A complete minimal surface in $\mathbf{R}^3$ between two parallel planes

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E. Calabi has asked if it is possible to have a complete minimal surface in $\mathbf{R}^3$ entirely contained in a half-space. We answer his question in the affirmative. In fact, we prove considerably more, namely, we show how to produce complete minimal surfaces contained in slabs of $\mathbf{R}^3$. The proof was motivated by Remmert's ingenious idea of using Runge's theorem to exhibit proper analytic embeddings of the unit disc $D \subset C$ in $\mathbf{C}^3$, as explained in [1], page 96.

It should be pointed out that M. Miranda [2] has shown that a surface that minimizes area globally and lies in a half-space of $\mathbf{R}^3$ is a plane. Here, however, we deal with surfaces that minimize area only locally. We also observe that B. Lawson [5] has given examples of complete surfaces of constant mean curvature which lie between two parallel planes.

We take the oportunity to call attention to [4] where a related conjecture is discussed.

The Example

LEMMA 1. (*Weierstrass representation of minimal surfaces*). *Let $f, g: D \to C$ be holomorphic and set*

$$\phi_1 = \frac{1}{2} f(1 - g^2)\ , \qquad \phi_2 = \frac{i}{2} f(1 + g^2)\ , \qquad \phi_3 = fg\ .$$

If f never vanishes then the function $x(x_1, x_2, x_3): D \to \mathbf{R}^3$, where $x_k = \mathrm{Re}\int \phi_k$ will define a (regular) minimal surface in $\mathbf{R}^3$ whose element of length is given by $ds = \lambda |dz|$ where

$$\lambda = \frac{|f|}{2}(1 + |g|^2)\ .$$

LEMMA 2. *Let $\{D_n\}$ be a sequence of closed discs centered at the origin, $D_n \subset \mathring{D}_{n+1}$, $\bigcup D_n = D$. Let $K_n \subset D_n$ be a compact set such that $K_n \cap D_{n-1} = \phi$*

0003-486X/80/0112-1/0203/004 $ 00.20/1

* Supported by UFC and FINEP (Brasil).
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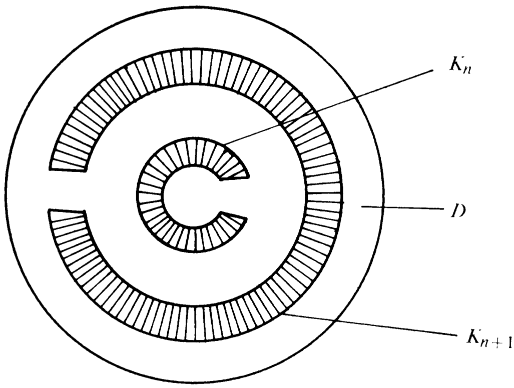
and $D\backslash K_n$ is connected. Let f be a holomorphic function defined in a neighborhood of the union of the K_n. Then it is possible to approximate f uniformly on that union by holomorphic functions defined in D.

For the proof of Lemma 1, see [3]. Lemma 2 is precisely exercise G of [1], page 96; there the reader will also find a sketch of its proof using Runge's theorem.

THEOREM. *There are nonflat complete minimal surfaces of $\mathbf{R}^3$ entirely contained in a slab.*

Proof. Consider the picture below.



As indicated, K_n is the compact region formed by deleting a piece from an annulus. Let r_n be the difference between the outer and inner radii of the annulus. For the next compact set, that is, K_{n+1}, we delete a piece from another annulus (disjoint from the first) but from the "other side", and so on. Let

$$E = \bigcup_{n \text{ even}} K_n \qquad \text{and} \qquad O = \bigcup_{n \text{ odd}} K_n .$$

We shall say that a path α *crosses* K_n if it intercepts both the inner and outer arcs of the annulus from which K_n is obtained. A little reflection shows that $\{K_n\}$ satisfies the following:

(*) *Any divergent path in D of finite Euclidean length will cross all but a finite number of the K_n in E or all but a finite number of the K_n in O.*

Note that if K_n is chosen so that its "*opening*" is to the right of 0 then the segment

$$\left\{(x, 0) \middle| ; \frac{1}{2} < x < 1\right\}$$

will meet the last requirement but not the first one.

Let $\{c_n\}$ be a sequence of positive numbers, to be specified later. By Lemma 2 there is a holomorphic function h on D such that $|h - c_n| < 1$ on K_n. Let e^h be the function g appearing in Lemma 1 and set $f = 1/g$. Consider the minimal surface determined by f and g, as in Lemma 1. An immediate consequence is that x_3 is bounded; that is, our minimal surface M is contained in a slab. We will show that $\{c_n\}$ can be chosen so as to make M complete. Let α be a divergent path in D, which may be supposed to have Euclidean speed one. We shall distinguish two cases.

1) Suppose that α has infinite Euclidean length, i.e., $\alpha: [0, \infty) \to D$. Since

$$\lambda = \frac{1}{2}\left(|g| + \frac{1}{|g|}\right) \geqq 1 ,$$

it follows that

$$l(\alpha) = \int_0^\infty \lambda(\alpha(t))\, dt = \infty .$$

2) Suppose now that α has finite Euclidean length, i.e., $\alpha: [0, b) \to D$, $b < \infty$. Consider the first alternative given in * and let m be an integer such that α crosses every $K_n \in E$ with $n \geqq m$. From $g = e^h = e^{c_n} e^{h - c_n}$ we have $|g| \geqq e^{c_n - 1}$ on K_n. Let

$$J_n = \{t \in [0, b) \,|\, \alpha(t) \in K_n\} .$$

Then

$$\begin{aligned} 2l(\alpha) &\geqq \int_0^b |g(\alpha(t))|\, dt \geqq \sum\nolimits_{n \geqq m, n \text{ even}} \int_{J_n} |g(\alpha(t))|\, dt \\ &\geqq \sum\nolimits_{n \geqq m, n \text{ even}} e^{c_n - 1} \int_{J_n} dt \geqq \sum\nolimits_{n \geqq m, n \text{ even}} r_n e^{c_n - 1} \end{aligned}$$

(recall that t is the Euclidean arc-length of α).

Similarly, if the other alternative in * holds we have

$$l(\alpha) \geqq \sum\nolimits_{n \geqq k, n \text{odd}} r_n e^{c_n - 1} .$$

Therefore, if c_n is chosen to grow fast enough, the curve α will have infinite length in M. In fact, it suffices to take $c_n = -\log r_n$.

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(Received September 4, 1979)