A Riemannian Bieberbach estimate

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Abstract. The Bieberbach estimate, a pivotal result in the classical theory of univalent functions, states that any injective holomorphic function f on the open unit disc D satisfies $|f''(0)| \leq 4|f'(0)|$. We generalize the Bieberbach estimate by proving a version of the inequality that applies to all injective smooth conformal immersions $f : D \to \mathbb{R}^n$, $n \geq 2$. The new estimate involves two correction terms. The first one is geometric, coming from the second fundamental form of the image surface $f(D)$. The second term is of a dynamical nature, and involves certain Riemannian quantities associated to conformal attractors. Our results are partly motivated by a conjecture in the theory of embedded minimal surfaces.

1 Introduction

A conformal orientation-preserving local diffeomorphism that is defined in the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ and takes values into $\mathbb{R}^2$ can be viewed as a holomorphic function $f : D \to \mathbb{C}$. Of special interest is the case when f is univalent, that is, injective. The class S of all holomorphic univalent functions in D satisfying $f(0) = 0$ and $f'(0) = 1$ was the object of much study in the last century, culminating with the solution by de Branges ([4], [22]) of the celebrated Bieberbach conjecture: for any $f \in S$, the estimate

$$|f^{(k)}(0)| \leq kk! \tag{1.1}$$

holds for all $k \geq 2$. Equivalently, $|f^{(k)}(0)| \leq kk!|f'(0)|$ for any injective holomorphic function on D. The case $k = 2$, due to Bieberbach, yields the so-called distortion theorems which, in turn, imply the compactness of the class S ([19], [22]). Thus, the basic estimate $|f''(0)| \leq 4$ for $f \in S$, most commonly written in the form $|a_2| \leq 2$ where $f(z) = z + a_2z^2 + \cdots$, already yields important qualitative information. In particular, it follows from the compactness of S that there are constants C_k such that $|f^{(k)}(0)| \leq C_k|f'(0)|$ for every $k \geq 2$ and injective holomorphic function f on D. The Bieberbach conjecture (the de Branges theorem) asserts that one can take C_k to be $kk!$.

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The aim of this paper is to establish a generalization of Bieberbach's fundamental estimate $|f''(0)| \leq 4|f'(0)|$ that applies to all injective smooth conformal immersions $f : D \to \mathbb{R}^n$, $n \geq 2$. The new estimate involves two correction terms. The first one is geometric, coming from the second fundamental form of the image surface $f(D)$. The second term is of a dynamical nature, and involves certain Riemannian quantities associated to conformal attractors.

Given a submanifold of $\mathbb{R}^n$, we denote by ∇ and σ its connection and second fundamental form, both extended by complex linearity. Recall also that a vector field on a Riemannian manifold is said to be conformal if it generates a local flow of conformal maps. We use subscripts to denote differentiation with respect to z, where $2\,\partial_z = \partial_x - i\partial_y$.

Theorem 1.1. *Let $f : (D, 0) \to (N, p)$ be an injective smooth conformal immersion, where $N \subset \mathbb{R}^n$ is a smooth embedded topological disc, $n \geq 3$. Let $\mathfrak{X}$ be the (non-empty) family of all normalized conformal attractors on (N, p), namely those smooth vector fields X on N satisfying*

i) X is conformal, $X(p) = 0$, $(\nabla X)_p = -I$.
ii) Every positive orbit of X tends to p.

Then

$$\sup_{X\in\mathfrak{X}} \left\| f_{zz}(0) - \sigma\big(f_z(0), f_z(0)\big) + (\nabla^2 X)_p\big(f_z(0), f_z(0)\big) \right\| \leq 4\|f_z(0)\|. \tag{1.2}$$

We expect to develop further the ideas in this paper and establish analogues of the higher order estimates $|f^{(k)}(0)| \leq kk!|f'(0)|$, $k \geq 3$, in the broader context of injective conformal immersions $f : D \to \mathbb{R}^n$. One would then have, in all dimensions, a geometric-conformal version of the de Branges theorem.

Example. Taking $N = \mathbb{R}^2 \subset \mathbb{R}^n$ as a totally geodesic submanifold and $X(w) = -(w-p)$, we have $\nabla X = -I$ and so $\nabla^2 X = 0$. Since $\sigma = 0$, one sees that the original Bieberbach estimate $|f''(0)| \leq 4|f'(0)|$ can be recovered from (1.2).

It follows from classical results that for any injective smooth immersion $g : D \to \mathbb{R}^n$ there is a diffeomorphism $h : \Omega \to D$, where Ω is either D or $\mathbb{C}$, such that $f = g \circ h$ is an injective conformal immersion. Restricting h to D in the case $\Omega = \mathbb{C}$, one then sees that the above theorem applies to every injective immersion, after a suitable reparametrization.

We observe that the original Bieberbach estimate does not carry over to the higher dimensional case. Indeed, if every injective conformal immersion $f : D \to \mathbb{R}^n$ were to satisfy $\|f_{zz}(0)\| \leq 4\|f_z(0)\|$, a contradiction could be reached as follows. Let $g : \mathbb{C} \to \mathbb{R}^n$ be an injective conformal harmonic immersion which is not totally geodesic, i.e., $g(\mathbb{C})$ is a parabolic simply-connected embedded minimal surface. A concrete example is provided by the helicoid given in coordinates $z = x + iy$ by $g(x, y) = (\sinh x \cos y, \sinh x \sin y, y)$ (see the Conjecture below). Applying the above estimate to $f : D \to \mathbb{R}^n$, $f(z) = g(Rz)$, and letting $R \to \infty$, one obtains

$$g_{zz}(0) = \frac{1}{4}\Big(g_{xx}(0) - 2ig_{xy}(0) - g_{yy}(0)\Big) = \frac{1}{2}\Big(g_{xx}(0) - ig_{xy}(0)\Big) = 0.$$

Replacing $g(z)$ by $g(z+z_o)$ in the above reasoning, we obtain $g_{zz} \equiv 0$, and so g_{xx}, g_{yy} and g_{xy} are also identically zero, contradicting the fact that $g(\mathbb{C})$ is not a plane.

It is tempting to believe that the contribution in (1.2) coming from the dynamic term can be made to vanish, regardless of the embedding, in which case one would obtain an estimate that depends solely on the geometry of the surface $f(D)$. To investigate this possibility, suppose that

$$\inf_{X \in \mathfrak{X}} \Big\| (\nabla^2 X)_p\big(f_z(0), f_z(0)\big) \Big\| = 0$$

for all conformal embeddings $f : D \to \mathbb{R}^n$ of the unit disc. Consider the map g above, parametrizing a helicoid. After applying (1.2) to $f : D \to \mathbb{R}^3$, $f(z) = g(Rz)$, and reasoning as above, one has

$$g_{zz} - \sigma\big(g_z, g_z\big) = \frac{1}{2}\Big(g^T_{xx} - ig^T_{xy}\Big) \equiv 0.$$

In particular, the coordinates curves of g are geodesics of $g(D)$. Since these curves are easily seen to be asymptotic lines of $g(D)$, one would conclude that the traces of the coordinates curves of g are (segments of) straight lines in $\mathbb{R}^3$. But this contradicts the fact that the coordinates curves $y \mapsto g(x, y)$ are helices. Hence, the dynamic term is essential for the validity of (1.2).

As the reader may have suspected by now, part of our motivation for proving a version of the Bieberbach estimate in the realm of conformal embeddings $D \to \mathbb{R}^n$ comes from the theory of minimal surfaces. We explain below how these two sets of ideas merge.

The past few years have witnessed great advances in the study of simply-connected embedded minimal surfaces (i.e., minimal surfaces without self-intersetions). From an analytic standpoint, these geometric objects correspond to conformal harmonic embeddings of either D or $\mathbb{C}$ into $\mathbb{R}^3$. In a series of groundbreaking papers, Colding and Minicozzi ([5]-[9]) were able to give a very detailed description of the structure of embedded minimal discs. Using their theory, as well as other tools, Meeks and Rosenberg showed in a landmark paper [14] that helicoids and planes are the only properly embedded simply-connected minimal surfaces in $\mathbb{R}^3$ (subsequently, properness was weakened to mere completeness).

The classical link between minimal surfaces and complex analysis has been explored, with great success, to tackle other fundamental geometric problems. Given the history of the subject, one is naturally inclined to look for a complex-analytic interpretation of the works of Colding-Minicozzi and Meeks-Rosenberg, with the hope that more could be revealed about the structure of embedded minimal discs. Although this effort is still in its infancy, one can already delineate the contours of a general programme. A central theme to be explored is the role of the conformal type in the embeddedness question for minimal surfaces. In particular, one would like to know, in the Meeks-Rosenberg theorem, if parabolicity alone suffices:

Conjecture. If $g : \mathbb{C} \to \mathbb{R}^3$ is a conformal harmonic embedding, then $g(\mathbb{C})$ is either a flat plane or a helicoid.

There is a compelling analogy between the theory of conformal harmonic *embeddings* of the open unit disc $D \subset \mathbb{C}$ into $\mathbb{R}^3$, and the very rich theory of holomorphic *univalent* functions on D. It is an easy matter to use a scaling argument, as it was done above, together with the (classical) Bieberbach estimate to establish the scarcity of univalent entire functions: they are all of the form $f(z) = az + b$, $a \neq 0$. One ought to regard this statement as the complex-analytic analogue of the above conjecture.

A major hurdle in trying to use similar scaling arguments to settle the above conjecture is that one does not have an *a priori* control on the dynamic term in (1.2). Nevertheless, a first step in the programme of using complex analysis towards studying embedded minimal discs has been taken in [12], where a new proof was given of the classical theorem of Catalan, characterizing (pieces of) planes and helicoids as the only ruled minimal surfaces. The proof of Catalan's theorem is reduced, after careful normalizations, to the uniqueness of solutions of certain holomorphic differential equations.

The present work is in fact part of a much larger programme, going well beyond minimal surface theory, whose aim is to identify the analytic, geometric and topological mechanisms behind the phenomenon of global injectivity; see, for instance, [1]-[3], [11], [13], [15]-[18], [21], [23]. The reader can find in [24], p.17, a short description of some of the recent results in the area of global injectivity.

In closing, we would like to point out that the main result in [23] – a rigidity theorem characterizing the identity map of $\mathbb{C}^n$ among injective local biholomorphisms –, is also based on ideas suggested by the original Bieberbach estimate.

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2 Proof of the generalized Bieberbach estimate

The proof of Theorem 1.1 will be split into a series of lemmas, reflecting the dynamical, complex-analytic and Riemannian aspects of the argument.

Lemma 2.1. *Let $\widetilde{X}$ be a smooth vector field defined in an open set $U \subset \mathbb{R}^n$. If $\widetilde{X}(p) = 0$ and $(d\widetilde{X})_p = -I$, then the local flow η_t of $\widetilde{X}$ satisfies*

(i) $(d\eta_t)_p = e^{-t} I,\ t > 0.$

(ii) $\lim_{t\to\infty} \|(d\eta_t)_p\|^{-1}(d^2\eta_t)_p(v, w) = (d^2\widetilde{X})_p(v, w), \quad v, w \in \mathbb{R}^n.$

Proof. **(i)** For all $x \in U$ and $v \in \mathbb{R}^n$,

$$\begin{aligned} \frac{d}{dt}(d\eta_t)_x(v) &= \frac{d}{dt}\left\{ \frac{d}{ds}\Big|_{s=0} \eta_t(x+sv) \right\} = \frac{d}{ds}\Big|_{s=0} \left(\frac{d}{dt}\eta_t(x+sv) \right) \\ &= \frac{d}{ds}\Big|_{s=0} \widetilde{X} \circ \eta_t(x+sv) = (d\widetilde{X})_{\eta_t(x)} \circ (d\eta_t)_x(v). \end{aligned} \tag{2.1}$$

Since $\widetilde{X}(p) = 0$, $\eta_t(p) = p$, for all t. Setting $x = p$ in (2.1), one obtains

$$\frac{d}{dt}(d\eta_t)_p(v) = (d\widetilde{X})_p \circ (d\eta_t)_p(v) = -(d\eta_t)_p(v). \tag{2.2}$$

Since $\eta_0(x) = x$ for all x, $(d\eta_0)_p = I$ and, by (2.2),

$$(d\eta_t)_p(v) = e^{-t}(d\eta_0)_p(v) = e^{-t}v, \quad v \in \mathbb{R}^n.$$

(ii) For all $x \in U$ and $v, w \in \mathbb{R}^n$,

$$(d^2\eta_t)_x(v,w) = \frac{d}{ds}\Big|_{s=0} \left\{ \frac{d}{dh}\Big|_{h=0} \eta_t(x+sv+hw) \right\}. \tag{2.3}$$

Thus,

$$\begin{aligned} \frac{d}{dt}(d^2\eta_t)_x(v,w) &= \frac{d}{ds}\Big|_{s=0} \left\{ \frac{d}{dh}\Big|_{h=0} \frac{d}{dt}\eta_t(x+sv+hw) \right\} \\ &= \frac{d}{ds}\Big|_{s=0} \left\{ \frac{d}{dh}\Big|_{h=0} \widetilde{X} \circ \eta_t(x+sv+hw) \right\} \\ &= d^2(\widetilde{X} \circ \eta_t)_x(v,w). \end{aligned} \tag{2.4}$$

It follows that

$$\frac{d}{dt}(d^2\eta_t)_x(v,w) = (d^2\widetilde{X})_{\eta_t(x)}\big((d\eta_t)_x v, (d\eta_t)_x w\big) + (d\widetilde{X})_{\eta_t(x)}\big((d^2\eta_t)_x(v,w)\big). \tag{2.5}$$

Taking $x = p$ in (2.5) and recalling that $(d\widetilde{X})_p = -I$, we have

$$\begin{aligned} \frac{d}{dt}(d^2\eta_t)_p(v,w) &= (d^2\widetilde{X})_p\big((d\eta_t)_p v, (d\eta_t)_p w\big) + (d\widetilde{X})_p\big((d^2\eta_t)_p(v,w)\big) \\ &= (d^2\widetilde{X})_p\big((d\eta_t)_p v, (d\eta_t)_p w\big) - (d^2\eta_t)_p(v,w). \end{aligned} \tag{2.6}$$

From (i) and (2.6), one has

$$\frac{d}{dt}(d^2\eta_t)_p(v,w) = e^{-2t}(d^2\tilde{X})_p(v,w) - (d^2\eta_t)_p(v,w). \tag{2.7}$$

Letting $V(t) = (d^2\eta_t)_p(v,w)$ and $W = (d^2\tilde{X})_p(v,w)$, the last equation becomes

$$\frac{d}{dt}V(t) = -V(t) + e^{-2t}W, \tag{2.8}$$

so that

$$\frac{d}{dt}(e^t V(t)) = e^{-t} W, \tag{2.9}$$

and, by integration,

$$e^t V(t) - V(0) = W(1 - e^{-t}). \tag{2.10}$$

Since $\eta_0(x) = x$, $x \in U$, it follows that $V(0) = (d^2\eta_0)_p(v, w) = 0$. Using this in (2.10), we have

$$e^t V(t) = W(1 - e^{-t}). \tag{2.11}$$

Taking the limit as $t \to \infty$ and using (i) one obtains (ii). □

Lemma 2.2. *Let $f : D \to \mathbb{R}^n$ and $g : D \to \mathbb{R}^n$ be two conformal embeddings of the unit disc into $\mathbb{R}^n$ such that $g(D) \subset f(D)$ and $f(0) = g(0)$. Assume that the orientations induced on $g(D)$ by f and g are the same. Then there exists $\zeta \in \mathbb{C}$, $|\zeta| = 1$, such that*

$$\left\| \frac{g_{zz}(0)}{\|g_z(0)\|^2} - \zeta \frac{f_{zz}(0)}{\|f_z(0)\|^2} \right\| \leq \frac{4}{\|g_z(0)\|}. \tag{2.12}$$

Proof. The function $\varphi = f^{-1} \circ g : D \to D$ is well defined, conformal and orientation preserving. Hence φ is holomorphic, $\varphi(0) = 0$. Using $g_z(0) = f_z(0)\varphi_z(0)$ and $g_{zz}(0) = f_{zz}(0)\varphi_z(0)^2 + f_z(0)\varphi_{zz}(0)$, one computes

$$\frac{g_{zz}(0)}{\|g_z(0)\|^2} = \frac{f_{zz}(0)}{\|f_z(0)\|^2} \frac{\varphi_z(0)^2}{|\varphi_z(0)|^2} + \frac{f_z(0)\,\varphi_{zz}(0)}{\|f_z(0)\|^2\,|\varphi_z(0)|^2}. \tag{2.13}$$

But

$$\left\| \frac{f_z(0)\,\varphi_{zz}(0)}{\|f_z(0)\|^2\,|\varphi_z(0)|^2} \right\| \leq \frac{4}{\|f_z(0)\| |\varphi_z(0)|} = \frac{4}{\|g_z(0)\|}, \tag{2.14}$$

by Bieberbach's inequality $|\psi''(0)| \leq 4|\psi'(0)|$, valid for all univalent holomorphic functions $\psi : D \to \mathbb{C}$. Hence

$$\left\| \frac{g_{zz}(0)}{\|g_z(0)\|^2} - \frac{\varphi_z(0)^2}{|\varphi_z(0)|^2} \frac{f_{zz}(0)}{\|f_z(0)\|^2} \right\| \leq \frac{4}{\|g_z(0)\|} \tag{2.15}$$

and the lemma follows from (2.15) with $\zeta = \frac{\varphi_z(0)^2}{|\varphi_z(0)|^2}$. □

Lemma 2.3. *Let $N \subset \mathbb{R}^n$, $n \geq 2$, be a smooth surface and $f : (D, 0) \to (N, p)$ an injective smooth conformal immersion. Let U be an open neighborhood of N in $\mathbb{R}^n$ and $\widetilde{X}$ a smooth vector field on U such that:*
(i) $\widetilde{X}(p) = 0$, $(d\widetilde{X})_p = -I$.
(ii) $\widetilde{X}$ *is tangent to* N, $\widetilde{X}|_N$ *is conformal and every positive orbit of* $\widetilde{X}|_N$ *converges to* p.
Then

$$\big\|(d^2\widetilde{X})_p\big(f_z(0), f_z(0)\big) + f_{zz}(0)\big\| \leq 4\|f_z(0)\|. \tag{2.16}$$

Proof. We may assume that $f(D)$ has compact closure in N (the general case follows by replacing $f(z)$ by $f(Rz)$, with $R < 1$, and letting $R \to 1$). Since every positive orbit of $\widetilde{X}|_N$ converges to p, there exists $t_0 > 0$ such that the positive flow η_t of $\widetilde{X}$ satisfies $\eta_t(f(D)) \subset f(D)$ for all $t \geq t_0$. Since $\widetilde{X}|_N$ is conformal, so is $g^{(t)} = g = \eta_t \circ f$. It follows easily from Lemma 2.1 (i) that f and g induce the same orientations on $g(D) \subset f(D)$, as required by Lemma 2.2. We have

$$g_z(z) = (\eta_t \circ f)_z(z) = (d\eta_t)_{f(z)}(f_z(z)) \tag{2.17}$$

and

$$g_{zz}(0) = (d^2\eta_t)_p(f_z(0), f_z(0)) + (d\eta_t)_p(f_{zz}(0)). \tag{2.18}$$

By (2.12),

$$\frac{\|g_{zz}(0)\|}{\|g_z(0)\|^2} \leq \frac{4}{\|g_z(0)\|} + \frac{\|f_{zz}(0)\|}{\|f_z(0)\|^2}. \tag{2.19}$$

From (2.17), (2.18) and (2.19), we then have

$$\frac{\|(d^2\eta_t)_p(f_z(0), f_z(0)) + (d\eta_t)_p(f_{zz}(0))\|}{\|(d\eta_t)_p\|^2\,\|f_z(0)\|^2} \leq \frac{4}{\|(d\eta_t)_p\|\,\|f_z(0)\|} + \frac{\|f_{zz}(0)\|}{\|f_z(0)\|^2} \tag{2.20}$$

Multiplying the last equation by $\|(d\eta_t)_p\|\|f_z(0)\|^2$,

$$\frac{\|(d^2\eta_t)_p(f_z(0), f_z(0)) + (d\eta_t)_p(f_{zz}(0))\|}{\|(d\eta_t)_p\|} \leq 4\|f_z(0)\| + \|(d\eta_t)_p\|\,\|f_{zz}(0)\|. \tag{2.21}$$

Taking the limit as $t \to \infty$ and using Lemma 2.1, one obtains (2.16). □

In the next lemma we denote by $\overline{\nabla}$ and ∇ the Riemannian connections of $\mathbb{R}^n$ and M, respectively, and by σ the second fundamental form of M.

Lemma 2.4. *Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold and X a smooth vector field on M such that $X(p) = 0$ and $\nabla_v X = -v$, for some $p \in M$ and all $v \in T_pM$. Then, for any extension $\widetilde{X}$ of X to an open neighborhood of p in $\mathbb{R}^n$ and all $v, w \in T_pM$, one has*

$$(d^2\widetilde{X})_p(v, w) = (\nabla^2 X)_p(v, w) - \overline{\nabla}_{\sigma(v,w)}\,\widetilde{X} - 2\sigma(v, w). \tag{2.22}$$

Proof. Writing $u_1, \ldots, u_n$ for the canonical basis of $\mathbb{R}^n$ one has, for all $v, w \in \mathbb{R}^n$,

$$(d^2\widetilde{X})_p(v,w) = \sum_{i,j=1}^{n} \frac{\partial^2 \widetilde{X}}{\partial x_i \partial x_j}(p)\langle v, u_i\rangle\langle w, u_j\rangle, \tag{2.23}$$

so that the coordinates of this vector are given by

$$\begin{aligned} \sum_{i,j=1}^{n} \frac{\partial^2 \widetilde{X}_k}{\partial x_i \partial x_j}(p)\langle v, u_i\rangle\langle w, u_j\rangle &= \sum_{i,j=1}^{n} \langle v, u_i\rangle\langle w, u_j\rangle \mathrm{hess}\widetilde{X}_k(p)(u_i, u_j) \\ &= \mathrm{hess}\widetilde{X}_k(p)(v,w), \quad k = 1, \ldots, n. \end{aligned} \tag{2.24}$$

Choose an orthonormal frame field $\{e_1, \ldots, e_n\}$ in an open neighborhood of p in such a way that $e_1, \ldots, e_m$ are tangent along M, $e_{m+1}, \ldots, e_n$ are normal along M and $\nabla_{e_i} e_j(p) = 0$, $i, j = 1, \ldots, m$. Hence, by (2.24),

$$\begin{aligned} \sum_{i,j=1}^{n} \frac{\partial^2 \widetilde{X}_k}{\partial x_i \partial x_j}(p)\langle v, u_i\rangle\langle w, u_j\rangle &= \sum_{i,j=1}^{n} v_i w_j \mathrm{hess}\widetilde{X}_k(p)(e_i, e_j) \\ &= \sum_{i,j=1}^{n} \langle \overline{\nabla}_{e_i} \mathrm{grad}\widetilde{X}_k(p), e_j(p)\rangle v_i w_j, \end{aligned} \tag{2.25}$$

where the $v_i's$ and $w_i's$ are the components of v and w in the basis $\{e_1, \ldots, e_n\}$. It follows that

$$\begin{aligned} \sum_{i,j=1}^{n} \frac{\partial^2 \widetilde{X}_k}{\partial x_i \partial x_j}(p)\langle v, u_i\rangle\langle w, u_j\rangle &= \sum_{i,j=1}^{n} v_i w_j \big(e_i\langle \mathrm{grad}\widetilde{X}_k, e_j\rangle - \langle \mathrm{grad}\widetilde{X}_k, \overline{\nabla}_{e_i} e_j\rangle\big)(p) \\ &= \sum_{i,j=1}^{n} v_i w_j \big(e_i e_j(\widetilde{X}_k) - \overline{\nabla}_{e_i} e_j(\widetilde{X}_k)\big)(p). \end{aligned} \tag{2.26}$$

From (2.23) and (2.26) we obtain,

$$(d^2\widetilde{X})_p(v,w) = \sum_{i,j=1}^{n} \big(\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}\widetilde{X}\big)(p) v_i w_j - \sum_{i,j=1}^{n} \big(\overline{\nabla}_{\overline{\nabla}_{e_i} e_j}\widetilde{X}\big)(p) v_i w_j. \tag{2.27}$$

Suppose now that v and w are tangent to M. Since the restriction to M of $\{e_1, \ldots, e_n\}$ is an adapted frame satisfying $\nabla_{e_i} e_j(p) = 0$, $i, j = 1, \ldots, m$, and $\widetilde{X}$ restricted to M is X, it follows from (2.27) and the Gauss equation [10]

$$\overline{\nabla}_Y X = \nabla_Y X + \sigma(X, Y) \tag{2.28}$$

that

$$\begin{aligned} (d^2\widetilde{X})_p(v,w) &= \sum_{i,j=1}^{m} \big(\overline{\nabla}_{e_i}\overline{\nabla}_{e_j} X\big)(p) v_i w_j - \sum_{i,j=1}^{m} \big(\overline{\nabla}_{\sigma(e_i,e_j)}\widetilde{X}\big) v_i w_j \\ &= \sum_{i,j=1}^{m} \big(\overline{\nabla}_{e_i}\overline{\nabla}_{e_j} X\big)(p) v_i w_j - \overline{\nabla}_{\sigma(v,w)}\widetilde{X}. \end{aligned} \tag{2.29}$$

We will now determine the first term on the right hand side of (2.29). From

$$\overline{\nabla}_{e_j} X = \nabla_{e_j} X + \sigma(e_j, X),$$

we have at p, for $1 \leq i, j \leq m$,

$$\overline{\nabla}_{e_i}\overline{\nabla}_{e_j} X = \overline{\nabla}_{e_i}\nabla_{e_j} X + \overline{\nabla}_{e_i}\sigma(e_j, X) = \nabla_{e_i}\nabla_{e_j} X + \sigma(e_i, \nabla_{e_j} X) + \overline{\nabla}_{e_i}\sigma(e_j, X).$$

Hence

$$\begin{aligned}\sum_{i,j=1}^{m}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_j} X)(p)v_i w_j &= \sum_{i,j=1}^{m}(\nabla_{e_i}\nabla_{e_j} X)(p)v_i w_j + \sum_{i,j=1}^{m}\sigma(e_i, \nabla_{e_j} X)(p)v_i w_j \\ &+ \sum_{i,j=1}^{m}\left(\overline{\nabla}_{e_i}\sigma(e_j, X)\right)(p)v_i w_j. \qquad (2.30)\end{aligned}$$

The third term on the right hand side of (2.30) involves the quantity $\overline{\nabla}_{e_i}\sigma(e_j, X)$ which can be computed using the normal connection $\nabla^{\perp}$:

$$\begin{aligned}\overline{\nabla}_{e_i}\sigma(e_j, X) &= \nabla^{\perp}_{e_i}\sigma(e_j, X) + (\overline{\nabla}_{e_i}\sigma(e_j, X))^T \\ &= (\nabla^{\perp}_{e_i}\sigma)(e_j, X) + \sigma(e_j, \nabla_{e_i} X) + \sum_{k=1}^{m}\left\langle(\overline{\nabla}_{e_i}\sigma(e_j, X))^T, e_k\right\rangle e_k \\ &= (\nabla^{\perp}_{e_i}\sigma)(e_j, X) + \sigma(e_j, \nabla_{e_i} X) + \sum_{k=1}^{m}\left\langle\overline{\nabla}_{e_i}\sigma(e_j, X), e_k\right\rangle e_k \\ &= (\nabla^{\perp}_{e_i}\sigma)(e_j, X) + \sigma(e_j, \nabla_{e_i} X) - \sum_{k=1}^{m}\left\langle\sigma(e_j, X), \overline{\nabla}_{e_i} e_k\right\rangle e_k \\ &= (\nabla^{\perp}_{e_i}\sigma)(e_j, X) + \sigma(e_j, \nabla_{e_i} X) - \sum_{k=1}^{m}\left\langle\sigma(e_j, X), \sigma(e_i, e_k)\right\rangle e_k. \qquad (2.31)\end{aligned}$$

From (2.30) and (2.31),

$$\begin{aligned}\sum_{i,j=1}^{m}(\overline{\nabla}_{e_i}\overline{\nabla}_{e_j} X)v_i w_j &= \sum_{i,j=1}^{m}(\nabla_{e_i}\nabla_{e_j} X)v_i w_j + \sigma(v, \nabla_w X) + \sum_{i,j=1}^{m}(\nabla^{\perp}_{e_i}\sigma)(e_j, X)v_i w_j \\ &+ \sum_{i,j=1}^{m}\sigma(e_j, \nabla_{e_i} X)v_i w_j - \sum_{i,j,k=1}^{m}\Big(\left\langle\sigma(e_j, X), \sigma(e_i, e_k)\right\rangle e_k\Big)v_i w_j \\ &= \sum_{i,j=1}^{m}(\nabla_{e_i}\nabla_{e_j} X)v_i w_j + \sigma(v, \nabla_w X) + (\nabla^{\perp}_{v}\sigma)(w, X) \\ &+ \sigma(w, \nabla_v X) - \sum_{k=1}^{m}\langle\sigma(w, X), \sigma(v, e_k)\rangle e_k. \qquad (2.32)\end{aligned}$$

Since, by assumption, $X(p) = 0$ and $(\nabla X)_p = -I$, we have

$$\sum_{i,j=1}^{m} (\overline{\nabla}_{e_i} \overline{\nabla}_{e_j} X)(p) v_i w_j = \sum_{i,j=1}^{m} (\nabla_{e_i} \nabla_{e_j} X)(p) v_i w_j - 2\sigma(v, w). \tag{2.33}$$

It follows from (2.29) and (2.33) that

$$(d^2 \widetilde{X})_p(v, w) = \sum_{j=1}^{m} (\nabla_v \nabla_{e_j} X)(p) w_j - 2\sigma(v, w) - \overline{\nabla}_{\sigma(v,w)} \widetilde{X}. \tag{2.34}$$

Let V and W be arbitrary smooth extensions of v and w to an open neighborhood of p in M. Using again $X(p) = 0$ and $(\nabla X)_p = -I$ and recalling that $\nabla_{e_j} e_i(p) = 0$, we have, from the definition of the curvature tensor R,

$$\begin{aligned} \sum_{j=1}^{m} (\nabla_v \nabla_{e_j} X)(p) w_j &= \sum_{j=1}^{m} \big(R(V, e_j)X + \nabla_{e_j} \nabla_V X + \nabla_{[V, e_j]} X\big)(p) w_j \\ &= \nabla_W \nabla_V X(p) + \nabla_W V(p). \end{aligned} \tag{2.35}$$

From (2.34) and (2.35) we obtain

$$(d^2 \widetilde{X})_p(v, w) = \nabla_W \nabla_V X(p) + \nabla_W V(p) - \overline{\nabla}_{\sigma(v,w)} \widetilde{X} - 2\sigma(v, w). \tag{2.36}$$

On the other hand,

$$\begin{aligned} (\nabla^2 X)_p(v, w) &= \big(\nabla_w \nabla X\big)(v) = \big(\nabla_W \nabla X(V)\big)(p) - \nabla X\big(\nabla_W V\big)(p) \\ &= \nabla_W \nabla_V X(p) - \nabla_{\nabla_W V} X(p). \end{aligned} \tag{2.37}$$

Since $(\nabla X)_p = -I$, we thus obtain

$$(\nabla^2 X)_p(v, w) = \nabla_W \nabla_V X(p) + \nabla_W V(p). \tag{2.38}$$

Formula (2.22) now follows from (2.34), (2.35) and (2.38). □

Proof of Theorem 1.1. We begin by showing that $\mathfrak{X} \neq \emptyset$. Being simply-connected, the Riemannian surface N is globally conformally flat and so it is isometric to $(\Omega, \widetilde{g})$, where Ω is either D or $\mathbb{C}$, $\widetilde{g} = e^{2\varphi} g$ for some smooth function φ, and g is the standard flat metric on Ω. Writing $\widetilde{\nabla}$ and ∇ for the Riemannian connections of $\widetilde{g}$ and g, respectively, one has ([20] p. 172):

$$\widetilde{\nabla}_Y X = \nabla_Y X + Y(\varphi) X + X(\varphi) Y - \langle X, Y \rangle \nabla \varphi. \tag{2.39}$$

Being holomorphic, $X(z) = -z$ generates a flow of conformal maps that clearly leaves Ω invariant. It now follows from (2.39) that $\widetilde{\nabla}_Y X = \nabla_Y X = -Y$ at 0, thus showing that $X \in \mathfrak{X}$.

We now proceed to establish the estimate (1.2). Since N is contractible, one can choose a global orthonormal frame $\{\xi_1, \ldots, \xi_{n-2}\}$ of the normal bundle of N in $\mathbb{R}^n$. Extend X to an open neighborhood of N in $\mathbb{R}^n$ by

$$\widetilde{X}\left(q + \sum_{i=1}^{n-2} s_i\xi_i(q)\right) = X(q) - \sum_{i=1}^{n-2} s_i\xi_i(q), \quad |s_i| < \epsilon(q), \tag{2.40}$$

with $\epsilon(q)$ sufficiently small, $q \in N$. For all $v \in T_pN$ we have

$$(d\widetilde{X})_p v = \overline{\nabla}_v \widetilde{X} = \overline{\nabla}_v X = \nabla_v X + \sigma(v, X(p)) = -v,$$

since $X(p) = 0$ and $(\nabla X)_p = -I$. On the other hand,

$$\begin{aligned}\overline{\nabla}_{\xi_i(p)}\widetilde{X} = (d\widetilde{X})_p\xi_i(p) &= \frac{d}{ds}\Big|_{s=0}\widetilde{X}\big(p + s\xi_i(p)\big) \\ &= \frac{d}{ds}\Big|_{s=0}\big(X(p) - s\xi_i(p)\big) = -\xi_i(p).\end{aligned}$$

It follows from the last two equations that $(\overline{\nabla}\widetilde{X})_p = (d\widetilde{X})_p = -I$. By Lemma 2.3, we have

$$\big\|(d^2\widetilde{X})_p\big(f_z(0), f_z(0)\big) + f_{zz}(0)\big\| \le 4\|f_z(0)\|. \tag{2.41}$$

Also, from Lemma 2.4 and $(\overline{\nabla}\widetilde{X})_p = -I$,

$$\begin{aligned}(d^2\widetilde{X})_p(v, w) &= (\nabla^2 X)_p(v, w) - \overline{\nabla}_{\sigma(v,w)}\widetilde{X} - 2\sigma(v, w), \\ &= (\nabla^2 X)_p(v, w) - \sigma(v, w), \quad v, w \in T_pN, \end{aligned} \tag{2.42}$$

which, in particular, implies that $(\nabla^2 X)_p$ is symmetric. From $f_z(0) = \frac{1}{2}\big(f_x(0) - if_y(0)\big)$ and

$$\begin{aligned}(d^2\widetilde{X})_p\big(f_z(0), f_z(0)\big) &= \frac{1}{4}(d^2\widetilde{X})_p\big(f_x(0) - if_y(0), f_x(0) - if_y(0)\big) \\ &= \frac{1}{4}\Big\{(d^2\widetilde{X})_p\big(f_x(0), f_x(0)\big) - (d^2\widetilde{X})_p\big(f_y(0), f_y(0)\big) \\ &- 2i(d^2\widetilde{X})_p\big(f_x(0), f_y(0)\big)\Big\},\end{aligned} \tag{2.43}$$

we obtain, applying (2.42) to each term in (2.43) and rearranging the terms,

$$(d^2\widetilde{X})_p\big(f_z(0), f_z(0)\big) = (\nabla^2 X)_p\big(f_z(0), f_z(0)\big) - \sigma\big(f_z(0), f_z(0)\big). \tag{2.44}$$

Formula (1.2) follows from (2.41) and (2.44). □

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