HOLOMORPHIC INJECTIVITY AND THE HOPF MAP

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Warmly dedicated to Joana

Abstract. We give sharp conditions on a local biholomorphism $F : X \to \mathbb{C}^n$ which ensure global injectivity. For $n \geq 2$, such a map is injective if for each complex line $l \subset \mathbb{C}^n$, the pre-image $F^{-1}(l)$ embeds holomorphically as a connected domain into $\mathbb{CP}^1$, the embedding being unique up to Möbius transformation. In particular, F is injective if the pre-image of every complex line is connected and conformal to $\mathbb{C}$. The proof uses the topological fact that the natural map $\mathbb{RP}^{2n-1} \to \mathbb{CP}^{n-1}$ associated to the Hopf map admits no continuous sections and the classical Bieberbach-Gronwall estimates from complex analysis.

1. Introduction

The study of univalence (injectivity) of holomorphic functions of one variable is a classical topic in complex analysis. One of the high points in the subject was the celebrated solution of the Bieberbach Conjecture by de Branges in 1984 [2, 5, 7]. As part of the effort to understand the conjecture, several authors introduced various univalence criteria for locally univalent holomorphic functions defined on the open unit disc in $\mathbb{C}$ [17]. In contrast, fewer injectivity criteria are known for local biholomorphisms in higher dimensions [5, chapter V]. In the simplest case when $F : \mathbb{C}^n \to \mathbb{C}^n$ is a *polynomial* local biholomorphism, it is not even known whether F is a priori injective (hence bijective [1, I, Theorem. 2.1]) without further hypotheses: this is the jacobian conjecture (see [1, 4] for general references), which remains open after more than 60 years.

In this paper we draw a connection between global injectivity of a local biholomorphism F and connectedness of certain pre-images of F. The naïve observation that F is injective if and only if the pre-image of each point is connected leads one naturally to ask whether there might be a similar criterion based on connectedness of pre-images of *positive* dimensional submanifolds. For example, it's easy to see that a local diffeomorphism $F : X \to \mathbb{R}^n$ is injective if for each real line l, the pre-image $F^{-1}(l)$ is connected [15, Example 3.3]. For a local biholomorphism $F : X \to \mathbb{C}^n$ and *complex* lines $l \subset \mathbb{C}^n$, it turns out that connectedness of the pre-images is not enough (Example 2.5), one also needs information about the conformal type of the pre-images $F^{-1}(l)$ (Corollaries1.2 and 1.3).

To state our injectivity criterion, we need the concept of a *rigid domain* of $\mathbb{CP}^1$: for us, these are the connected open sets $i : U \hookrightarrow \mathbb{CP}^1$ such that any holomorphic embedding $f : U \hookrightarrow \mathbb{CP}^1$ differs from i by an automorphism M of $\mathbb{CP}^1$, that is $f = M \circ i$. A rigid domain $U \subset \mathbb{CP}^1$ is necessarily dense (otherwise apply an automorphism of $\mathbb{CP}^1$ that takes U into the unit disc $D \subset \mathbb{C}$. By the Riemann mapping theorem, there are many holomorphic

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embeddings $F : D \hookrightarrow \mathbb{C} \subset \mathbb{CP}^1$ which are not the restriction of a Möbius transformation. By unicity of continuation, the restriction $F|_U$ is not either). Typical examples include $\mathbb{C} \subset \mathbb{CP}^1$, the complement of $\mathbb{C}$ by finitely many points (a connected rational curve, in the sense of algebraic geometry) or, more generally, the complement in the Riemann sphere of a removable closed set (a closed subset $E \subset \mathbb{CP}^1$ is *removable* if for each open set $U \subset \mathbb{CP}^1$, the bounded holomorphic functions $f : U - E \to \mathbb{C}$ extend holomorphically to U).

Theorem 1.1. *Let X be a connected complex manifold of dimension $n \geq 2$, $F : X \to \mathbb{C}^n$ a local biholomorphism. Fix $q \in F(X)$ and suppose that $F^{-1}(l)$ is conformal to a rigid domain $D_l \subset \mathbb{CP}^1$ for every complex line l passing through q. Then q is assumed exactly once by F.*

The statement is sharp in all respects, starting with the obvious condition $n \geq 2$. Its conclusion may fail if even a pre-image of a single line is disconnected (Example 2.4) or if the pre-images are conformal to punctured compact Riemann surfaces of *positive* genera (Example 2.5). The rigidity condition also cannot be removed (Example 2.7). The proof is a blend of analytic and topological ideas: if $F^{-1}(q)$ contains *two* points, we can construct a *continuous* section to the natural map $\pi : \mathbb{RP}^{2n-1} \to \mathbb{CP}^{n-1}$ associated to the Hopf map, an impossibility. The continuity of this section follows from the classical Bieberbach-Gronwall estimates for univalent functions on the unit disc. Globalizing Theorem 1.1 yields:

Corollary 1.2. *Let $F : X \to \mathbb{C}^n$ be a local biholomorphism with $n \geq 2$. If each non-empty pre-image of a complex line is connected and conformal to $\mathbb{C}$, then F is injective.*

Applying Serre's GAGA principle [18] gives an algebro-geometric variant:

Corollary 1.3. *Let $F : X \to \mathbb{A}^n_{\mathbb{C}}$ be an étale morphism of schemes with $n \geq 2$. Then the following are equivalent.*

(1) *F is injective.*
(2) *For each line $l \subset \mathbb{C}^n$ meeting $F(X)$, the pre-image $F^{-1}(l)$ is connected and rational.*

The forward direction is clear because the pre-image of a line l is identified with a Zariski open subset of l; the converse is immediate from Theorem 1.1 above. Example 2.5 shows that rationality is needed in condition (2) above.

Finally, applying generic smoothness gives a projective version:

Corollary 1.4. *Let $F : Z \to \mathbb{P}^n_{\mathbb{C}}$ be a generically finite morphism of schemes with $n \geq 2$ and Z smooth. Then the following are equivalent.*

(1) *F is birational.*
(2) *There is an open set $U \subset \mathbb{P}^n$ such that for each line $l \subset \mathbb{P}^n$, the pre-image $F^{-1}(l \cap U)$ is either empty or irreducible and rational.*

Proof. Since $\mathbb{C}$ has characteristic zero, generic smoothness [8, III, Cor. 10.7] gives an open set $V \subset \mathbb{P}^n$ for which the restricted map $F : F^{-1}(V) \to V$ is smooth. Generic finiteness of F implies that (a) the set V is non-empty (hence dense) and (b) the relative dimension is zero so that $F : F^{-1}(V) \to V$ is étale. Further restriction to a standard open affine subset $\mathbb{A}^n \cong U_i \subset \mathbb{P}^n$ gives an étale morphism $F^{-1}(V \cap \mathbb{A}^n) \to V \cap \mathbb{A}^n \hookrightarrow \mathbb{A}^n$.
$\Rightarrow$: If F is birational, then $\deg F = 1$ and we may take $U = V \cap \mathbb{A}^n$.
$\Leftarrow$: The second condition continues to hold if we replace U with $U \cap V \cap \mathbb{A}^n$. Composing

with the inclusion into $\mathbb{A}^n$, Corollary 1.3 implies that $F|_{F^{-1}(U\cap V\cap \mathbb{A}^n)}$ is injective, hence F is birational. □

In section two we prove Theorem 1.1, modulo a delicate continuity proof. We also give examples which show that the hypotheses cannot easily be removed. Section three is devoted to the continuity proof. We direct the interested reader to [12, 15, 16, 19, 20] and the references therein for further results on global injectivity in the differentiable context.

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2. Main ideas and examples

In this section we prove Theorem 1.1 except for a lengthy technical matter which is delayed to section three. We also give examples to show that the hypotheses cannot be weakened.

Recall that the Hopf map $S^{2n-1} \to \mathbb{CP}^{n-1}$ sends a unit vector $u \in \mathbb{C}^n$ to the complex one-dimensional subspace containing it. Clearly this induces a map $\pi : \mathbb{RP}^{2n-1} \to \mathbb{CP}^{n-1}$ between real and complex projective spaces.

Fact 2.1. *Neither π nor the Hopf map admits a continuous section for $n \geq 2$.*

For instance, the composite map in cohomology

$$H^2(\mathbb{CP}^{n-1}) \to H^2(\mathbb{RP}^{2n-1}) \to H^2(\mathbb{CP}^{n-1})$$

induced by a continuous section of π must be the identity, but this is impossible since $H^2(\mathbb{CP}^{n-1}) \neq 0$ and $H^2(\mathbb{RP}^{2n-1}) = 0$.

This topological fact can sometimes be used to prove injectivity of local biholomorphisms. The following example illustrates the idea.

Example 2.2. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a local bihilomorphism with $n \geq 2$ such that the pre-image of every complex line is connected and simply connected. Then F is injective.

Proof. If F is not injective, we may suppose that $F(p) = F(q) = 0$ with $p \neq q$, hence $F^{-1}(l)$ contains both p and q for all one-dimensional complex subspaces l. It follows from the inverse function theorem that the complex curve $F^{-1}(l)$ is properly embedded in $\mathbb{C}^n$ (whether F is a proper map or not), hence with respect to the induced Riemannian metric it is a complete simply connected real surface of non-positive curvature [6, p. 79].

It follows from Hadamard's theorem [3, Ch. 7, Theorem 3.1] that any two points in $F^{-1}(l)$ can be joined by a *unique* geodesic of $F^{-1}(l)$. Given $l \in \mathbb{P}^{n-1}$, let $w(l)$ denote the initial vector of the (unique) unit-speed geodesic along $F^{-1}(l)$ joining p to q and set $v(l) = dF(0)w(l) \in T_{l,0}$. Notice that all geodesic segments are contained in a fixed compact subset of $\mathbb{C}^n$. The map v is continuous because geodesics converge to geodesics in the C^2

topology (which, after passing to subsequences, is a consequence of uniform C^3 boundedness) and from the uniqueness of the geodesic along $F^{-1}(l)$. Since v is nonvanishing, it is clear that $\frac{v(l)}{|v(l)|}$ defines a continuous section to the Hopf map, a contradiction in view of 2.1. $\square$

Remark 2.3. Notice in the previous example that if the pre-images of complex lines are not simply connected, then the geodesics along them from p to q may not be unique, so the construction above does not work. On the other hand, because the curvature is non-positive, there is a unique geodesic for each homotopy class of paths from p to q, so there is at least a local finite multisection in this case: potentially pieces of these could be glued to construct a global continuous section. We conjecture that for local biholomorphisms $F : \mathbb{C}^n \to \mathbb{C}^n$ that connectedness of the pre-images alone is enough to ensure injectivity of F.

The proof of our main theorem is similar in spirit to Example 2.2, except that we use tangent vectors to rational curves to produce a section to the map π. The construction of the map is given below, the proof of continuity being delayed to section three.

Proof of Theorem 1.1

Taking $q = 0$, suppose that $F^{-1}(0)$ contains at least two distinct points $z_1 \neq z_2$. Fix the two points $w_1 = 0$ and $w_2 = 1$ in $\mathbb{CP}^1$ along with two nonzero real tangent vectors $0 \neq \alpha_i \in T_{\mathbb{CP}^1, w_i}$. Let l be a complex line through 0. We claim that

(i) There are exactly two holomorphic embeddings $T_l = T_l^1, T_l^2 : F^{-1}(l) \to \mathbb{CP}^1$ such that $T_l(z_i) = w_i$ $(i = 1, 2)$ and

$$dF(z_1)(dT_l(z_1))^{-1}\alpha_1 = dF(z_2)(dT_l(z_2))^{-1}\alpha_2. \tag{1}$$

(ii) Letting v_1 (resp. v_2) be the vector in equation (1) for T_l^1 (resp. T_l^2), we have $v_1 = -v_2$.

To see this, fix a holomorphic embedding $S_l : F^{-1}(l) \hookrightarrow \mathbb{CP}^1$ that takes z_i to w_i. By rigidity, any such embedding $F^{-1}(l) \to \mathbb{CP}^1$ is of the form $T_l = U \circ S_l$ for an automorphism $U : \mathbb{CP}^1 \to \mathbb{CP}^1$ which fixes $w_1 = 0$ and $w_2 = 1$. Any such U has the form

$$U_a(z) = \frac{z}{az + 1 - a}$$

for some $a \neq 1$. Setting $b = 1 - a$, and using $U_a'(0) = b^{-1}$, $U_a'(1) = b$ we substitute $(dT_l(z_1))^{-1}\alpha_1 = b\,dS_l^{-1}(0)\alpha_1$ and $(dT_l(z_2))^{-1}\alpha_2 = b^{-1}dS_l^{-1}(1)\alpha_2$ into Equation (1) to obtain

$$b\,dF(z_1)dS_l^{-1}(0)\alpha_1 = b^{-1}dF(z_2)dS_l^{-1}(1)\alpha_2$$

in the tangent space $T_{l,0} \subset T_{\mathbb{C}^n,0}$, which we identify with the subset $\mathbb{C} \cong l \subset \mathbb{C}^n$. Thus $b^2 \in \mathbb{C} - \{0\}$ and we find two values for b, one being the negative of the other. This proves (i) and (ii) above.

In particular, both choices of T_l yield the same *real* line $v(l) \subset l \subset \mathbb{C}^n$, hence the map $v : \mathbb{CP}^{n-1} \to \mathbb{RP}^{2n-1}$ is a set-theoretic section of the map π above. Using the Bieberbach-Gronwall estimate for univalent functions on the unit disc [7, Theorem 1.3], we will show in section three that v is a *continuous* section: this contradiction finishes the proof. $\square$

Example 2.4. The conclusion of Theorem 1.1 may fail if the pre-image of even a single line is disconnected. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with nowhere zero derivative and which assumes the value 0 more than once. Let F be the local biholomorphism of $\mathbb{C}^2$ defined by $F(z, w) = (f(z), w)$. If $l \subset \mathbb{C}^2$ is the complex line $aw_1 + bw_2 = 0$, then $F^{-1}(l)$ is conformal to $\mathbb{C}$ if $b \neq 0$, which is a rigid domain. For $b = 0$, however, the pre-image $F^{-1}(l)$ is homeomorphic to the disconnected space $f^{-1}(0) \times \mathbb{C}$.

Example 2.5. This example shows that the rationality condition in part (2) of Corollary 1.3 cannot be removed, hence the condition that $F^{-1}(l)$ be conformal to a rigid domain of $\mathbb{CP}^1$, rather than a compact Riemann surface of positive genus, cannot be removed from Theorem 1.1 or Corollary 1.2. Let $Y \subset \mathbb{P}^3$ be a general smooth surface of degree $d > 2$. Then Y is neither ruled nor a Steiner surface, hence has no two-dimensional families of reducible plane sections [10, Lemma II.2.4]. It follows that for a general point $p \in \mathbb{P}^3$, only a *finite* number of planes H_i containing p intersect Y in a reducible curve. Let $F : Y \to \mathbb{P}^2$ be the projection from such a point p onto a plane $\mathbb{P}^2 \subset \mathbb{P}^3$. If $q \in \mathbb{P}^2$ and $q \notin \bigcup_i H_i$, then *every* line l through q has irreducible pre-image $F^{-1}(l) = Y \cap H$ (H spanned by l and p) of arithmetic genus $p_a(Y \cap H) = \frac{1}{2}(d-1)(d-2)$ [8, I, Ex. 7.2(b)]. Moreover, $Y \cap H$ is smooth and irreducible for general l containing q by Bertini's theorem [9, Thm. 6.10], so the geometric (or topological) genus is also $p_g(F^{-1}(l)) = \frac{1}{2}(d-1)(d-2)$ [8, III, Remark 7.12.2].

There is an open subset $U \subset Y - \bigcup_i H_i$ such that the restriction map $F|_U : U \to \mathbb{P}^2$ is étale [13, I, Prop. 3.8]. Removing a line $L \subset \mathbb{P}^2$ gives an étale morphism $F : X = U - \pi_p^{-1}(L) \to \mathbb{C}^2$ of degree d such that the pre-image of every line meeting $F(X)$ is an irreducible curve. For the general such line l, the pre-image $F^{-1}(l)$ is a Zariski open subset of a smooth projective plane curve ($Y \cap H$ in the discussion above) of geometric genus $\frac{1}{2}(d-1)(d-2) > 0$.

Example 2.6. Let $S \subset \mathbb{R}^3$ be the cubic surface $x^2 + y^2 = z^2(1 - z)$ obtained by rotating a nodal cubic curve about the z-axis and consider the projection $F : S \to \mathbb{R}^2$ to the xy-plane. The Jacobian determinant vanishes along the set W consisting of the origin and the circle $z = \frac{2}{3}$, so the restricted map $X := S - W \to \mathbb{R}^2$ becomes a local diffeomorphism. The map is not globally injective (points on the disk $D = \{(x, y) : x^2 + y^2 \leq \frac{4}{27}\}$ have three pre-images) and the pre-image of a line l is connected if and only if l misses the disk D (compare [15, Example 3.3]).

Now we complexify the above example. Take $S \subset \mathbb{C}^3$ to be the *complex* surface of the same equation and remove the locus W consisting of the origin and the complex curve $z = \frac{2}{3}$ to obtain $X = S - W$. Projection to the xy-plane now gives a non-injective local biholomorphism $F : X \to \mathbb{C}^2$. Consider the complex line $l : y = ax$ through the origin: the equations of the pre-image under F are $y = ax$ and $(1 + a^2)x^2 = z^2(1 - z)$. If $a \neq \pm i$, then the pre-image $F^{-1}(l)$ is a connected rational curve (a parametrization is given by $x = \frac{t^2-1-a^2}{t^3},\ y = ax,\ z = tx$), but for $a = \pm i$ the pre-image is not connected, consisting of two disjoint lines $z = 0$ and $z = 1$. The pre-images of the lines through the point $(0, 1)$ are all connected (being zero sets of irreducible polynomials), but most of them are the complement of a smooth elliptic cubic curve by finitely many points.

Example 2.7. Theorem 1.1 fails if we remove the rigidity condition. Consider the map $F : D \times D \to \mathbb{C}^2$ given by $F(z, w) = ((w + 2)e^{7z}, w)$, where $D \subset \mathbb{C}$ is the open unit disc.

Then F is a local biholomorphism since $\det JF = 7(w+2)e^{7z} \neq 0$ on $D \times D$. Now let $\mathcal{U} \subset D \times D$ be the union of all connected components through $(0,0)$ of all pre-images of complex lines l passing through $(2,0)$. It follows from the inverse function theorem that the interior $\mathcal{U}^0$ of $\mathcal{U}$ is closed in $\mathcal{U}$. Since $(0,0) \in \mathcal{U}^0$ and $\mathcal{U}$ is connected, one has that $\mathcal{U} = \mathcal{U}^0$ is open. The restriction $F|_{\mathcal{U}}$ is not injective, since $F(0,0) = F(\frac{2\pi i}{7}, 0) = (2,0)$. The line $w = 0$ pulls back to $D \times \{0\}$, a non-rigid domain in $\mathbb{CP}^1$. Restricting F to a sufficiently small neighborhood $\mathcal{U}' \subset \mathcal{U}$ of $D \times \{0\}$, it is clear that the pull-back of every line l through $(2,0)$ is a connected set which is conformal to an open non-rigid subset of $\mathbb{CP}^1$.

3. The continuity proof

In this section we complete the proof of Theorem 1.1 by showing continuity of the map $v : \mathbb{CP}^{n-1} \to \mathbb{RP}^{2n-1}$ there, constructed as follows. Starting with a local biholomorphism $F : X \to \mathbb{C}^n$ such that $z_1 \neq z_2$ and $F(z_1) = F(z_2) = 0$, we assumed that for each complex line $l \in \mathbb{CP}^{n-1}$ through the origin, $F^{-1}(l)$ is a rigid domain in $\mathbb{CP}^1$, hence a connected rational curve. Fixing $w_1 = 0$ and $w_2 = 1$ in $\mathbb{CP}^1$ and real tangent vectors $\alpha_i \in T_{\mathbb{P}^1, w_i}$, we showed that there are exactly two embeddings $T_l^1, T_l^2 : F^{-1}(l) \to \mathbb{CP}^1$ sending z_i to w_i $(i = 1,2)$ and satisfying

$$dF(z_1)(dT_l^j(z_1))^{-1}\alpha_1 = dF(z_2)(dT_l^j(z_2))^{-1}\alpha_2 \in \mathbb{C}^n$$

for $j = 1, 2$. Furthermore, both embeddings give rise to the same *real* line through 0 (we now drop the superscript and write T_l for either of the two embeddings). In particular, we have defined a map $v : \mathbb{CP}^{n-1} \to \mathbb{RP}^{2n-1}$. such that $v(l) = p(\kappa(l))$, where $p : \mathbb{C}^n - \{0\} \to \mathbb{RP}^{2n-1}$ is the natural projection and $\kappa : \mathbb{CP}^{n-1} \to \mathbb{C}^n - \{0\}$ is given by $\kappa(l) = dF(z_1)(dT_l(z_1))^{-1}\alpha_1$. We will now show that v is continuous; in other words, for any convergent sequence $l_q \to l$ in $\mathbb{CP}^{n-1}$, we will show that $v(l_q) \to v(l)$.

The salient feature is that the submanifolds $F^{-1}(l_q)$ converge to $F^{-1}(l)$ uniformly over compact subsets of X. Since we are proving that $v(l_q) \to v(l)$ for *any* sequence $l_q \to l$, it suffices to show that $v(l_q)$ has a *subsequence* converging to $v(l)$. Indeed, this shows that $v(l)$ is the only accumulation point. We will achieve this by essentially (though not literally) writing the map $T_l : F^{-1}(l) \to \mathbb{CP}^1$ as a limit of the maps $T_q = T_{l_q}$. To this end, let S_q be a sequence in the unitary group $U(n)$ such that $S_q \to I_n$ and $S_q(l) = l_q$.

Fix a complete Riemannian metric on X. Since $F^{-1}(l)$ is connected, there is a closed geodesic ball $K_0 = \overline{B_R(z_1)}$ of some radius $R > 0$ centered at z_1 such that z_1, z_2 lie in the same connected component C_0 of the compact domain $K_0 \cap F^{-1}(l)$ for the Riemann surface $F^{-1}(l)$. Setting $K_m = \overline{B_{R+m}(z_1)}$, we obtain a countable increasing exhaustion $\{K_m\}_{m \geq 1}$ of X. Clearly z_1, z_2 lie in the same connected component C_m of $K_m \cap F^{-1}(l)$ for each $m \geq 1$. Since K_m is compact, there is $\delta = \delta(m) > 0$ such that

(1) $F|_{B_\delta(z)}$ is a biholomorphism onto its image for all $z \in K_m$.
(2) $F|_{B_\delta(z) \cup B_\delta(z')}$ is injective whenever $B_\delta(z) \cap B_\delta(z')$ is nonempty.

Indeed, for $x \in K_m$, there exists $\delta_x > 0$ such that $F|_{B_{\delta_x}(x)}$ is a biholomorphism onto its image. The open cover $\{B_{\frac{\delta_x}{4}}(x)\}_{x \in K_m}$ for K_m has a finite subcover $\{B_{\frac{\delta_k}{4}}(x_k)\}_{k=1}^r$ and we set $\delta = \delta(m) = \min_{1 \leq k \leq r}\{\frac{\delta_k}{4}\}$. For $z \in B_\delta(x_k) \cap K_m$, we have $B_\delta(z) \subset B_{2\delta}(x_k)$ which implies condition 1 since F takes the larger ball biholomorphically onto its image. If $B_\delta(z')$ meets $B_\delta(z)$, then $B_\delta(z') \subset B_{4\delta}(x_k)$ so that $B_\delta(z) \cup B_\delta(z') \subset B_{4\delta}(x_k)$ and the restriction of F to the latter ball is injective, which checks condition (2). Setting $\delta_m = \min_{k \le m} \delta(k)$ we obtain a nonincreasing sequence with the same properties.

Now fix $m \geq 1$. The set

$$S = \{(a, z) \in X \times (F^{-1}(l) \cap K_m) : d(a, z) = \frac{\delta_m}{2}\}$$

is compact, hence the continuous function $G(a, z) = |F(a) - F(z)|$ achieves its minimum value λ on S; moreover $\lambda > 0$ because $F|_{B_{\delta_m}(z)}$ is injective for each $z \in F^{-1}(l) \cap K_m$. For $\mu = \frac{\lambda}{2}$ we have $B_\mu(F(z)) \subset F(B_{\delta_m}(z))$ for all $z \in F^{-1}(l) \cap K_m$. The convergence $S_q \to I$ yields $N_m > 0$ such that the absolute values of the entries of the matrix $S_q - I_n$ are all less than $\frac{\mu}{n^2 M}$ for $q \geq N_m$, where

$$M = M(m) = \sup_{z \in F^{-1}(l) \cap K_m} |F(z)|.$$

With these choices, we have that $|S_q(F(z)) - F(z)| < \mu$ for $z \in F^{-1}(l) \cap K_m$ and $q \geq N_m$ so that $S_q(F(z)) \in F(B_{\delta_m}(z))$. As above, we may assume that the N_m are nondecreasing.

For δ_m and N_m as above, we have thus constructed well-defined holomorphic injections $\phi_{m,q} : C_m \hookrightarrow F^{-1}(l_q)$ for $q \geq N_m$ given by

$$\phi_{m,q}(z) = [(F|_{B_{\delta_m(z)}})^{-1} \circ S_q \circ F](z).$$

By construction it is clear that $\phi_{m,q}$ fixes z_1, z_2 and that $\phi_{m',q}|_{C_m} = \phi_{m,q}$ for $m' \geq m$ and $q \geq N_m$. Injectivity follows from condition (2) on $\delta(m)$ above. Composition with T_q yields injective holomorphic maps

$$\psi_{m,q} = T_q \circ \phi_{m,q} : C_m \hookrightarrow \mathbb{CP}^1$$

for $q \geq N_m$ which satisfy $\psi_{m,q}(z_i) = w_i$ and $\psi_{m',q}|_{C_m} = \psi_{m,q}$ for $m' \geq m$. Note that we will make a convenient choice of identification $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ later in the proof.

We will need the following compactness result in the proof of Lemma 3.2 below:

Lemma 3.1. *Let $D \subset \mathbb{C}$ be the open unit disc, $0 \neq a \in D$ and $0 \neq b \in \mathbb{C}$. Then*

$$\mathcal{F}_{a,b} = \{\text{holomorphic injections } f : D \hookrightarrow \mathbb{C} \mid f(0) = 0 \text{ and } f(a) = b\}$$

is a normal family of functions.

Proof. Any univalent holomorphic function f on D that vanishes at 0 satisfies the well-known estimate of Bieberbach and Gronwall (see, for example [7, Theorem 1.3] for the normalized estimate)

$$\frac{|f'(0)|r}{(1+r)^2} \leq |f(z)| \leq \frac{|f'(0)|r}{(1-r)^2},$$

where $r = |z|$. Evaluating the inequality on the left at $z = a$ gives an upper bound on $|f'(0)|$. The inequality on the right now shows that $|f(z)|$ is uniformly bounded over compact subsets of D, with a bound independent of f. By Montel's theorem $\mathcal{F}_{a,b}$ is normal, that is $\overline{\mathcal{F}_{a,b}}$ is compact in the space of holomorphic functions on D, endowed with the topology of uniform convergence over compact subsets. □

Lemma 3.2. *Fix $m \geq 1$ and an identification $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Let $\{\varphi_k\}$ be a subsequence of $\{\psi_{m,q}\}_{q \geq N_m}$ such that each φ_k has a pole p_k and $\{p_k\}$ converges to a point x in the interior of C_m. Then $\{\varphi_k\}$ has a further subsequence converging uniformly over compact subsets of the interior of $C_m - \{x\}$ to a holomorphic injection of the interior of $C_m - \{x\}$ into $\mathbb{C} = \mathbb{CP}^1 - \{\infty\}$.*

Proof. To set notation, write $\varphi_k = \psi_{m,s_0(k)}$ for some strictly increasing function $s_0 : \mathbb{N} \to \mathbb{N}$ with $s_0(1) \geq N_m$. Cover $C_m - \{x\}$ with countably many embedded holomorphic disks $\xi_r : D \hookrightarrow C_m - \{x\}$ in such a way that $\xi_r(0) = z_1,\ z_2 \in \xi_r(D)$ but $x \notin \overline{\xi_r(D)}$.

Consider the rth embedded holomorphic disk $\xi_r : D \hookrightarrow C_m - \{x\}$. Then $p_k \notin \xi_r(D)$ for k sufficiently large because $x \notin \overline{\xi_r(D)}$, hence the maps $\varphi_k \circ \xi_r$ belong to class $\mathcal{F}_{\xi_r^{-1}(z_2),w_2}$ of Lemma 3.1 (their images lie in $\mathbb{C}$ via the identification $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$). In particular, there exists a subsequence $\varphi_{s_1(k)} \circ \xi_r$ converging uniformly over the compact sets of D by normality of $\mathcal{F}_{\xi_r^{-1}(z_2),w_2}$. Composing with ξ_r^{-1}, the subsequence $\varphi_{s_1(k)}$ converges uniformly over the compact subsets of the topological disc $\xi_r(D) \subset C_m - \{x\}$ to a holomorphic map taking values in $\mathbb{C} = \mathbb{CP}^1 - \{\infty\}$. Since $\varphi_{s_1(k)}(z_i) = w_i\ \ (i = 1, 2)$, we see that the nonconstant limit function is the local uniform limit of a sequence of injective holomorphic functions. By a well-known theorem of Hurwitz, the limit function is itself injective.

Taking $r = 1$ above, φ_k has a subsequence $\varphi_{s_1(k)}$ which converges uniformly over compact sets in $\xi_1(D)$ ($s_1 : \mathbb{N} \to \mathbb{N}$ strictly increasing as above). Applying the previous paragraph with $r = 2$ and the subsequence $\varphi_{s_1(k)}$, we obtain a further subsequence $\varphi_{s_2(k)}$ with the same property with respect to both $\xi_1(D)$ and $\xi_2(D)$. Continuing in this fashion, we obtain a countable sequence of subsequences $\varphi_{s_r(k)}$ (each uniformly convergent on compact subsets of $\xi_p(D)$ for $1 \leq p \leq r$), each being extracted from the previous: $s_1(\mathbb{N}) \supset\ s_2(\mathbb{N}) \supset\ s_3(\mathbb{N}) \ldots$. Thus the diagonal subsequence $\varphi_{s_r(r)}$ converges uniformly to an injective holomorphic function on every compact subset of $C_m - \{x\}$. This proves Lemma 3.2. □

Taking $m = 1$, we now make a choice of both $\infty \in \mathbb{CP}^1$ and a subsequence φ_k of $\psi_{1,q}$ for which the hypotheses of Lemma 3.2 holds. Start with any point $a \in \mathbb{CP}^1$. If there happens to be a subsequence ψ_{1,q_k} and $p_k \in C_1$ with $\psi_{1,q_k}(p_k) = a$ and $p_k \to x$ with x in the interior of C_1, then we simply take $\infty = a$ and $\varphi_k = \psi_{1,q_k}$. The alternatives are that (1) $a \in \psi_{1,q}(C_1)$ for only *finitely* many q or (2) $a \in \psi_{1,q}(C_1)$ for infinitely many q, but for all subsequences $p_k,\ q_k$ with $\psi_{1,q_k}(p_k) = a$ and $p_k \to x$, we have $x \in \partial C_1$. In either case, we can find a subsequence q_k and an embedded holomorphic disk $\xi : D \hookrightarrow C_1$ such that $\xi(0) = z_1,\ z_2 \in \xi(D)$ and $a \notin \psi_{1,q_k}(\xi(D))$ for k sufficiently large. Thus $\psi_{1,q_k} \circ \xi : D \hookrightarrow \mathbb{C} = \mathbb{CP}^1 - \{a\}$ for $k >> 0$ and by Lemma 3.1 we can find a further subsequence $\psi_{1,q_{k_l}} \circ \xi$ converging uniformly over compact subsets. Now fix $y \in D$, set $x = \xi(y)$ and take $\infty = \lim_{l \to \infty} \psi_{1,q_{k_l}}(x)$. Thus we obtain a subsequence $\varphi_k = \psi_{1,q_{k_l}}$ of $\psi_{1,q}$ for which every further subsequence forcibly has poles which converge to x in the interior of C_1 and we may apply Lemma 3.2.

Finally we vary $m \geq 1$. Starting with $m = 1$, choose $\infty \in \mathbb{CP}^1$ and φ_k as above and apply Lemma 3.2 to obtain a subsequence $\varphi_{s_1(k)}$ which converges uniformly to an injective holomorphic function on compact subsets of the interior of $C_1 - \{x\}$. Setting $B = \min\{k : s_1(k) \geq N_2 - 1\}$ and $t_1(k) = s_1(k + B)$ we have a subsequence $\varphi_{t_1(k)}$ of $\psi_{2,n}$ whose restrictions to C_1 form a subsequence of $\varphi_{s_1(k)}$; moveover, the hypotheses of Lemma 3.2

hold automatically by the compatibility condition on the maps $\psi_{m,q}$, that is, $\psi_{m',q}|_{C_m} = \psi_{m,q}$ for $m' \geq m$. Thus applying Lemma 3.2 with $m = 2$ gives a further subsequence that converges uniformly to an injective holomorphic function on compact subsets of the interior of $C_2 - \{x\}$. Continuing in this fashion, we obtain a countable sequence $\{\varphi_{s_m(k)}\}_{m\geq 1}$ of subsequences, each being extracted from the restriction of the next: $\varphi_{s_m(k)}$ converges uniformly over compact subsets to an injective holomorphic function on the interior of $C_m - \{x\}$. Taking the diagonal subsequence $\varphi_{s_m(m)}$ yields

Lemma 3.3. *There is an injective holomorphic map $\eta : F^{-1}(l) - \{x\} \hookrightarrow \mathbb{C} = \mathbb{CP}^1 - \{\infty\}$ such that for each fixed $m \geq 1$, the restriction of η to the interior of $C_m - \{x\}$ is the local uniform limit of a subsequence ψ_{m,q_l} of $\psi_{m,q}$.*

The map η extends to a holomorphic injection $\tilde{\eta} : F^{-1}(l) \to \mathbb{CP}^1$ such that $\tilde{\eta}(z_i) = w_i$ for $i = 1, 2$. For z in the interior of $\in C_1 - \{x\}$ (say z near z_1 or z_2), we use Lemma 3.3 to write

$$\tilde{\eta}(z) = \lim_{k\to\infty} \psi_{1,q_k}(z)$$

as a local uniform limit for some subsequence q_k. Since the sequence of derivatives also converges, we have

$$dF(z_1) \circ (d\tilde{\eta}(z_1))^{-1}(\alpha_1) = \lim_{k\to\infty} dF(z_1) \circ (d\psi_{1,q_k}(z_1))^{-1}(\alpha_1).$$

Since $\psi_{1,q} = T_q \circ \phi_{1,q}$ and $\phi_{1,q}(z) = [(F|_{(B_{\delta_1(z)})})^{-1} \circ S_q \circ F](z)$, this becomes

$$\lim_{k\to\infty} [(S_{q_k})^{-1} \circ dF(z_1) \circ (dT_{q_k}(z_1))^{-1}(\alpha_1)],$$

but $\lim_{k\to\infty} S_{q_k} = I = n \times n$ identity matrix, so we conclude that

$$dF(z_1) \circ (d\tilde{\eta}(z_1))^{-1}(\alpha_1) = \lim_{k\to\infty} dF(z_1) \circ (dT_{q_k}(z_1))^{-1}(\alpha_1).$$

Analogously,

$$dF(z_2) \circ (d\tilde{\eta}(z_2))^{-1}(\alpha_2) = \lim_{k\to\infty} dF(z_2) \circ (dT_{q_k}(z_2))^{-1}(\alpha_2).$$

By the definition of T_{q_k}, one has

$$dF(z_1) \circ (dT_{q_k}(z_1))^{-1}(\alpha_1) = dF(z_2) \circ (dT_{q_k}(z_2))^{-1}(\alpha_2),$$

so that

$$dF(z_1) \circ (d\tilde{\eta}(z_1))^{-1}(\alpha_1) = dF(z_2) \circ (d\tilde{\eta}(z_2))^{-1}(\alpha_2).$$

Together with $\tilde{\eta}(z_i) = w_i$ $(i = 1, 2)$, the last relation shows that $\tilde{\eta}$ is equal to one of the two maps T_l^j. Taking images in $\mathbb{RP}^{2n-1}$ one now sees that $v(l_{q_k}) \to v(l)$. This shows that the section $v : \mathbb{CP}^{n-1} \to \mathbb{RP}^{2n-1}$ of the natural map $\pi : \mathbb{RP}^{2n-1} \to \mathbb{CP}^{n-1}$ is continuous. Following the earlier part of the proof, this contradiction proves Theorem 1.1. □

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