The Gauss Map of a Complete Non-Flat Minimal Surface Cannot Omit 7 Points of the Sphere

Frederico Xavier

The Annals of Mathematics, 2nd Ser., Vol. 113, No. 1 (Jan., 1981), 211-214.

Annals of Mathematics, **113** (1981), 211-214

The Gauss map of a complete non-flat minimal surface cannot omit 7 points of the sphere

By FREDERICO XAVIER*

A well-known theorem of Osserman ([2], [3], [4]) states that the Gauss map of a complete minimal surface $M^2 \subset \mathbf{R}^3$ cannot omit a set of positive logarithmic capacity unless the surface is a plane. In this paper we improve Osserman's theorem by showing that the Gauss map of M^2 omits at most 6 points (provided M^2 is not flat). It should be pointed out, however, that no example is known where the omitted set has 5 points. Therefore the problem of determining the exact size of the omitted set remains unsolved (we refer to [2] and [3] for a historical account as well as for general facts pertaining to minimal surfaces).

We take the opportunity to thank L. Petrola for sharing his insights with us and Professor J. Shah for some very enlightening conversations on related matters.

The Theorem

For the sake of clarity we shall state our result more precisely.

THEOREM. *The complement of the image of the Gauss map of a non-flat complete minimal surface in* $\mathbf{R}^3$ *contains at most* 6 *points of* S^2.

We need the following two results

THEOREM ([5], Theorem 1). *Let M be a complete Riemannian manifold of infinite volume and u a non-negative function satifying* $\Delta \log u = 0$ *almost everywhere. Then* $\int_M u^p = \infty$ *for* $p > 0$.

LEMMA. *Let f be a holomorphic function in the unit disk D and let* $f \neq 0, a$. *Let* $\alpha = 1 - 1/k$, $k \in \mathbf{Z}^+$. *Then we have*

$$\frac{|f'|}{|f|^{\alpha} + |f|^{2-\alpha}} \in L^p(D)$$

for every p *with* $0 < p < 1$.

0003-486X/81/0113-1/0211/004 $ 00.20/1

* Supported by FINEP, CNPq and UFPe (BRAZIL).

Proof. Let us recall that a function g on D is called normal if the family $\{g(S(z))\}$, where S is a conformal transformation of D into itself, is normal in Montel's sense. Since $f^{1/k}$ omits two values, it is normal (see [1], page 169). By Theorem 6.5 of [1], there is a constant C such that

$$\frac{|g'|}{1+|g|^2} \leqq \frac{C}{1-|z|^2}\ .$$

Applying this estimate on the spherical derivative to $f^{1/k}$, we have

$$\frac{|f'|}{k|f|^{1-1/k}(1+|f|^{2/k})} \leqq \frac{C}{1-|z|^2}$$

so that

$$\frac{|f'|}{|f|^{1-1/k}+|f|^{2-(1-1/k)}} \leqq \frac{kC}{1-|z|^2}\ .$$

In particular, $|f'|/(|f|^{\alpha}+|f|^{2-\alpha}) \in L^p(D)$, $0 < p < 1$.

Proof of the theorem

Suppose that M is a complete non-flat minimal surface whose Gauss map misses 7 points. By arguing as in the proof of the theorem of Osserman quoted above (see [3], for details), we may assume that $M = D$ and the metric is $\lambda(z)|dz|^2$ with $\lambda = |f|^2(1+|g|^2)^2$, where the functions f and g are holomorphic and $|f| > 0$. They come from the so-called Weierstrass representation of minimal surfaces. The nice thing about it is that after composition with the inverse stereographic projection, the map g becomes the Gauss map of the surface. The fact that g has no poles means that the north pole is among the omitted points. In view of this we are reduced to proving the following:

(*) *Let f, g be holomorphic functions on D, $|f| > 0$. Suppose that for six distinct complex numbers $a_1, a_2, \cdots, a_6$ the equation $g(z) = a_i$ has no solution ($i = 1, 2, \cdots, 6$). Then the metric $|f|^2(1+|g|^2)^2|dz|^2$ on D is not complete.*

For the proof consider the function

$$h = f^{-2/p} g' \textstyle\prod_{i=1}^{6} (g-a_i)^{-\alpha}\ ,$$

where $5/6 < \alpha < 1$ is as in the lemma and $p = 5/6\alpha$. Note that $f^{-2/p}$ is well-defined because $|f| > 0$. The Laplace-Beltrami operator Δ of the metric

$$\lambda|dz|^2 \quad (\lambda = |f|^2(1+|g|^2)^2)$$

is given by $(1/\lambda)(\partial/\partial z)(\partial/\partial \bar{z})$. Hence the function $u = |h|$ satisfies $\Delta \log u = 0$ almost everywhere in D (there may be a discrete set where g' vanishes). We assert that $u \notin L^p(M)$. Indeed, if u is a (necessarily non-zero) constant, this follows from the fact that complete simply-connected surfaces of nonpositive curvature have infinite area. If u is not constant this follows from Yau's theorem. Since the area element is $\lambda\, dxdy$, the condition $u \notin L^p(M)$ can be written

$$\int_D \frac{|g'|^p(1 + |g|^2)^2}{\prod_{i=1}^{6} |g - a_i|^{p\alpha}}\, dxdy = \infty \,.$$

The contradiction will be achieved by showing that this integral is actually finite. Let

$$D_j = \{z \in D \,|\, |g(z) - a_j| \leqq l\} \,,$$

where $0 < l < (1/4) \min_{i \neq k; i,k=1,\cdots,6} |a_i - a_k|$. Also, let $D' = D \backslash \bigcup_{j=1}^{6} D_j$. Denoting by $H(z)$ the integrand of the last integral we have

$$\int_D H\, dxdy = \sum_{j=1}^{6} \int_{D_j} H\, dxdy + \int_{D'} H\, dxdy \,.$$

On each D_j we have an estimate $H \leqq C(|g'|^p / |g - a_j|^{p\alpha})$. We may also assume $l < 1$, so that

$$\frac{|g'|^p}{|g - a_j|^{p\alpha}} \leqq 2^p \frac{|g'|^p}{(|g - a_j|^{\alpha} + |g - a_j|^{2-\alpha})^p} \,.$$

Hence $\int_{D_j} H\, dxdy < \infty$ by the lemma. The integral over D' can be handled in a similar fashion. We observe that

$$\frac{(1 + |g|^2)^2}{\prod_{j=1}^{5} |g - a_j|^{p\alpha}} = \frac{(1 + |g|^2)^2}{\prod_{j=1}^{5} |g - a_j|^{5/6}}$$

is bounded over D'. Hence

$$\int_{D'} H\, dxdy \leqq C \int_{D'} \frac{|g'|^p}{|g - a_6|^{p\alpha}}\, dxdy < \infty \,,$$

as before. This concludes the proof of (*).

Besides the people named in the introduction, we wish to thank the referee for suggesting an improvement in the paper which allowed us to bring down our original bound from 11 to 7. Also, we thank R. Osserman for clarifying certain aspects of the proof.

UNIVERSIDADE FEDERAL DE PERNAMBUCO, RECIFE, BRAZIL

REFERENCES

[1] W. K. HAYMAN, *Meromorphic Functions*, Oxford Mathematical Monographs, 1964.
[2] B. LAWSON, Lectures on minimal submanifolds, Instituto de Matemática Pura e Aplicada, Rio de Janeiro (1970).
[3] R. OSSERMAN, *A Survey of Minimal Surfaces*, Van Nostrand-Reinhold, N.Y., 1969.
[4] ———, Minimal surfaces in the large, Comm. Math. Helv. **35** (1961), 65–76.

[5] S.-T. YAU, Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry, Indiana U. Math. J. **25** (1976), 659-670.

(Received December 14, 1979)
(Revised September 8, 1980)