Good shadows, dynamics and convex hulls of complete submanifolds

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Abstract. Any non-empty open convex subset of $\mathbb{R}^n$ is the convex hull of a complete submanifold M, of any codimension, but there are obstructions if the geometry of M is, *a priori*, suitably controlled at infinity. In this paper we develop tools to explore the geometry of $\partial[\text{Conv}(M)]$ when the Grassmanian-valued Gauss map of M is *uniformly continuous*, a condition that, in the C^2 case, is weaker than bounding the second fundamental form of M. Our proofs are based on the Ekeland variational principle, and on a conceptual refinement of the Omori-Yau asymptotic maximum principle that is of interest in its own right. If the Ricci (sectional) curvature of M is bounded below and f is a C^2 function on M that is bounded above, then not only there exists *some* maximizing sequence for f with good properties, as predicted by the Yau (Omori) principle but, in fact, *every* maximizing sequence for f can be *shadowed* by a maximizing sequence that has good properties. This abundance of *good shadows* allows for *information to be localized at infinity,* revealing in our geometric setting the relation between the asymptotic behavior of M and the supporting hyperplanes of $\partial[\text{Conv}(M)]$ in general position that pass through some fixed boundary point. We also use ideas from dynamics to prove a special case of a conjecture meant to extend our refinement of the Yau maximum principle to manifolds that satisfy a property weaker than $\inf \text{Ric} > -\infty$. The authors expect that this new understanding of the Omori-Yau principle – in terms of good shadows and localization at infinity – will lead to applications in contexts other than convexity.

1 Introduction.

Any non-empty open convex subset $\mathcal{O}$ of $\mathbb{R}^n$ is the convex hull of a C^∞ complete submanifold M, of any codimension. To see this when $n \geq 3$, take a smooth embedded curve $\Gamma \subset \mathcal{O}$, of infinite length on both ends, whose convex hull is $\mathcal{O}$. Let M be the union over all $p \in \Gamma$ of smoothly varying k-dimensional spheres $S^{(k)}_{r(p)}$, $1 \leq k \leq n-2$, centered at p and contained in the normal space of Γ at p. Taking $r(p)$ to decay fast enough one can make sure that the resulting manifold M, which is automatically complete, is contained in $\mathcal{O}$. Since p is in the convex hull of $S^{(k)}_{r(p)}$ for any $p \in \Gamma$, it follows that the convex hull of M satisfies $\text{Conv}\,(M) = \mathcal{O}$.

Despite the examples of the previous paragraph, one expects that not every $\mathcal{O}$ can be realized as $\text{Conv}\,(M)$ if the complete submanifold M has a geometry that is, *a priori*, suitably controlled at infinity. More generally, we study $\partial[\text{Conv}h(M)]$, where $h : M \to \mathbb{R}^n$ is an immersion, $\dim M = m$, and the induced metric is complete. Along the way, we introduce new tools that may be useful in other problems as well.

A natural way to control the geometry of a submanifold is to bound its second fundamental form, but this requires the immersion to be at least of class C^2. Instead, we work here with a weaker condition that makes sense even in the C^1 case: the Grassmanian-valued Gauss map $\mathcal{G} : M \to G(n-m, n)$, given by $\mathcal{G}(p) = [h_* T_p M]^\perp$, is *uniformly continuous.* Indeed, if the immersion is C^2 then boundedness of the second fundamental form is equivalent to the Gauss map being Lipschitz, a condition that is stronger than uniform continuity (this can be seen using a Plücker-like embedding; see Section 4).

It is shown in Theorem 2.1 that the number of supporting hyperplanes in general position, at any point at the boundary of the convex hull of $h(M)$, is at most the codimension of the submanifold. This generalizes to these non-compact submanifolds the statement that if $M^{n-1} \subset \mathbb{R}^n$ is a C^1 compact hypersurface, then each point in $\partial[\text{Conv}\,(M^{n-1})]$ admits a unique supporting hyperplane and the resulting map $\partial[\text{Conv}\,(M^{n-1})] \to G(n-1, n)$ is continuous. Our main geometric result, Theorem 2.4, valid for the convex hull of certain non-compact immersed submanifolds in arbitrary codimension, ultimately generalizes the classical statement that if a compact convex body in $\mathbb{R}^n$ has a C^2 boundary, then the second fundamental form at any boundary point is semi-definite.

On the technical side, our results on the convex hull of $h(M)$ spring from two sources: the Ekeland variational principle for the C^1 case, and a conceptual refinement of the Omori-Yau maximum principle if the immersion is of class C^r, $r \geq 2$. In order to convey the flavor of these new results, which are of interest in their own right, we recall that the original Yau (Omori) maximum principle (see Section 3) guarantees, under the appropriate curvature restrictions, the existence of a maximizing sequence along which the gradient is small and the Laplacian (Hessian) of the function is almost negative.

We show in Theorem 3.4 that there is actually an abundance of these good maximizing sequences: *every* maximizing sequence has a *good shadow*, a second sequence which is good, in the above sense, and is such that the distance between the general terms of the two sequences tends to zero.

The advantage of this result over the various forms of the Omori-Yau principle to be found in the literature is that the new understanding allows for *information to be localized at infinity.* In terms of our applications to convexity, this translates into one being able to draw conclusions about the relation between the asymptotic behavior of $h(M)$, and the supporting hyperplanes of $\partial[\text{Conv}(M)]$ in general position that pass through some fixed boundary point.

In the first version of this paper our refinement of the Yau maximum principle (one of the halves of Theorem 3.4) was proved using ideas from dynamics, under the additional hypothesis that the Hessian of the function is bounded, and we conjectured that this assumption is superfluous. Afterwards, the possibility was raised to us that perhaps our new ideas, together with the traditional approach to the Omori-Yau principle, using arguments from Riemannian geometry, might be modified to yield a proof of our original conjecture on the abundance of good shadows when $\inf \text{Ric} > -\infty$. This is implemented in Theorem 3.4, the scope being broadened so as to cover also the Omori principle. On the other hand, most applications to convexity given in Section 2 were already present in the first version.

Our original technique using dynamics can be adapted to prove Theorem 5.1, still under the assumption that the Hessian of the function is bounded, but where the condition $\inf \mathrm{Ric} > -\infty$ has been replaced by the much weaker and flexible hypothesis that the manifold satisfies LVP (*Local Volume Property*).

The latter condition means that there exist $a > 0$, $b > 1$, such that for any $p \in M$ and $0 < r < a$, one has Vol $B(p,r) \leq b$Vol $B(p, \frac{r}{2})$. This class of manifolds is rather large, containing all complete manifolds that are quasi-isometric to manifolds satisfying $\inf \mathrm{Ric} > -\infty$. More generally, if $f : (M, g) \to (N, h)$ and there exist $c_1, c_2 > 0$ such that $c_1|v| \leq |df(v)| \leq c_2|v|$ for all tangent vectors v, then (M, g) is LVP if and only if (N, h) is. We conjecture that the conclusion in the Yau maximum principle should hold for all complete LVP manifolds.

The main idea behind the proof of Theorem 5.1 is to establish estimates that control the volume compression under the gradient flow. In principle, a similar technique should also work for any differential operator given in divergence form ([14]).

The authors expect that this new understanding of the Omori-Yau principle – in terms of good shadows and localization at infinity – will lead to applications in contexts other than convexity. In this regard, a specially interesting question is whether the classical Ahlfors-Schwarz Lemma (or its far-reching generalization, the Yau-Schwarz Lemma [21]) can be sharpened using Theorem 3.4.

2 Convex hulls and controlled submanifold geometry.

In this section we state our geometric results. The proofs, which are based on the Ekeland variational principle ([5]) and on our refinements of the asymptotic maximum principles of Omori ([12]) and Yau ([4],[20]), will be given in Section 4.

Isometric C^1 immersions are plentiful, thanks to the Nash-Kuiper theorem [9], but their geometry is hard to control since one cannot make sense of extrinsic curvatures. Nevertheless, under the hypothesis that the Grassmanian-valued Gauss map is uniformly continuous, one has a fairly good description of their convex hulls (an immersion into $\mathbb{R}^n$ is *substantial* if its image is not contained in a proper affine subspace):

Theorem 2.1. *Let M be a complete m-dimensional Riemannian manifold, $n > m$, and $h : M \to \mathbb{R}^n$ a substantial C^1 isometric immersion for which the Grassmanian-valued Gauss map $\mathcal{G} : M \to G(n-m, n)$, given by $\mathcal{G}(p) = [h_* T_p M]^\perp$, is uniformly continuous. If $Conv(h(M)) \neq \mathbb{R}^n$, then each point in $\partial[Conv(h(M))]$ admits at most $n - m$ supporting hyperplanes in general position. In particular, in the hypersurface case each point in $\partial[Conv(h(M^{n-1}))]$ admits a unique supporting hyperplane, and the resulting map $\partial[Conv(h(M^{n-1}))] \to G(n-1, n)$ is continuous.*

Although the result below is probably known, we were unable to locate a reference in the literature:

Corollary 2.2. *If $h : M^m \to \mathbb{R}^n$ is a substantial C^1 immersion and M is compact, then each point in $\partial[Conv(h(M))]$ admits at most $n-m$ supporting hyperplanes in general position. In particular, if M^{n-1} is compact and $h : M^{n-1} \to \mathbb{R}^n$ is a C^1 immersion, then h is substantial, each point in $\partial[Conv(h(M^{n-1}))]$ admits a unique supporting hyperplane, and the resulting map $\partial[Conv(h(M^{n-1}))] \to G(n-1,n)$ is continuous.*

Before proceeding to discuss our results on convex hulls of C^2 immersions, we need to establish some terminology first. Given a substantial convex set $A \subset \mathbb{R}^n$, a supporting hyperplane H containing $p \in A$, and a unit vector e that is normal to H, we say that e is a *positive normal* for H if A intersects the component of $\mathbb{R}^n - H$ determined by e.

Near the intersection of supporting hyperplanes that pass through the same point of $\partial[\text{Conv}\,(h(M^m))]$ one expects, if h is at least C^2 and the geometry of $h(M^m)$ is not too wild, that asymptotically the submanifold will be bending mostly towards the positive side of the associated half-spaces. The theorem below confirms this intuition.

Theorem 2.3. *Let M^m be a complete manifold with Ricci curvature bounded from below, $n > m$, and $h : M^m \to \mathbb{R}^n$ a substantial C^2 isometric immersion for which the Gauss map $\mathcal{G} : M \to G(n-m,n)$ and the mean curvature vector field $\overrightarrow{H} : M \to \mathbb{R}^n$ are uniformly continuous. Suppose $Conv(h(M^m)) \neq \mathbb{R}^n$ and let $H_1, \ldots, H_s$ be supporting hyperplanes in general position that pass through the same point in $\partial[Conv(h(M^m))]$, with corresponding positive normals $e_1, \ldots, e_s$. Then $s \leq n-m$, $d(h(M^m), H_1 \cap \cdots \cap H_s) = 0$, and*

$$\liminf_{d(h(p),H_1\cap\cdots\cap H_s)\to 0} \left\langle \overrightarrow{H}(p), e_i \right\rangle \geq 0, \quad i = 1, \ldots, s.$$

The previous theorem admits a version where the hypothesis on the Ricci curvature is replaced by the sectional curvatures being bounded below. Accordingly, the conclusion will be about the second fundamental form, instead of the mean curvature vector. But before we can state this result, we need to explain what it means to say, in our context, that the second fundamental form σ of an immersion $h : M^m \to \mathbb{R}^n$ is uniformly continuous. Denote by $Q_\sigma(x)$ the associated quadratic form, $Q_\sigma(x)(v) = \sigma(x)(v,v) \in [h_* T_x M]^\perp$ for every $v \in T_x M$.

Definition. The second fundamental form σ is said to be *uniformly continuous* if for every $\epsilon > 0$ there is $\delta > 0$ such that, for all $x, y \in M$ with $d(x,y) < \delta$, the Hausdorff distance in $\mathbb{R}^n$ between the sets given by the images under $Q_\sigma(x)$ and $Q_\sigma(y)$ of the unit spheres in the respective tangent spaces is less than ϵ. □

Remarks. In concrete terms, the above condition means that for all x, y in M with $d(x,y) < \delta$, and $v \in T_x M$, $|v| = 1$, there exists $w \in T_y M$, $|w| = 1$, such that

$$||\sigma(x)(v,v) - \sigma(y)(w,w)|| < \epsilon. \tag{2.1}$$

In particular, if σ is uniformly continuous, (x_n, v_n) is a sequence in the unit tangent bundle of M and $d(x_n, y_n) \to 0$, then there exist unit vectors $w_n \in T_{y_n}M$ such that

$$||\sigma(x_n)(v_n, v_n) - \sigma(y_n)(w_n, w_n)|| \to 0. \tag{2.2}$$

The Riemannian connection on M, together with the connection $\nabla^\perp$ in the normal bundle, given by $\nabla^\perp_X \xi = (\overline{\nabla}_X \overline{\xi})^\perp$, where $\overline{\nabla}$ is the Euclidean connection and $\overline{\xi}$ is a local extension of ξ, can be used to define the covariant derivative $\widetilde{\nabla}_u \sigma$ of the second fundamental form, with respect to a vector u tangent to M ([8], p. 25). It is possible to give a differential condition that implies uniform continuity of the second fundamental form. Indeed, it will be shown in Section 4 that σ is uniformly continuous if $|\sigma| + |\widetilde{\nabla}\sigma|$ is uniformly bounded on M. □

When the sectional curvatures are bounded below the conclusion in Theorem 2.3 can be sharpened. Asymptotically, the submanifold will bend towards the positive sides of the half-spaces determined by the supporting hyperplanes in general position, and not just in an average sense:

Theorem 2.4. *Let M^m be a complete manifold with sectional curvatures bounded from below, $n > m$, and $h : M^m \to \mathbb{R}^n$ a substantial C^2 isometric immersion for which the Gauss map $\mathcal{G} : M \to G(n-m, n)$ and the vector-valued second fundamental form σ are uniformly continuous. Suppose Conv$(h(M^m)) \neq \mathbb{R}^n$ and let $H_1, \ldots, H_s$ be supporting hyperplanes in general position that pass through the same point in $\partial[\text{Conv}(h(M^m))]$, with corresponding positive normals $e_1, \ldots, e_s$. Then $s \le n - m$, $d(h(M^m), H_1 \cap \cdots \cap H_s) = 0$, and*

$$\liminf_{d(h(p), H_1\cap\cdots\cap H_s)\to 0} \left[\min_{v\in T_pM,\ |v|=1} \langle \sigma(p)(v,v), e_i \rangle \right] \ge 0, \quad i = 1, \ldots, s.$$

In particular, the theorem applies when M^m is complete, $h(M^m)$ is contained in a half-space, and $|\sigma| + |\widetilde{\nabla}\sigma|$ is uniformly bounded.

One should note that, albeit somewhat technical in its formulation, the above theorem extends to the non-compact setting and to arbitrary codimension the following classical result about compact convex bodies:

Corollary 2.5. *If M^{n-1} is compact and $h : M^{n-1} \to \mathbb{R}^n$ is a C^2 immersion such that $h(M^{n-1})$ bounds a convex body, then the second fundamental forms of h are semi-definite.*

Examples. It is easy to illustrate Theorem 2.1, already in low dimensions:

i) Let l be a line in $\mathbb{R}^3$, and P_1, P_2 planes such that $P_1 \cap P_2 = l$. Let $\mathcal{O}$ be a component of $\mathbb{R}^3 - (P_1 \cup P_2)$. One can construct a complete C^∞ embedded curve $\Gamma \subset \mathcal{O}$ such that Conv $(\Gamma) = \mathcal{O}$ and Γ has bounded curvature. The last condition ensures that the Gauss map $\mathcal{G} : \Gamma \to G(2,3)$ is uniformly continuous. Along l, the maximum number of supporting hyperplanes to $\partial\mathcal{O}$ that are in general position is two, which is also the codimension of Γ. This gives the equality case in Theorem 2.1.

We give an informal description of how Γ can be constructed. Start with oriented line segments l_n parallel to l, $n \geq 1$, contained in $\mathcal{O}$, getting longer as $n \to \infty$, and accumulating onto the entire oriented line l. One obtains Γ by connecting for all $n \geq 1$ the last point of l_n, in a smooth way, to the first point of l_{n+1}, by means of a curve γ_n of curvature less than one. The curve γ_n is supposed to be very long, going deep inside $\mathcal{O}$ and turning slowly, so that the curvature can be kept smaller than one. Once γ_n is far from l, one can also make γ_n twist around, with controlled curvature, so as to make its convex hull bigger. It is now clear that a sequence of curves γ_n can be created so that Γ has curvature less than one and Conv $(\Gamma) = \mathcal{O}$.

Observe that such a construction is impossible if, instead of a curve, one takes Γ to be a complete surface. Indeed, as the surface gets closer and closer to l, in order for Γ to remain in $\mathcal{O}$ it has to fold abruptly, thus violating the condition that the Gauss map is uniformly continuous.

ii) Let M be an open hemisphere in $S^2 \subset \mathbb{R}^3$. Its convex hull is, of course, the solid hemisphere. At points along the great circle, Conv (M) has two supporting hyperplanes in general position, whereas the codimension of M is one. This shows that Theorem 2.1 fails, as expected, if the submanifold is not complete. □

Remarks. The uniform continuity condition on the Gauss map allows for the Ricci curvature of the submanifold to be unbounded from below. In fact, it is easy to construct smooth complete graphs in $\mathbb{R}^3$ with these properties. This shows that the Omori-Yau minimum principle cannot be applied to prove Theorem 2.1, even if the submanifold in question is of class C^∞. We also note that Theorem 2.1 applies to these examples, whereas Theorems 2.3 and 2.4 do not.

To put these matters in perspective we note that, by the Gauss equation, the natural way to force the intrinsic curvatures of a submanifold to be bounded is simply to require that the second fundamental form has bounded length. Although this is not obvious in codimension greater than one, as it was already observed in the Introduction the last condition simply means that the Gauss map is globally Lipschitzian, which is stronger than merely requiring the Gauss map to be uniformly continuous (see Section 4).

We stress that, in Theorem 2.1, even if the submanifold is C^∞ and has bounded second fundamental form, the original form of the Omori-Yau minimum principle cannot be applied. Indeed, as it will be clear from the proof, one needs to find good shadows (that are provided in the C^1 context by Theorem 3.3) for *arbitrary* minimizing sequences. The Omori-Yau minimum principle, on the other hand, guarantees the existence of *some* minimizing sequence with good properties.

A natural question that arises is whether the condition in Theorem 2.1, stating that the boundary points of the convex hull admits at most $n - m$ supporting hyperplanes in general position, is also sufficient for the construction of examples. We thank J. Fu for pointing out that the work of Alberti [1] may be relevant to this question.

The Grassmanian-valued Gauss map $\mathcal{G} : M \to G(n-m, n)$, $\mathcal{G}(p) = [h_* T_p M]^\perp$, can be used in other contexts to retrieve geometric information. In this regard, we refer the reader to [17] for its role in the problem of detecting when a family of compact submanifolds with boundary, perhaps of different dimensions, has a non-empty stable interior intersection. In turn, this problem is a facet of the more general question of deciding when maps are globally invertible ([19]). □

3 Abundance of good minimizing sequences.

If M^m is a Riemannian manifold and $f : M \to \mathbb{R}$ is a C^2-function that attains a minimum at a point $p \in M$, then $||\nabla f(p)|| = 0$ and $\text{Hess}f(p)(v, v) \geq 0$ for all $v \in T_p M$. Such a point p always exists in case M^m is compact but clearly it may not exist if M^m is non-compact, even if $\inf f > -\infty$.

For a C^2-function $f : \mathbb{R}^m \to \mathbb{R}$ which is bounded from below, there always exists a sequence (p_n) in $\mathbb{R}^m$ such that

$$f(p_n) \to \inf_M f, \quad ||\nabla f(p_n)|| \to 0, \quad \liminf_{n\to\infty} \text{Hess}f(p_n) \geq 0. \tag{3.1}$$

The last condition means that for every $\varepsilon > 0$ there exists $n_o \in \mathbb{N}$ such that

$$\text{Hess}f(p_n)(v, v) > -\varepsilon ||v||^2, \quad v \in \mathbb{R}^m,\ n > n_o. \tag{3.2}$$

In case M^m is an arbitrary complete non-compact Riemannian manifold, and $f : M \to \mathbb{R}$ is a C^2-function satisfying $\inf_M f > -\infty$, a sequence (p_n) satisfying (3.1) may not exist. This is the case, for instance, if M is a complete bounded minimal surface in $\mathbb{R}^3$ (see the discussion at the end of Section 5). However, Omori [12] proved the following.

Theorem 3.1. *Let M be a complete manifold whose sectional curvature is bounded from below, and $f : M \to \mathbb{R}$ of class C^2 such that $\inf_M f > -\infty$. Then, there exists a sequence (p_n) in M satisfying (3.1).*

Subsequently, Yau ([4],[20]) obtained the following version for complete manifolds with Ricci curvature bounded from below:

Theorem 3.2. *Let M be a complete manifold whose Ricci curvature is bounded from below, and $f : M \to \mathbb{R}$ of class C^2 such that $\inf_M f > -\infty$. Then, there exists a sequence (p_n) in M satisfying $f(p_n) \to \inf_M f$, $||\nabla f(p_n)|| \to 0$ and $\liminf_{n\to\infty} \Delta f(p_n) \geq 0$.*

Theorems 3.1 and 3.2, which together are known in the literatute as the Omori-Yau maximum (minimum) principle, became powerful tools in geometric analysis (see, for instance, [7],[15],[21]).

The following result can be regarded as a strong version, in the C^1 category, of the Omori-Yau minimum principle:

Theorem 3.3. *Let M be a complete manifold, and $f : M \to \mathbb{R}$ of class C^1 such that $\inf_M f > -\infty$. Then, for every sequence (p_n) in M such that $f(p_n) \to \inf_M f$, there exists a sequence (q_n) in M such that $d(p_n, q_n) \to 0$, $f(q_n) \to \inf_M f$ and $||\nabla f(q_n)|| \to 0$.*

In the proof of Theorem 3.3 we will use a well known result in control theory and non-linear analysis ([5], [6], [16]):

The Ekeland Variational Principle. *Let (X, d) be a complete metric space, and $f : X \to \mathbb{R}$ a function which is lower semi-continuous and bounded from below. Then for any $\varepsilon, \delta > 0$, and $x \in X$ with $f(x) \leq \inf_X f + \varepsilon$, there is $y \in X$ satisfying*
i) $d(x, y) \leq \delta$
ii) $f(y) \leq f(x)$
iii) $f(y) < f(z) + \frac{\varepsilon}{\delta} d(y, z)$, for all $z \in X$ with $z \neq y$.

Proof of Theorem 3.3. For each $n \in \mathbb{N}$, let $\varepsilon_n = f(p_n) - \inf_M f$, $\delta_n = \sqrt{\varepsilon_n}$. In the sequel we will prove the existence of a sequence (q_n) in M satisfying

$$f(q_n) \leq f(p_n), \quad d(p_n, q_n) \leq \delta_n \tag{3.3}$$

and

$$||\nabla f(q_n)|| \leq \delta_n. \tag{3.4}$$

Since $\delta_n \to 0$ as $n \to \infty$, (q_n) will have the desired properties. If $f(p_n) = \inf_M f$, take $q_n = p_n$. Otherwise, we have $\varepsilon_n > 0$, $\delta_n > 0$, and applying the Ekeland Variational Principle with $\varepsilon = \varepsilon_n$, $\delta = \delta_n$ and $x = p_n$, we obtain $q_n \in M$ satisfying (3.3) and

$$f(q_n) < f(z) + \frac{\varepsilon_n}{\delta_n} d(q_n, z) = f(z) + \delta_n \, d(q_n, z), \tag{3.5}$$

for all $z \in M$ with $z \neq q_n$. To show (3.4), take an arbitrary unit vector $v \in T_{q_n}M$ and let $\gamma : (-c, c) \to M$ be the unit speed geodesic in M so that $\gamma(0) = q_n$ and $\gamma'(0) = v$. Reducing c if necessary, we can suppose that the image of γ is contained in a normal neighborhood of q_n in M. From (3.5) we have, with $z = \gamma(t)$,

$$f\big(\gamma(t)\big) - f(q_n) > -\delta_n \, d\big(\gamma(t), q_n\big) = -\delta_n \, |t|, \quad 0 < |t| < c, \tag{3.6}$$

which implies

$$\frac{f\big(\gamma(t)\big) - f(q_n)}{t} < \delta_n, \quad -c < t < 0. \tag{3.7}$$

Since f is of class C^1, it follows that

$$\langle \nabla f(q_n), v \rangle = \frac{d}{dt}\Big|_{t=0} f \circ \gamma(t) = \lim_{t \to 0^-} \frac{f\big(\gamma(t)\big) - f(q_n)}{t} \leq \delta_n, \tag{3.8}$$

for all $v \in T_{q_n}M$ with $||v|| = 1$. Therefore,

$$\big|\langle \nabla f(q_n), v\rangle\big| \le \delta_n, \tag{3.9}$$

for all unit vector $v \in T_{q_n}M$, so that

$$||\nabla f(q_n)|| \le \delta_n, \quad n \in \mathbb{N}. \tag{3.10}$$

The sequence (q_n) satisfies (3.3) and (3.4) and thus the conditions of the theorem. $\square$

The result below strengthens the original asymptotic maximum principles in [12] and [20]. It can also be understood as a version of Theorem 3.3 in the C^2 setting of the Omori-Yau maximum principle.

Theorem 3.4. *Let M^m be a complete manifold with Ricci curvature (sectional curvature) bounded from below, and $f : M \to \mathbb{R}$ a function of class C^2 such that* $\inf f > -\infty$. *Then, for every minimizing sequence (p_n) of f there exists a sequence (q_n) such that*

$$d(p_n, q_n) \to 0, \;\; f(q_n) \to \inf_M f, \;\; ||\nabla f(q_n)|| \to 0 \tag{3.11}$$

and

$$\liminf_{n\to\infty} \Delta f(q_n) \ge 0 \quad \Big(\liminf_{n\to\infty} \mathit{Hess} f(q_n) \ge 0\Big). \tag{3.12}$$

Proof. Given the fundamental nature of the result, we provide full details. For each $n \in \mathbb{N}$, let

$$r_n = f(p_n) - \inf_M f, \quad \delta_n = r_n^{1/4}, \quad \varepsilon_n = r_n^{1/2}. \tag{3.13}$$

We will construct a sequence (q_n) satisfying

$$\mathrm{d}(q_n, p_n) \le \delta_n, \;\; f(q_n) \le f(p_n), \;\; ||\nabla f(q_n)|| \le 2r_n^{3/4} \tag{3.14}$$

and

$$\Delta f(q_n) \ge -2mC\varepsilon_n \quad \Big(\mathrm{Hess} f(q_n)(v, v) \ge -4C\varepsilon_n ||v||^2\Big), \tag{3.15}$$

for some $C > 1$. Since $r_n, \delta_n, \varepsilon_n \to 0$ as $n \to \infty$, the sequence (q_n) will then satisfy (3.11) and (3.12).

If $f(p_n) = \inf_M f$, take $q_n = p_n$. If $f(p_n) > \inf_M f$, define a function $f_n : M \to \mathbb{R}$ by

$$f_n(x) = f(x) + \varepsilon_n d_n^2(x), \tag{3.16}$$

where $d_n(x) = d(p_n, x)$ is the distance in M^m from x to p_n.

We claim that $\inf_M f_n$ is attained. If M^m is compact, this follows from the continuity of f_n. If M^m is not compact, we have, since f is bounded below and $\varepsilon_n > 0$,

$$\lim_{x\to\infty} f_n(x) = +\infty, \tag{3.17}$$

and $\inf_M f_n$ is attained as well.

Choose q_n to be any point for which $f_n(q_n) = \inf_M f_n$. Using (3.13) and (3.16), and noting that $f_n(p_n) = f(p_n)$, we have, for every $x \notin \overline{B}(p_n, \delta_n)$,

$$f_n(x) > f(x) + \varepsilon_n \delta_n^2 = f(x) + r_n = f(x) + f(p_n) - \inf_M f \geq f(p_n) = f_n(p_n), \quad (3.18)$$

which implies $d(q_n, p_n) \leq \delta_n$. From (3.16) we also obtain

$$f(p_n) = f_n(p_n) \geq f_n(q_n) = f(q_n) + \varepsilon_n d_n^2(q_n) \geq f(q_n). \quad (3.19)$$

We will need to prove that f_n is differentiable in a neighborhood of q_n, but before doing this let us first show how this fact can be used to obtain the inequality (3.15) and the third inequality in (3.14). We can suppose $q_n \neq p_n$, since in the case $q_n = p_n$ these inequalities are easily obtained from $\nabla d_n^2(p_n) = 0$ and $\mathrm{Hess}(d_n^2)(p_n)(v,v) = 2||v||^2$. From (3.16), we have

$$0 = \nabla f_n(q_n) = \nabla f(q_n) + 2\varepsilon_n d_n(q_n) \nabla d_n(q_n). \quad (3.20)$$

Recalling that $||\nabla d_n|| \equiv 1$, it follows from (3.20) and $d_n(q_n) \leq \delta_n$ that

$$||\nabla f(q_n)|| = 2\varepsilon_n d(p_n, q_n) \leq 2\varepsilon_n \delta_n = 2r_n^{3/4}. \quad (3.21)$$

Using that q_n is a minimum point for f_n, we also have

$$0 \leq \mathrm{Hess} f_n(q_n)(v,v) = \mathrm{Hess} f(q_n)(v,v) + \varepsilon_n \mathrm{Hess}\,(d_n^2)(q_n)(v,v), \quad (3.22)$$

which implies

$$\mathrm{Hess} f(q_n)(v,v) \geq -\varepsilon_n \mathrm{Hess}\,(d_n^2)(q_n)(v,v). \quad (3.23)$$

Taking the trace in (3.23), we obtain

$$\Delta f(q_n) \geq -\varepsilon_n \Delta\,(d_n^2)(q_n). \quad (3.24)$$

If the sectional curvature of M^m is bounded from below and k_o is a positive number such that $\inf_M K \geq -k_o^2$ we have, by the Hessian comparison theorem [13],

$$\mathrm{Hess}\,(d_n^2)(q_n)(v,v) \leq 4d_n(q_n)k_o \coth(k_o d_n(q_n))||v||^2 \leq 4\delta_n k_o \coth(\delta_n k_o)||v||^2, \quad (3.25)$$

where the last inequality follows from $d(p_n, q_n) \leq \delta_n$ and from the fact that $t \mapsto t\coth(t)$ is increasing. Since $\delta_n \to 0$ as $n \to \infty$, there exists $C > 1$ such that $\delta_n k_o \coth(\delta_n k_o) \leq C$ for all $n \in \mathbb{N}$. Thus

$$\mathrm{Hess}\,(d_n^2)(q_n)(v,v) \leq 4C||v||^2, \quad (3.26)$$

and (3.15) follows from (3.23) and (3.26).

If the Ricci curvature of M^m is bounded from below and k_o is a positive number such that $\inf_M \mathrm{Ric} \geq -k_o^2$, we have, by the Laplacian comparison theorem [13],

$$\Delta\,(d_n^2)(q_n) \leq 2mk_o d_n(q_n)\coth(k_o d_n(q_n)) \leq 2mk_o\delta_n\coth(k_o\delta_n) \leq 2mC \tag{3.27}$$

and (3.15) follows from (3.24) and (3.27).

To complete the proof it remains to show that f_n is differentiable at q_n. The argument we will present here is a slight modification of an argument presented by Borbély [2]. In view of (3.16), it is enough to prove that d_n is differentiable at q_n. If not, q_n is on the cut locus of p_n and we have two possibilities ([13]):

(i) There are two distinct minimizing geodesic segments $\gamma, \sigma : [0, t_n] \to M$ joining p_n to q_n, $t_n = d_n(q_n) = d(p_n, q_n)$.

(ii) There is a minimizing geodesic segment $\gamma : [0, t_n] \to M$ from p_n to q_n along which q_n is conjugate to p_n.

Suppose first that we have (i), and let $v = \gamma'(t_n)$, $w = \sigma'(t_n)$. Since f_n attains a minimum at q_n, from (3.16) we have

$$\begin{aligned} 0 &\leq \liminf_{s\to 0^+} \frac{f_n(\gamma(t_n+s)) - f_n(\gamma(t_n))}{s} \\ &= \liminf_{s\to 0^+}\{\frac{f(\gamma(t_n+s)) - f(\gamma(t_n))}{s} + \varepsilon_n \frac{d_n^2(\gamma(t_n+s)) - d_n^2(\gamma(t_n))}{s}\} \\ &= v(f) + \varepsilon_n \liminf_{s\to 0^+} \frac{d_n^2(\gamma(t_n+s)) - d_n^2(\gamma(t_n))}{s}. \end{aligned} \tag{3.28}$$

Using that $\gamma|_{[0,t_n]}$ is minimizing, we also have

$$\begin{aligned} 0 &\leq \liminf_{s\to 0^+} \frac{f_n(\gamma(t_n-s)) - f_n(\gamma(t_n))}{s} \\ &= \liminf_{s\to 0^+}\{\frac{f(\gamma(t_n-s)) - f(\gamma(t_n))}{s} + \varepsilon_n \frac{d_n^2(\gamma(t_n-s)) - d_n^2(\gamma(t_n))}{s}\} \\ &= -v(f) + \varepsilon_n \liminf_{s\to 0^+} \frac{(t_n-s)^2 - t_n^2}{s} \\ &= -v(f) - 2\varepsilon_n t_n. \end{aligned} \tag{3.29}$$

From (3.28) and (3.29), we obtain

$$\liminf_{s\to 0^+} \frac{d_n^2(\gamma(t_n+s)) - d_n^2(\gamma(t_n))}{s} \geq 2t_n. \tag{3.30}$$

On the other hand, since $v \neq w$ there exists $0 < c < 1$ such that, for all $s > 0$ sufficiently small,

$$d(\sigma(t_n - s), \gamma(t_n + s)) < 2cs, \tag{3.31}$$

which implies

$$d_n(\gamma(t_n+s)) < d_n(\sigma(t_n-s)) + 2cs = t_n - s + 2cs = t_n + (2c-1)s. \tag{3.32}$$

Recalling that $t_n = d_n(q_n) = d_n(\gamma(t_n))$ and $0 < c < 1$, it follows from (3.32) that

$$\liminf_{s\to 0^+} \frac{d_n^2(\gamma(t_n+s)) - d_n^2(\gamma(t_n))}{s} \leq 2t_n(2c-1) < 2t_n, \tag{3.33}$$

which contradicts (3.30).

Suppose now that we have (ii). From (3.29) we have $\nabla f(q_n) \neq 0$ and so the level set $\Gamma = \{x \in M : f(x) = f(q_n)\}$ is a smooth hypersurface in a neighborhood of q_n. Let Γ_t be the hypersurface parallel to Γ and passing through the point $\gamma(t)$. There exists $\bar{t} \in (0, t_n)$ such that, for all $t \in (\bar{t}, t_n)$, Γ_t is smooth near $\gamma(t)$.

Since $\gamma : [0, t_n] \to M$ is minimizing, d_n is differentiable at $\gamma(t)$ and $\nabla d_n(\gamma(t)) = \gamma'(t)$ for all $t \in (0, t_n)$. Thus the geodesic sphere S_t with center at p_n and radius t is smooth in a neighborhood of $\gamma(t)$, for all $0 < t < t_n$.

Also, since $q_n = \gamma(t_n)$ is the first point that is conjugate to $p_n = \gamma(0)$ along γ, there exists a Jacobi field J along γ satisfying $J(0) = J(t_n) = 0$, $J(t) \neq 0$ for $t \in (0, t_n)$, $\langle J, \gamma' \rangle \equiv 0$. From

$$\operatorname{Hess} d_n(J(t), J(t)) = \langle J(t), J'(t) \rangle = \frac{1}{2}(|J(t)|^2)' \tag{3.34}$$

and $J'(t_n) \neq 0$ (because of the uniqueness of the solution of the initial value problem for the Jacobi equation), it follows that

$$\lim_{t\to t_n} \operatorname{Hess} d_n \left(\frac{J(t)}{|J(t)|}, \frac{J(t)}{|J(t)|} \right) = -\infty. \tag{3.35}$$

We claim that for some $t \in (\bar{t}, t_n)$ there exists $q_t \in \Gamma_t$, sufficiently close to $\gamma(t)$, that lies inside the open ball $B(p_n, t)$ with center at p_n and radius t.

If $\gamma'(t)$ is not normal to Γ_t for some $t \in (\bar{t}, t_n)$, Γ_t is transversal to S_t and the claim follows trivially. Thus we may suppose $\gamma'(t) \perp \Gamma_t$ for all $t \in (\bar{t}, t_n)$, in which case Γ_t and S_t are tangent at $\gamma(t)$.

Given a smooth function $g : M \to \mathbb{R}$ and a level set S of g through a point p with $\nabla g(p) \neq 0$, one has

$$\operatorname{Hess} g(w, w) = -\langle \sigma(w, w), \nabla g(p) \rangle, \quad w \in T_pS, \tag{3.36}$$

where σ denotes the second fundamental form of S. Indeed, denoting by ∇° the connection on S and extending w to a local field on M that is tangent to S, one computes

$$0 = w\langle \nabla g, w \rangle = \operatorname{Hess} g(w, w) + \langle \nabla g, \nabla_w w \rangle, \tag{3.37}$$

and (3.36) follows since

$$\langle \nabla g, \nabla_w w \rangle = \langle \nabla g, \nabla_w^\circ w + \sigma(w, w) \rangle = \langle \nabla g, \sigma(w, w) \rangle. \tag{3.38}$$

Applying (3.36) to $g = d_n$ and recalling that $\nabla d_n(\gamma(t)) = \gamma'(t)$, we obtain

$$\operatorname{Hess} d_n(w, w) = \langle \sigma_t(w, w), -\gamma'(t) \rangle, \tag{3.39}$$

for all $w \in T_{\gamma(t)} S_t = T_{\gamma(t)} \Gamma_t$, where σ_t is the second fundamental form of S_t.

From (3.35) and (3.39), one has

$$\lim_{t \to t_n} \big\langle \sigma_t\big(w(t), w(t)\big), -\gamma'(t) \big\rangle = -\infty, \tag{3.40}$$

where $w(t) = J(t)/|J(t)|$. Denoting by $\overline{\sigma}_t$ the second fundamental form of Γ_t, it follows from (3.40) that there exists $t \in (\bar{t}, t_n)$ satisfying

$$\big\langle \overline{\sigma}_t\big(w(t), w(t)\big), -\gamma'(t) \big\rangle > \big\langle \sigma_t\big(w(t), w(t)\big), -\gamma'(t) \big\rangle. \tag{3.41}$$

Let $\alpha : (-\delta, \delta) \to \Gamma_t$ be a smooth curve such that $\alpha(0) = \gamma(t)$ and $\alpha'(0) = w(t)$, and let $f(s) = d(p_n, \alpha(s)) = d_n(\alpha(s))$. We have $f(0) = d_n(\gamma(t)) = t$ and

$$f'(s) = \langle \nabla d_n(\alpha(s)), \alpha'(s) \rangle. \tag{3.42}$$

In particular, $f'(0) = \langle \gamma'(t), w(t) \rangle = 0$. It follows from (3.39) and (3.42) that

$$\begin{aligned} f''(0) &= \langle \nabla_{w(t)} \nabla d_n, w(t) \rangle + \langle \gamma'(t), \nabla_{\alpha'} \alpha'(\gamma(t)) \rangle \\ &= \operatorname{Hess} d_n(w(t), w(t)) + \langle \gamma'(t), \overline{\sigma}_t\big(w(t), w(t)\big) \rangle \\ &= \langle \sigma_t\big(w(t), w(t)\big), -\gamma'(t) \rangle + \langle \overline{\sigma}_t\big(w(t), w(t)\big), \gamma'(t) \rangle, \end{aligned} \tag{3.43}$$

and using (3.41) one obtains $f''(0) < 0$. Since $f(0) = t$ and $f'(0) = 0$, we conclude that $d(p_n, \alpha(s)) = f(s) < t$ for all $s \neq 0$ sufficiently small, and the claim is proved.

Since Γ_t is parallel to Γ, there is a point $q \in \Gamma$ such that $d(q_t, q) \leq t_n - t$. Combining this with $d(q_t, p_n) < t$ yields

$$d_n(q) = d(p_n, q) \leq d(p_n, q_t) + d(q_t, q) < t + t_n - t = t_n = d_n(q_n). \tag{3.44}$$

From (3.16), (3.44) and the definition of Γ, one obtains

$$f_n(q) = f(q) + \varepsilon_n d_n^2(q) < f(q_n) + \varepsilon_n d_n^2(q_n) = f_n(q_n), \tag{3.45}$$

contradicting the fact that f_n attains a minimum at q_n.

Since both (i) and (ii) lead to contradictions, f_n must be differentiable at q_n. The proof of the theorem is now complete. □

The Omori-Yau minimum principle guarantees the existence of a minimizing sequence with good properties. By contrast, Theorems 3.3 and 3.4 guarantee the existence of a good minimizing sequence asymptotically close to any given minimizing sequence.

Definition. We refer to any sequence that has the properties of (q_n) in either Theorem 3.3 or 3.4 as being a *good shadow* of the minimizing sequence (p_n). □

As we will see in the next section, the abundance of good shadows plays a fundamental role in the proofs of Theorems 2.1, 2.3 and 2.4.

4 Proofs of the geometric theorems.

In the Introduction, as well as in the Remarks at the end of Section 2, we alluded to the fact that the Grassmanian-valued Gauss map is Lipschitzian if and only if the length of the second fundamental form is uniformly bounded. Although this statement is not needed in order to establish Theorems 2.1, 2.3 and 2.4 below, for the sake of completeness we provide its proof here, especially because it does not seem to have been recorded before. In fact, we will prove only that boundedness of the second fundamental form implies that the Gauss map is Lipschitzian, leaving the converse to the reader.

Let $h : M^n \to \mathbb{R}^N$ be an immersion, $r = N - n$ its codimension, and σ its second fundamental form. Passing to the orientable double cover, if necessary, we may assume that M^n is orientable. Also, for notational simplicity, we assume $M^n \subset \mathbb{R}^N$ and that h is the inclusion map.

The Grassmanian $G^\circ(r, N)$ of oriented r-planes in $\mathbb{R}^N$ can be identified with a subset of the r-th exterior power $\Lambda^r(\mathbb{R}^N)$. Indeed, the Plücker map $\mathcal{P} : G^\circ(r, N) \to \Lambda^r(\mathbb{R}^N)$, which assigns to W the pre-dual of its volume form, is well-defined and injective. More concretely, if $\{v_1, \cdots, v_r\}$ is any positive orthonormal basis of W, then $\mathcal{P}(W) = v_1 \wedge \cdots \wedge v_r$. A distance function can be given on $G^\circ(r, N)$ by setting $d(W_1, W_2) = ||\mathcal{P}(W_1) - \mathcal{P}(W_2)||$.

The orientations of T_pM^n and $\mathbb{R}^N$ induce an orientation on $[T_pM^n]^\perp$, and so one has the (oriented) Gauss map $\mathcal{G} : M^n \to G^\circ(r, N)$, $\mathcal{G}(p) = [T_pM^n]^\perp$. Let $\alpha : [0, l(\alpha)] \to M^n$ be an unit-speed curve joining p and q, and $\{N_1(t), \cdots, N_r(t)\}$, $0 \le t \le l(\alpha)$, a smoothly varying orthonormal frame of $\mathcal{G}(\alpha(t))$. For a fixed unit vector $u \in \mathcal{G}(p)$, let S_u be the linear endomorphism of T_pM^n given by $\langle S_u(X), Y\rangle = \langle \sigma(X, Y), u\rangle$. Denoting by $\overline{\nabla}$ the connection in $\mathbb{R}^N$, it is easy to see that $S_u(X) = -[\overline{\nabla}_X \overline{u}]^T$, where $\overline{u}$ is an unitary local section of the normal bundle that extends u and the superscript T stands for the tangential component. Suppose now that $||\sigma||$ is uniformly bounded on M^n. In particular, $||S_u||$ is also uniformly bounded, independently of $p \in M^n$ and the unitary normal vector u.

From

$$\begin{aligned}
\frac{d}{dt}[N_1(t) \wedge \cdots \wedge N_r(t)] &= \sum_j N_1(t) \wedge \cdots \wedge [\overline{\nabla}_{\alpha'} N_j(t)] \wedge \cdots \wedge N_r(t) \\
&= \sum_j N_1(t) \wedge \cdots \wedge [\overline{\nabla}_{\alpha'} N_j(t)]^T \wedge \cdots \wedge N_r(t) \\
&= -\sum_j N_1(t) \wedge \cdots \wedge [S_{N_j(t)} \alpha'(t)] \wedge \cdots \wedge N_r(t), \qquad (4.1)
\end{aligned}$$

one has

$$||\frac{d}{dt}[N_1(t) \wedge \cdots \wedge N_r(t)]|| \le C$$

for an absolute constant C. Integrating 4.1 over $[0, l(\alpha)]$ and taking the infimum over α,

$$d((\mathcal{G}(q), \mathcal{G}(p)) = ||N_1(q) \wedge \cdots \wedge N_r(q) - N_1(p) \wedge \cdots \wedge N_r(p)|| \le Cd(p, q),$$

showing that $\mathcal{G} : M^n \to G^\circ(r, N)$ is Lipschitzian. □

In our comments in Section 2, following the definition of uniform continuity for the second fundamental form, we remarked that σ is uniformly continuous if $|\sigma|$ and $|\widetilde{\nabla}\sigma|$ are uniformly bounded, say by constants D and C.

To see this, let $x, y \in M \subset \mathbb{R}^n$ and $v \in T_xM$, $|v| = 1$. Consider a normalized minimizing geodesic $\gamma : [0, l] \to M$ joining x and y, and denote by $V(s)$ the parallel vector field along γ such that $V(0) = v$. Denote $w = V(l)$, and let $\xi_1, ..., \xi_r$ be parallel normal vector fields along γ such that $\{\xi_1(s), ..., \xi_r(s)\}$ is an orthonormal basis of $[T_{\gamma(s)}M]^\perp$ for all $s \in [0, l]$. Define $g : [0, l] \to \mathbb{R}^n$ by

$$g(s) = \sum_{i=1}^{r} \langle \sigma(V(s), V(s)), \xi_i(s) \rangle \xi_i(s).$$

Decomposing $\overline{\nabla}$ at various stages of the computation below into its tangential and normal components, using $\nabla^\perp_{\gamma'}\xi_i = 0$, $\nabla_{\gamma'}V = 0$, and the definition of S given in the above proof, we have

$$\begin{aligned}\overline{\nabla}_{\gamma'(s)} g(s) &= \sum_{i=1}^{r} \left\langle \nabla^\perp_{\gamma'(s)} \sigma(V, V), \xi_i(s) \right\rangle \xi_i(s) + \sum_{i=1}^{r} \langle \sigma(V(s), V(s)), \xi_i(s) \rangle [\overline{\nabla}_{\gamma'} \xi_i]^T \\ &= \sum_{i=1}^{r} \left\langle (\widetilde{\nabla}_{\gamma'(s)} \sigma)(V(s), V(s)), \xi_i(s) \right\rangle \xi_i(s) - \sum_{i=1}^{r} \langle \sigma(V(s), V(s)), \xi_i(s) \rangle S_{\xi_i(s)} \gamma'(s),\end{aligned}$$

so that

$$||g'(s)|| \le \sum_{i=1}^{r} |\widetilde{\nabla}_{\gamma'(s)} \sigma| + \sum_{i=1}^{r} |\sigma_{\gamma(s)}|^2 \le (C + D^2) r.$$

Thus

$$||\sigma(w, w) - \sigma(v, v)|| = ||g(l) - g(0)|| \le \int_0^l ||g'(s)|| ds \le (C + D^2) r d(x, y). \tag{4.2}$$

It is now clear that (4.2) implies (2.1), showing that σ is uniformly continuous. □

Proof of Theorem 2.1. Suppose $\operatorname{Conv}(h(M)) \neq \mathbb{R}^n$ and let $H_1, \cdots, H_s$ be supporting hyperplanes in general position through a point p_o in the boundary of $\operatorname{Conv}(h(M))$. We want to show that $s \le n - m$. To this end, for $i = 1, ..., s$, denote by e_i the unit vector that is normal to H_i and points inside $\operatorname{Conv}(h(M))$, and let $f_i : \mathbb{R}^n \to \mathbb{R}$ be the height function with respect to H_i, i.e.,

$$f_i(y) = \langle y - p_o, e_i \rangle. \tag{4.3}$$

The fact that e_i points inside Conv $[h(M)]$, $i = 1, \dots, s$, means that

$$h(M) \subset \{y \in \mathbb{R}^n : f_i(y) \geq 0\}, \quad i = 1, \dots, s. \tag{4.4}$$

By our assumption that the immersion is substantial, one has that $h(M)$ is not contained in $H_1 \cap \cdots \cap H_s$.

We claim that there is a sequence (p_k) in M, $h(p_k) \notin H_1 \cap \cdots \cap H_s$, such that the distance between $h(p_k)$ and $H_1 \cap \cdots \cap H_s$ tends to zero as $k \to \infty$ (the sequence $h(p_k)$ may actually go to infinity in $\mathbb{R}^n$).

To prove the claim we will need a formula for computing the distance to the intersection $H_1 \cap \cdots \cap H_s$ of the affine hyperplanes $H_1, \dots, H_s$. Let y be a fixed point in $\mathbb{R}^n$. Suppose first $y \notin H_1 \cap \cdots \cap H_s$ and let z be the unique point in $H_1 \cap \cdots \cap H_s$ realizing the distance between y and $H_1 \cap \cdots \cap H_s$. Since $y - z \perp H_1 \cap \cdots \cap H_s$, there exist unique real numbers $a_1, ..., a_s$ such that $y - z = a_1 e_1 + \cdots + a_s e_s$. Hence,

$$\sum_{i=1}^{s} a_i \langle e_i, e_j \rangle = \langle y - z, e_j \rangle = \langle y - p_o, e_j \rangle + \langle p_o - z, e_j \rangle = \langle y - p_o, e_j \rangle, \tag{4.5}$$

which implies

$$a_j = \sum_{i=1}^{s} \langle y - p_o, e_i \rangle g^{ij}, \quad j = 1, ..., s, \tag{4.6}$$

where $(g^{ij})_{i,j=1,...,s}$ is the inverse of the matrix $(\langle e_i, e_j \rangle)_{i,j=1,...,s}$. From (4.5) and (4.6),

$$\begin{aligned}
\mathrm{d}(y, H_1 \cap \cdots \cap H_s) &= ||y - z|| \\
&= \left\langle \sum_{i=1}^{s} a_i e_i, \sum_{j=1}^{s} a_j e_j \right\rangle^{\frac{1}{2}} \\
&= \left[\sum_{j=1}^{s} a_j \sum_{i=1}^{s} a_i \langle e_i, e_j \rangle \right]^{\frac{1}{2}} = \left[\sum_{j=1}^{s} a_j \langle y - p_o, e_j \rangle \right]^{\frac{1}{2}} \\
&= \left[\sum_{i,j=1}^{s} \langle y - p_o, e_i \rangle \langle y - p_o, e_j \rangle g^{ij} \right]^{\frac{1}{2}},
\end{aligned} \tag{4.7}$$

If $y \in H_1 \cap \cdots \cap H_s$, we have $\langle y - p_o, e_i \rangle = 0$, $i = 1, ..., s$, and (4.7) holds in the same way.

Assuming the claim is not true, there is $\varepsilon > 0$ so that

$$d\big(h(x), H_1 \cap \cdots \cap H_s\big) \geq \varepsilon, \quad x \in M. \tag{4.8}$$

Let H be the hyperplane of $\mathbb{R}^n$ that contains p_o and is orthogonal to the vector $e_1 + \cdots + e_s$, and $f : \mathbb{R}^n \to \mathbb{R}$ the corresponding height function with respect to $(e_1 + \cdots + e_s)/a$, $a = ||e_1 + \cdots + e_s||$, so that

$$f(y) = \Big\langle y - p_o, \frac{e_1 + \cdots + e_s}{a} \Big\rangle. \tag{4.9}$$

If $f_i(y) \geq 0,\ i = 1, ..., s$, and $f(y) < \delta$, it follows from (4.3) and (4.9) that

$$0 \leq \langle y - p_o, e_i \rangle < a\delta, \quad i = 1, ..., s, \tag{4.10}$$

which implies, with the aid of (4.7),

$$d(y, H_1 \cap \cdots \cap H_s) < naC\delta, \quad C^2 := \max_{i,j} |g^{ij}|. \tag{4.11}$$

Choosing $\delta = \varepsilon / naC$, we conclude that $d(y, H_1 \cap \cdots \cap H_s) < \varepsilon$ for all $y \in \mathbb{R}^n$ satisfying $f(y) < \delta$ and $f_i(y) \geq 0,\ i = 1, ..., s$. It follows from the above and (4.8) that $f\big(h(x)\big) \geq \delta$, for all $x \in M$. Since the set $\{y \in \mathbb{R}^n : f(y) \geq \delta\}$ is convex, we conclude that

$$\mathrm{Conv}\,(h(M)) \subset \{y \in \mathbb{R}^n : f(y) \geq \delta\}, \tag{4.12}$$

contradicting the fact that p_o belongs to H and also to the boundary of $\mathrm{Conv}\,(h(M))$. Hence (4.8) cannot occur, and the claim is proved.

Since

$$\lim_{k \to \infty} f_i\big(h(p_k)\big) = \inf_M (f_i \circ h) = 0, \ \ 1 \leq i \leq s,$$

we can use Theorem 3.3 to obtain s sequences $q_k^{(i)} \in M,\ 1 \leq i \leq s,\ k \geq 1$, such that the distance between $q_k^{(i)}$ and p_k goes to zero and $\nabla(f_i \circ h)(q_k^{(i)}) \to 0$ when $k \to \infty$. Since $\nabla(f_i \circ h)(x)$ is the tangential component of $\nabla f_i\big(h(x)\big)$ in T_xM for all $x \in M$, and $e_i = \nabla f_i(y)$ for all $y \in \mathbb{R}^n$, this last condition means that the angle between e_i and the normal space $\mathcal{G}(q_k^{(i)})$ is tending to zero.

Passing to a subsequence, we may assume that $\mathcal{G}(p_k) \to W$ for some $W \in G(n-m, n)$. Since the distance between $q_k^{(i)}$ and p_k is going to zero as $k \to \infty$, and the Gauss map is uniformly continuous, it follows that $\mathcal{G}(q_k^{(i)})$ is also converging to W. But, as remarked before, the limit of $\mathcal{G}(q_k^{(i)})$ in $G(n-m, n)$ contains e_i. This proves that W contains the s linearly independent vectors $e_1, \ldots, e_s$. In particular, $\mathrm{codim}\, h(M) = \dim W = n - m \geq s$. To conclude the proof of Theorem 2.1 we observe that, in the hypersurface case, the assertion about continuity follows from the fact that the limit of supporting hyperplanes is itself a supporting hyperplane. □

Proofs of Theorems 2.3 and 2.4. The first assertion in Theorem 2.3, $s \leq n - m$, follows from Theorem 2.1 and our hypothesis that the Grassmanian-valued Gauss map is uniformly continuous.

Defining $f_i : \mathbb{R}^n \to \mathbb{R}$ by (4.3), it follows from our assumption on the vectors $e_1, \ldots, e_s$ that the functions $f_i \circ h,\ i = 1, \ldots, s$, are all nonnegative on M^m. A simple calculation shows, for $i = 1, \ldots, s$, that

$$\mathrm{Hess}(f_i \circ h)_x(v, v) = \langle \sigma(v, v), e_i \rangle, \quad x \in M,\ v \in T_xM, \tag{4.13}$$

and so

$$\Delta(f_i \circ h)(x) = m\langle \overrightarrow{H}(x), e_i \rangle, \ \ x \in M,\ i = 1, \ldots, s. \tag{4.14}$$

Let (p_k) be a sequence in M^m such that $\mathrm{d}(h(p_k), H_1 \cap \cdots \cap H_s) \to 0$ as $k \to \infty$. That such a sequence exists is a consequence of the proof of Theorem 2.1. It is immediate that (p_k) is a minimizing sequence for each one of the functions $f_i \circ h$.

Since, by assumption, the Ricci curvature of M is bounded from below, we can apply Theorem 3.4 to obtain s sequences $\big(q_k^{(i)}\big)$ in M, $1 \le i \le s$, $k \ge 1$, such that

$$\lim_{k\to\infty} \mathrm{d}\big(q_k^{(i)}, p_k\big) = 0, \tag{4.15}$$

$$\liminf_{k\to\infty} \Delta(f_i \circ h)\big(q_k^{(i)}\big) \ge 0. \tag{4.16}$$

It follows from (4.14) and (4.16) that

$$\liminf_{k\to\infty} \Big\langle \overrightarrow{H}\big(q_k^{(i)}\big), e_i \Big\rangle \ge 0, \quad i = 1, \ldots, s. \tag{4.17}$$

Using (4.15) and our assumption that the mean curvature vector field $\overrightarrow{H}$ is uniformly continuous on M, we obtain $||\overrightarrow{H}(p_k) - \overrightarrow{H}(q_k^{(i)})|| \to 0$, which implies, with the aid of (4.17), that $\liminf_{k\to\infty} \Big\langle \overrightarrow{H}(p_k), e_i \Big\rangle \ge 0$, $i = 1, \ldots, s$. This concludes the proof of Theorem 2.3.

The proof of Theorem 2.4 runs along similar lines. Let (p_k) be a sequence in M^m such that $\mathrm{d}(h(p_k), H_1 \cap \cdots \cap H_s) \to 0$ as $k \to \infty$, and choose unit vectors $v_k \in T_{p_k}M$. Using the half of Theorem 3.4 that improves on the Omori theorem, one can construct sequences $(q_k^{(i)})$ in M, $i = 1, \ldots, s$, such that $\lim_{k\to\infty} \mathrm{d}\big(q_k^{(i)}, p_k\big) = 0$ and, from (4.13),

$$\liminf_{k\to\infty} \min_{|w|=1} \Big\langle \sigma(q_k^{(i)})(w, w), e_i \Big\rangle \ge 0, \quad i = 1, \ldots, s. \tag{4.18}$$

Since σ is uniformly continuous, it follows from (2.2) that there exists $w_k^{(i)} \in T_{q_k^{(i)}}M$, $|w_k^{(i)}| = 1$, such that

$$||\sigma(p_k)(v_k, v_k) - \sigma(q_k^{(i)})(w_k^{(i)}, w_k^{(i)})|| \to 0. \tag{4.19}$$

Combining (4.18) and (4.19) one has $\liminf_{k\to\infty} \langle \sigma(p_k)(v_k, v_k), e_i \rangle \ge 0$, as desired. □

5 Dynamics and good shadows.

Definition. A complete Riemannian manifold M^m satisfies the *Local Volume Property* (LVP) if there exist $a > 0$, $b > 1$, such that for any $p \in M$ and $0 < r < a$, one has

$$\mathrm{Vol}\ B(p, r) \le b \mathrm{Vol}\ B(p, \frac{r}{2}).$$

□

Examples. If $f : (M, g) \to (N, h)$ and there exist $c_1, c_2 > 0$ such that $c_1|v| \leq |df(v)| \leq c_2|v|$ for all tangent vectors v, then (M, g) is LVP if and only if (N, h) is. Homogeneous manifolds are clearly LVP.

Suppose now that the Ricci curvature of M is bounded from below. If M^c stands for the space form of curvature $c < 0$, where c is sufficiently negative as compared to the lower bound of the Ricci curvature of M, Gromov's theorem on monotonicity of volume ratios ([3], [13]) implies that for all $x \in M$ the quotient

$$\frac{\mathrm{Vol}^c(r)}{\mathrm{Vol}(x, r)},$$

between the volumes of balls of radius r in M^c and M, is a nondecreasing function. In particular,

$$\frac{\mathrm{Vol}\big(B(p, r)\big)}{\mathrm{Vol}\big(B(p, \frac{r}{2})\big)} \leq \frac{\mathrm{Vol}^c(r)}{\mathrm{Vol}^c(\frac{r}{2})}, \tag{5.1}$$

showing that complete manifolds with Ricci curvature bounded from below are LVP.

The previous observation about quasi-isometric manifolds indicates that an LVP manifold need not have Ricci curvature bounded from below. Two dimensional examples can be easily constructed by taking a conformal metric $\lambda(z)|dz|^2$ on $\mathbb{R}^2$, with λ varying between positive constants to ensure the LVP property, and for which the curvature

$$K = -\frac{1}{2\lambda}\Delta \log \lambda$$

is unbounded below. Here, Δ stands for the flat Laplacian and so

$$\Delta \log \lambda = \frac{1}{\lambda}\Delta\lambda - \frac{1}{\lambda^2}|\nabla\lambda|^2.$$

In particular, $\inf K = -\infty$ if λ is chosen so as to have a sequence p_n of critical points that satisfy $\sup \Delta\lambda(p_n) = \infty$. □

The hypothesis in the Yau maximum principle that the Ricci curvature is bounded below has been weakened by several authors (e.g., [2], [14]). Here we propose a conjecture that weakens the hypothesis on the Ricci curvature, while strengthening the conclusion. Recall the definition of good shadows, given at the end of Section 3.

Conjecture. If M is an LVP manifold and $f \in C^2(M)$ is bounded below, then any minimizing sequence of f admits a good shadow (relative to Δ). □

The result below verifies the above conjecture for a special class of functions:

Theorem 5.1. *Let M^m be a complete manifold that satisfies LVP, and $f : M \to \mathbb{R}$ a function of class C^2 such that $\inf f > -\infty$ and $\sup ||Hess f|| < \infty$. Then every minimizing sequence of f admits a good shadow (relative to Δ). Explicitly, if $f(p_n) \to \inf f$, then there exists (q_n) such that $f(q_n) \to \inf f$, $d(p_n, q_n) \to 0$, $||\nabla f(q_n)|| \to 0$, and $\liminf \Delta f(q_n) \geq 0$.*

Proof. Let ϕ_t be the local flow of $X = -\nabla f$ on M, so that

$$\frac{d}{dt}\phi_t(p) = -\nabla f(\phi_t(p)), \quad \phi_0(p) = p, \tag{5.2}$$

where $t \in [0, \tau(p))$ = the maximal interval of existence of the forward solution.

For each $n \in \mathbb{N}$, set

$$\delta_n = \sqrt{f(p_n) - \inf_M f}, \quad r_n = \sqrt{\delta_n}. \tag{5.3}$$

Since $\delta_n \to 0$ as $n \to \infty$, we may suppose, without loss of generality, that $\delta_n < r_n$ for all $n \in \mathbb{N}$. We will construct a sequence (q_n) in M satisfying

$$\mathrm{d}(p_n, q_n) \le r_n, \quad \liminf_{n\to\infty} \Delta f(q_n) \ge 0. \tag{5.4}$$

For every $n \in \mathbb{N}$ for which $\delta_n = 0$, we have $\Delta f(p_n) \ge 0$ and we choose $q_n = p_n$. For each $n \in \mathbb{N}$ so that $\delta_n > 0$, we have two possibilities:

a) Every positive orbit originating in $\overline{B}(p_n, r_n^2)$ remains in the open ball $B(p_n, r_n)$.

b) There is at least one trajectory that joins the boundaries of $B(p_n, r_n^2)$ and $B(p_n, r_n)$ in finite time.

In the first alternative, $\tau(p) = \infty$ for every $p \in \overline{B}(p_n, r_n^2)$. Let μ denote the Riemannian measure of M. By Liouville's formula for the change of volume under a flow [10], one has, for all $t > 0$, since $\Delta = \mathrm{div}\nabla$,

$$\mu\big(B(p_n, r_n)\big) \ge \mu\big(\phi_t(B(p_n, r_n^2))\big) = \int_{B(p_n, r_n^2)} \exp\Big(\int_0^t -\Delta f(\phi_s(p))ds\Big) d\mu(p). \tag{5.5}$$

If there exists $\varepsilon > 0$ such that $\Delta f(q) \le -\varepsilon$ for all $q \in B(p_n, r_n)$, a contradiction can be easily established by letting $t \to \infty$ in the above formula. Hence one can choose $q_n \in \overline{B}(p_n, r_n)$ such that $\Delta f(q_n) \ge 0$.

We now work under the conditions of alternative b). Consider the quantity τ_n which gives the shortest time to travel from $\partial B(p_n, r_n^2)$ to $\partial B(p_n, r_n)$, along a trajectory of X. Formally,

$$\tau_n = \inf\ \big\{t : t \in (0, \tau(p)),\ p \in \partial B(p_n, r_n^2) \text{ and } \phi_t(p) \in \partial B(p_n, r_n)\big\}. \tag{5.6}$$

In particular,

$$\phi_{\tau_n}\big(\overline{B}(p_n, r_n^2)\big) \subset \overline{B}(p_n, r_n). \tag{5.7}$$

We want to estimate the first exit time τ_n. Let $x_n \in \partial B(p_n, r_n^2)$ and $t_n \in (0, \tau(x_n))$ be such that $\phi_{t_n}(x_n) \in \partial B(p_n, r_n)$ and $t_n < 2\tau_n$.

The last integral of

$$\begin{aligned} f(x_n) - f(\phi_{t_n}(x_n)) = -\int_0^{t_n} (f \circ \phi_s)' ds &= \int_0^{t_n} ||\nabla f||^2(\phi_s(x_n))ds \\ &\ge \frac{1}{t_n}\left[\int_0^{t_n} ||\nabla f||(\phi_s(x_n))ds\right]^2 \end{aligned} \tag{5.8}$$

gives the length of the portion of the orbit of $X = -\nabla f$ over the time interval $[0, t_n]$, through x_n. Since the last point of this orbit segment lies in $\partial B(p_n, r_n)$, its length is at least $r_n(1 - r_n)$. Collecting this information, and observing (5.8), we obtain

$$\frac{\delta_n(1-r_n)^2}{2\tau_n} \le \frac{r_n^2(1-r_n)^2}{t_n} \le f(x_n) - f(\phi_{t_n}(x_n)) \le f(x_n) - \inf_M f. \tag{5.9}$$

We will now estimate the last term in (5.9). Define $h : M \to \mathbb{R}$ by $h(x) = ||\nabla f(x)||^2 + \varepsilon$, where ε is a positive real number. Given $p, q \in M$, consider an unit speed minimizing geodesic $\gamma : [0, \overline{a}] \to M$ joining p to q. If K is an upper bound for the norm of the Hessian operator of f, we have

$$\Big|\frac{d}{dt}h(\gamma(t))\Big| = 2|\langle \nabla_{\gamma'}\nabla f, \nabla f\rangle| \le 2K||\nabla f|| \le 2K\sqrt{h(\gamma(t))}, \tag{5.10}$$

and so

$$\big|\sqrt{h(\gamma(t))} - \sqrt{h(\gamma(0))}\,\big| \le Kt, \quad t > 0. \tag{5.11}$$

Setting $t = \overline{a}$ and letting $\varepsilon \to 0$,

$$\big|\, ||\nabla f(p)|| - ||\nabla f(q)||\,\big| \le K\overline{a} = Kd(p,q), \quad p, q \in M. \tag{5.12}$$

On the other hand, from the proof of Theorem 3.3, there exists $y_n \in \overline{B}(p_n, \delta_n)$ so that $||\nabla f||(y_n) \le \delta_n$. Using this fact and (5.12), we obtain

$$||\nabla f(z)|| \le ||\nabla f(y_n)|| + K\mathrm{d}(y_n, z) \le \delta_n(1 + 2K), \quad z \in \overline{B}(p_n, \delta_n). \tag{5.13}$$

Considering an unit speed minimizing geodesic segment $\gamma : [0, \delta_n] \to M$ joining p_n to x_n, it follows from (5.13) that

$$\begin{aligned} |f(x_n) - f(p_n)| &\le \int_0^{\delta_n} |(f\circ\gamma)'(t)|dt = \int_0^{\delta_n} |\langle \nabla f(\gamma(t)), \gamma'(t)\rangle|dt \\ &\le \int_0^{\delta_n} ||\nabla f(\gamma(t))||dt \le \delta_n^2(1 + 2K). \end{aligned} \tag{5.14}$$

From (5.9) and (5.14), we obtain

$$\begin{aligned} \frac{\delta_n(1-r_n)^2}{2\tau_n} &\le f(x_n) - f(p_n) + f(p_n) - \inf_M f \\ &\le \delta_n^2(1+2K) + \delta_n^2 = 2\delta_n^2(1+K), \end{aligned} \tag{5.15}$$

and so

$$\frac{(1-r_n)^2}{\tau_n} \le 4\delta_n(1+K). \tag{5.16}$$

Since $\delta_n \to 0$ and $r_n \to 0$ as $n \to \infty$, one has, in particular, $\lim_{n\to\infty} \tau_n = +\infty$.

From Liouville's formula,

$$\mu\big(\phi_{\tau_n}(B(p_n, r_n^2))\big) = \int_{B(p_n,r_n^2)} \exp\left(\int_0^{\tau_n} -\Delta f(\phi_s(p))ds\right) d\mu(p). \tag{5.17}$$

Jensen's inequality applied to the probability measure $\nu/\nu(\Omega)$, where ν is a finite measure on Ω, gives

$$\psi\left(\frac{1}{\nu(\Omega)} \int_\Omega g d\nu\right) \leq \frac{1}{\nu(\Omega)} \int_\Omega (\psi \circ g) d\nu, \tag{5.18}$$

whenever ψ is convex and g is integrable.

Applying (5.18) to (5.17),

$$\frac{\mu(\phi_{\tau_n}(B(p_n, r_n^2)))}{\mu(B(p_n, r_n^2))} \geq \exp\left[\frac{1}{\mu(B(p_n, r_n^2))} \int_{B(p_n,r_n^2)} \left(\int_0^{\tau_n} -\Delta f(\phi_s(p))ds\right) d\mu(p)\right]. \tag{5.19}$$

Using our assumption that M^m satisfies LVP, it is not difficult to see that

$$\frac{\mu(B(p,d))}{\mu(B(p,c))} \leq b^{\frac{d}{c}} \tag{5.20}$$

for all $p \in M$ and $0 < c < d < a$. Indeed, from

$$\text{Vol } B(p,r) \leq b\text{Vol } B(p, \frac{r}{2}),\ r < a,$$

one obtains inductively, for any positive integer k,

$$\text{Vol } B(p,r) \leq b^k \text{Vol } B(p, \frac{r}{2^k}).$$

Hence, if $0 < c < d < a$ and k is the least positive integer such that $\frac{d}{c} \leq 2^k$, one has $2^{k-1} < \frac{d}{c}$ and so $k \leq 2^{k-1} < \frac{d}{c}$. In particular, since $b > 1$,

$$\text{Vol } B(p,d) \leq b^k \text{Vol } B(p, \frac{d}{2^k}) \leq b^k \text{Vol } B(p,c) < b^{\frac{d}{c}} \text{Vol } B(p,c).$$

Recalling that $\phi_{\tau_n}(\overline{B}(p_n, r_n^2)) \subset \overline{B}(p_n, r_n)$, it follows from (5.19) and (5.20) that

$$\exp\left[\frac{1}{\mu(B(p_n, r_n^2))} \int_{B(p_n,r_n^2)} \left(\int_0^{\tau_n} -\Delta f(\phi_s(p))ds\right) d\mu(p)\right] \leq \frac{\mu(B(p_n, r_n))}{\mu(B(p_n, r_n^2))} \leq b^{\frac{1}{r_n}}, \tag{5.21}$$

which implies

$$\frac{1}{\mu(B(p_n, r_n^2))} \int_{B(p_n,r_n^2)} \left(\int_0^{\tau_n} -\Delta f(\phi_s(p))ds\right) d\mu(p) \leq \frac{\log b}{r_n} \tag{5.22}$$

Next, we take $\Omega_n = B(p_n, r_n^2) \times [0, \tau_n]$, endowed with the probability measure ν given by the normalization of the product measure on $B(p_n, r_n^2) \times [0, \tau_n]$. Dividing (5.22) by τ_n, and using (5.16), we obtain

$$\int_{\Omega_n} -\Delta f(\phi_s(q)) d\nu(q, s) \leq \frac{\log b}{\tau_n r_n} \leq \frac{4(1+K) r_n \log b}{(1 - r_n)^2}. \tag{5.23}$$

Since Ω_n has mass one, it follows from (5.23) that there are $s_n \in [0, \tau_n]$ and $q_n' \in B(p_n, r_n^2)$ such that, with $q_n = \phi_{s_n}(q_n')$, one has

$$d(p_n, q_n) \leq r_n, \quad \Delta f(q_n) \geq \frac{-4 r_n (1+K) \log b}{(1 - r_n)^2}, \tag{5.24}$$

and (5.4) follows from the fact that $r_n \to 0$ as $n \to \infty$.

Next we will prove that (q_n) is a minimizing sequence of f. Employing the same argument that was used to obtain (5.13), one obtains

$$||\nabla f(z)|| \leq r_n (1 + 2K), \quad z \in \overline{B}(p_n, r_n). \tag{5.25}$$

Using the above inequality and reasoning as in (5.14), we arrive at

$$f(q_n) \leq f(p_n) + r_n^2 (1 + 2K). \tag{5.26}$$

In view of (5.4) and (5.26), to complete the proof that (q_n) is a good shadow of (p_n), it remains to show that $||\nabla f||(q_n) \to 0$ as $n \to \infty$. By Theorem 3.3, there exists a minimizing sequence (q_n') with $\mathrm{d}(q_n, q_n') \to 0$, $||\nabla f||(q_n') \to 0$. Applying (5.12) to q_n and q_n', one sees that $||\nabla f||(q_n)$ also tends to zero. □

Adjusting the proof of Theorem 5.1 one can prove a result which, in the terminology of [14], represents a "weak minimum principle":

Theorem 5.2. *Let M be a complete manifold that satisfies LVP, and $f : M \to \mathbb{R}$ a function of class C^2 satisfying $\inf_M f > -\infty$. Let (p_n) be a sequence in M that is strongly minimizing for f, in the sense that there exists $\delta > 0$ such that the oscillation of f on $B(p_n, \delta)$ tends to zero, i.e.,*

$$\lim_{n \to \infty} \left[\max_{B(p_n, \delta)} f - \min_{B(p_n, \delta)} f \right] = 0. \tag{5.27}$$

Then there exists a minimizing sequence (q_n) in M for f such that

$$\lim_{n \to \infty} d(p_n, q_n) = 0, \quad \liminf_{n \to \infty} \Delta f(q_n) \geq 0. \tag{5.28}$$

Proof. For each $n \in \mathbb{N}$, set

$$r_n = \left[\max_{B(p_n, \delta)} f - \min_{B(p_n, \delta)} f \right]^{\frac{1}{4}}. \tag{5.29}$$

We will construct a sequence q_n such that

$$\mathrm{d}(p_n, q_n) \leq r_n, \quad \liminf \Delta f(q_n) \geq 0. \tag{5.30}$$

From (5.27), one sees that such a sequence (q_n) will satisfy (5.28). Since $r_n \to 0$ as $n \to \infty$, we may suppose, without loss of generality, that $r_n \in [0, \delta)$ for all $n \in \mathbb{N}$, where δ is as in the statement of the theorem. If $r_n = 0$, f is constant in $B(p_n, \delta)$, and we take $q_n = p_n$. Fix a real number $\kappa \in (0, 1)$ and denote by ϕ_t the local flow of $X = -\nabla f$ on M. For each $n \in \mathbb{N}$ for which $r_n > 0$, we have two possibilities: either every positive orbit originating in $\overline{B}(p_n, \kappa r_n)$ remains in the open ball $B(p_n, r_n)$ or there is at least one trajectory that joins the boundaries of $B(p_n, \kappa r_n)$ and $B(p_n, r_n)$ in finite time. In the first case we obtain, as in the proof of Theorem (5.1), a point $q_n \in \overline{B}(p_n, r_n)$ such that $\Delta f(q_n) \geq 0$. In the second case, reasoning as in the proof of Theorem 5.1, with r_n^2 replaced by κr_n, we obtain

$$\frac{(1-\kappa)^2 r_n^2}{2\tau_n} \leq \max_{B(p_n,\delta)} f - \min_{B(p_n,\delta)} f, \tag{5.31}$$

which implies, in view of (5.29),

$$\frac{(1-\kappa)^2}{2\tau_n} \leq r_n^2. \tag{5.32}$$

In particular, $\tau_n \to \infty$ as $n \to \infty$. Continuing as in the proof of Theorem 5.1, we arrive at

$$\frac{\mu(B(p_n, r_n))}{\mu(B(p_n, \kappa r_n))} \geq \exp\left[\frac{1}{\mu(B(p_n, \kappa r_n))} \int_{B(p_n,\kappa r_n)} \left(\int_0^{\tau_n} -\Delta f(\phi_s(p))ds\right) d\mu(p)\right]. \tag{5.33}$$

Since M satisfies the local volume doubling condition, and $r_n \to 0$ as $n \to \infty$, it follows from (5.20) and (5.33) that

$$\frac{1}{\mu(B(p_n, \kappa r_n))} \int_{B(p_n,\kappa r_n)} \left(\int_0^{\tau_n} -\Delta f(\phi_s(p))ds\right) d\mu(p) \leq \frac{\log b}{k}. \tag{5.34}$$

Using (5.34) and (5.32), and arguing as in Theorem 5.1, we conclude that there exists $q_n \in \overline{B}(p_n, r_n)$ so that

$$\Delta f(q_n) \geq -\frac{2r_n^2 \log b}{k(1-k)^2}. \tag{5.35}$$

Now (5.30) follows from (5.35) and from the fact that $r_n \to 0$ as $n \to \infty$. That (q_n) is minimizing is an immediate consequence of (5.27) and (5.28). $\square$

Corollary 5.3. *Let $B(0, 1)$ be the open unit ball in $\mathbb{R}^3$, (M, g) a complete surface and $X : (M, g) \to B(0, 1)$ a proper minimal isometric immersion. Then (M, g) does not satisfy LVP.*

(Examples of complete minimal surfaces that are properly immersed in the open unit ball in $\mathbb{R}^3$ have been constructed by Martin-Morales ([11])). If $X : M^2 \to B(0,1) \subset R^3$ is such an immersion, let $f(p) = -|X(p)|^2$. Since X is proper, $\lim_{p\to\infty} f(p) = -1$ uniformly on p, and condition (5.27) holds. On the other hand, from the minimality of M we obtain $\Delta f = -4$. It now follows from Theorem 5.2 that these surfaces are not LVP. $\square$

If true, the conjecture at the beginning of this section would imply that no complete bounded minimal surface, properly immersed or not, is LVP. On the other hand, it has long been known that such surfaces must have unbounded Gaussian curvature (see [18] for a more general result).

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