially important because its integral with respect to the Liouville measure gives the entropy (the Pesin formula-see [Y] and [KH]). By contrast, no relation between $\mathcal{B}^{(j)}$ and the geodesic flow of M seems to have been known in the case $j \geq 2$. The object of this note is to prove the following result about the bilaplacian cocycle $\mathcal{B}^{(2)}$. First recall that $\phi_1, \phi_2 : UM \to \mathbb{R}$ are called cohomologous if there exists $\phi_3 : UM \to \mathbb{R}$ such that $(\phi_2 - \phi_1)(x) = \frac{d}{dt}(\phi_3(g_t(x)))|_{t=0}$; here g_t stands for the geodesic flow.

Theorem. *$\mathcal{B}^{(2)}$ is not cohomologous to a negative continuous function. In particular, the topological pressure of $\mathcal{B}^{(2)}$ is non-negative.*

A current theme in differential geometry and dynamical systems is the question of finding dynamical properties associated to geometrically defined objects that might allow for a caracterization of locally symmetric spaces of negative curvature. This is the subject of extensive research. A major rigidity result of G. Besson, G. Courtois and S. Gallot [B-C-G] states that locally symmetric spaces minimize entropy when the volume is prescribed

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(for the case $n = 2$ this was proved by A. Katok). Combining this with results of P. Foulon and F. Laburie [F-L], and F. Ledrappier [F], one has that a compact orientable Riemannian manifold whose Bowen-Margulis measure and harmonic measure coincide is locally symmetric. Let us mention also the entropy conjecture (due to Katok and proved in [K] for surfaces) that a compact orientable Riemannian manifold whose Bowen-Margulis measure and Liouville measure coincide must be locally symmetric.

In light of the remarks above and motivated by our theorem we propose the following characterization of locally symmetric spaces of negative curvature.

Conjecture. *A compact manifold M of negative curvature is locally symmetric if and only if its bilaplacian cocycle is cohomologous to a non-positive function.*

The "only if" part of the conjecture follows from the fact that symmetric spaces with negative curvature are asymptotically harmonic, i.e., their horospheres have constant mean curvature. From the results of [FL] and [BCG] it is known that asymptotically harmonic spaces with compact quotients are necessarily symmetric. (The conclusion may fail if the manifold in question does not admit a compact quotient-see [DR]).

There is evidence to suggest that the techniques of the present paper can be pushed to prove that if $\mathcal{B}^{(2)}$ is cohomologous to a non-positive function, then $\mathcal{B}^{(2)}$ must in fact be a trivial cocycle. Suppose that one can go further and prove that the Busemann functions are actually biharmonic. It appears that a simple argument involving ergodicity and Fatou's theorem on non-tangential limits of bounded harmonic functions (see [M], for instance) would show then that $\widetilde{M}$ is asymptotically harmonic. As observed above, this would imply that M is locally symmetric.

The statement in the theorem about the topological pressure can be established as follows. It follows directly from the definition that if the Hölder continuous function $\psi : UM \to \mathbb{R}$ has negative pressure then, for sufficiently large n, the function $\psi_n(x) := \frac{1}{n}\int_0^n \psi(g_t(x))dt$ has a negative upper bound (see [KH], page 623). On the other hand, by the Livsic theorem (see [L]), the functions ψ and ψ_n are cohomologous. Putting these two facts together one sees that if the pressure of the the function $\psi = \mathcal{B}^{(2)}$ is negative, the function must be cohomologous to a negative cocycle.

The proof of the above theorem is based on the examination of the spectral properties of certain Schrodinger operators defined on the universal cover of M. If the conclusion of the theorem were false one could construct a Schrodinger operator with empty spectrum, an impossibility for self-adjoint operators.

§2. Commutators of Hilbert space operators

An important task in the spectral theory of self-adjoint operators is to formulate conditions to ensure that the spectral measures of a given operator H are absolutely continuous with respect to the Lebesgue measure. One such technique is Putnam's theorem. It involves producing an auxiliary operator A for which the commutator $i[H, A]$ is positive in the sense of quadratic forms. As pointed out in [CFKS], page 60, a byproduct of the proof of Putnam's theorem is the fact that the estimate

$$\langle i[H, A]x, x\rangle = 2\mathrm{Im}\langle Hx, Ax\rangle \geq c|x|^2, \ c > 0,$$

is impossible for a pair of *bounded* self-adjoint operators H and A. Below we formulate an extension of this observation which is valid for unbounded self-adjoint operators.

Lemma 1. *Let H be a self-adjoint operator on a Hilbert space $\mathcal{H} \neq 0$, and let $\mathcal{C} \subset D(H)$ be a core of H, i.e., $\mathcal{C}$ is dense in $D(H)$ in the graph norm $|x|_H^2 = |Hx|^2 + |x|^2$ of H. Let A be a symmetric operator with domain $\mathcal{C}$. It is impossible for H and A to satisfy simultaneously the two properties below:*

1. $|Im\langle Hx, Ax\rangle| \geq c|x|^2,\ c > 0,\ x \in \mathcal{C}$;
2. $|Ax| \leq C(|\langle Hx, x\rangle|^{\frac{1}{2}} + |x|),\ x \in \mathcal{C}$.

Proof. The proof consists in showing that if H and A satisfy (i) and (ii) the spectrum of H has to be empty. This is impossible since the Hilbert space has non-zero dimension. We observe that the spectrum of H is the union of the set of eigenvalues of finite multiplicity and the essential spectrum. A point $\lambda \in \mathbb{R}$ belongs to the essential spectrum if there is a sequence of unit vectors $x_n \in D(H)$ which converge weakly to 0 and satisfy $|(H - \lambda I)x_n| \to 0$. Since $\mathcal{C}$ is dense in $D(H)$ in the graph norm, we may actually take x_n to belong to $\mathcal{C}$, $|x_n| \geq \delta > 0$. If, on the other hand, λ is an eigenvalue (of finite or infinite multiplicity), $Hx = \lambda x$, $|x| = 1$, we set $x_n = x$. In either case we may compute, using (i),(ii) and $\mathrm{Im}\langle (H-\lambda)x, Ax\rangle = \mathrm{Im}\langle Hx, Ax\rangle$:

$$c\delta^2 \leq c|x_n|^2 \leq |(H-\lambda I)x_n||Ax_n| \leq C|(H-\lambda I)x_n|\Big(|x_n| + \big(|\langle (H-\lambda I)x_n, x_n\rangle| + |\lambda||x_n|^2\big)^{\frac{1}{2}}\Big).$$

Clearly the quantity on the right side goes to zero as $n \to \infty$. This contradiction establishes the lemma. $\square$

§3. An integral identity involving the bilaplacian

The object of this section is to establish the integral identity (1) below. The case $q = 0$ was the starting point in [X] of an investigation of the absolute continuity of the spectral measure of the Laplacian on complete manifolds. In the present sign convention, Δu is the trace of the Hessian of u.

Lemma 2. *Let M be a riemannian manifold and f, u smooth real-valued functions on M, at least one of which has compact support. Let also q be smooth on M. Then*

$$\int(-\Delta u + qu)\left(\nabla f \cdot \nabla u + \frac{\Delta f}{2}u\right) = \int Hessf(\nabla u, \nabla u) - \frac{1}{4}\int(\Delta^2 f + 2\nabla f \cdot \nabla q)u^2. \quad (1)$$

Proof. We treat the case $q = 0$ first. Consider the following identity ([KW], page 32).

$$2\Delta u(\nabla f \cdot \nabla u) = \mathrm{div}(2(\nabla f \cdot \nabla u)\nabla u - |\nabla u|^2 \nabla f) - 2\mathrm{Hess}f(\nabla u, \nabla u) + (\Delta f)|\nabla u|^2.$$

After integration we have

$$2\int(\Delta u)(\nabla f \cdot \nabla u) = -2\int \mathrm{Hess}f(\nabla u, \nabla u) + \int(\Delta f)|\nabla u|^2.$$

Using

$$\int (\Delta f)|\nabla u|^2 = \frac{1}{2}\int (\Delta f)\Delta(u^2) - \int u(\Delta u)\Delta f$$

we obtain

$$\int (\Delta u)(\nabla f \cdot \nabla u) = -\int \mathrm{Hess} f(\nabla u, \nabla u) + \frac{1}{4}\int (\Delta f)\Delta(u^2) - \frac{1}{2}\int u\Delta u \Delta f$$

so that

$$\int (\Delta u)(\nabla f \cdot \nabla u + \frac{1}{2}u\Delta f) = -\int \mathrm{Hess} f(\nabla u, \nabla u) + \frac{1}{4}\int (\Delta f)\Delta(u^2).$$

The result in the case $q = 0$ then follows by transforming the second integral on the right hand side.

To deal with the case $q \neq 0$, we start by computing the (formal) L^2 adjoint X^* of a vector field X on M. Since $\int \mathrm{div} Y = 0$ whenever Y has compact support, we have with $u, v \in C_0^\infty(M)$:

$$0 = \int \mathrm{div}(uvX) = \int u(\mathrm{div}X)v + \int uXv + \int vXu,$$

so that $X^* = -X - \mathrm{div}X$. In particular, $\langle Xu, u\rangle + \frac{1}{2}\langle (\mathrm{div}X)u, u\rangle = 0$. Taking $v = qu$, one sees that $\langle qXu, u\rangle + \langle uXq, u\rangle = \langle X(uq), u\rangle = \langle uq, X^*u\rangle$. Using the formula for X^* one then has

$$\langle Xu + \frac{1}{2}(\mathrm{div}X)u, qu\rangle = -\frac{1}{2}\langle (Xq)u, u\rangle$$

and (1) follows by taking $X = \nabla f$. $\square$

§4. Proof of the Theorem

We argue by contradiction and write $\mathcal{B}^{(2)} = W_1 + W_2$, with $W_1 < 0$ and

$$W_2(v) = \frac{d}{dt}W_3(g_t(v))|_{t=0}, \ v \in UM.$$

Fix $z \in N(\infty)$ and let $\alpha : UN \to UM$ be the natural map. Next, we define $q : N \to \mathbb{R}$ by

$$q(x) = -\frac{1}{2}W_3(\alpha(\nabla B_z(x))).$$

Note that q is bounded. Since $-\Delta$ is essentially self-adjoint on $C_0^\infty(N, \mathbb{C})$ (recall that N is complete), the same is true of $-\Delta + q$ (see [Ka], page 288). Taking $f = B_z$, the quantity $\Delta^2 f + 2\nabla f \nabla q$ which figures in Lemma 2 becomes

$$\begin{aligned}
&(\Delta^2 B_z + 2\nabla B_z \cdot \nabla q)(x) \\
&= \mathcal{B}^{(2)}(\alpha(\nabla B_z(x))) - \nabla B_z(x) \cdot \nabla(W_3(\alpha(\nabla B_z(x)))) \\
&= \mathcal{B}^{(2)}(\alpha(\nabla B_z(x))) - \frac{d}{dt}(W_3(g_t(\alpha(\nabla B_z(x)))))|_{t=0} \\
&= \mathcal{B}^{(2)}(\alpha(\nabla B_z(x))) - W_2(\alpha(\nabla B_z(x))) \\
&= W_1(\alpha(\nabla B_z(x))) < 0.
\end{aligned}$$

Since W_1 is continuous and B_z is convex, Lemma 2 implies for every $u \in C_0^\infty(N,\mathbb{R})$, and some $\delta > 0$:

$$\int(-\Delta u + qu)(\nabla f \cdot \nabla u + \frac{\Delta f}{2}u) \geq \delta \int u^2. \tag{2}$$

Next, let H be the unique self-adjoint extension of $(-\Delta + q)|_{C_0^\infty(N,\mathbb{C})}$. Let also A be defined on $C_0^\infty(N,\mathbb{C})$ by $Au = i(\nabla f \cdot \nabla u + \frac{\Delta f}{2}u)$. It is easy to see that A so defined is symmetric. Indeed, the skew-symmetry of iA amounts to the identity

$$\int(\nabla f \nabla u + \frac{\Delta f}{2}u)u = 0, \ u \in C_0^\infty(N,\mathbb{R}),$$

which was essentially established at the end of the proof of Lemma 1. Applying (2) to u_1, u_2 and adding we have

$$\mathrm{Im}\langle Hu, Au\rangle \geq \delta|u|^2, \ u = u_1 + iu_2 \in C_0^\infty(N,\mathbb{C}).$$

In other words, the pair H, A obeys (i) of Lemma 1. To reach a contradiction we now show that (ii) of Lemma 1 is also satisfied. The key here is the fact that q is bounded from below. Since the curvature is bounded, so is ΔB_z. Hence, for $u \in C_0^\infty(N,\mathbb{C})$,

$$\begin{aligned}|Au|^2 = \int|\nabla f \cdot \nabla u + \frac{\Delta f}{2}u|^2 &\leq C\int(|\nabla u|^2 + |u|^2)\\ &\leq C\int(-\Delta u + qu)\bar{u} + C\int(1-q)|u|^2\\ &\leq C'(|\langle Hu, u\rangle| + |u|^2).\end{aligned}$$

Taking square roots one obtains (ii) of lemma 1. Hence the pair H, A violates the conclusion of Lemma 1, and this represents a contradiction to the assumption that $\mathcal{B}^{(2)}$ is cohomologous to a negative function. $\square$

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