KÄHLER PARABOLICTY AND THE EULER NUMBER OF COMPACT MANIFOLDS OF NON-POSITIVE SECTIONAL CURVATURE

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Abstract

Let M^{2n} be a compact Riemannian manifold of *non-positive sectional* curvature. It is shown that if M^{2n} is homeomorphic to a Kähler manifold, then its Euler number satisfies the inequality $(-1)^n\chi(M^{2n}) \geq 0$.

Introduction

The results of this paper are related to a well-known problem, attributed sometimes to Hopf and sometimes to Chern, to the effect that the Euler number $\chi(M^{2n})$ of a compact Riemannian manifold M^{2n} of negative sectional curvature must satisfy the inequality $(-1)^n\chi(M^{2n}) > 0$. This conjecture is true in dimensions 2 and 4 [Ch] and it has been verified in the Kähler case for all n by Gromov [G] and Stern [S] (the work in [S] also uses results of Greene and Wu; see [GW], p.183-215).

Gromov's arguments are rather general and establish the following result: "Let M^{2n} be a compact Riemannian manifold of negative curvature. If M^{2n} is homotopy equivalent to a compact Kähler manifold then $(-1)^n\chi(M^{2n}) > 0$"; see [G], Theorem 0.4.A and Example (a), p.265.

A companion conjecture asserts that if the sectional curvature of a Riemannian manifold M^{2n} is assumed to be only *non-positive,* then the Euler number must satisfy $(-1)^n\chi(M^{2n}) \geq 0$. Again, this second conjecture is known to be true in dimensions two and four [Ch]. The aim of this paper is to establish its validity for all n in the Kähler case, thus complementing the above result of Gromov:

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Main Theorem. *Let M^{2n} be a compact Riemannian manifold of non-positive curvature. If M^{2n} is homeomorphic to a Kähler manifold, then the Euler number of M^{2n} satisfies the inequality $(-1)^n\chi(M^{2n}) \geq 0$.*

Remark: The proof of Theorem 2.1 works if the two manifolds are assumed to be only homotopy equivalent but, in view of Farrell and Jones [FJ], the manifolds are actually homeomorphic if $\dim M \geq 6$ (the case $\dim M = 4$ follows from [Ch]).

Theorem 2.1 is related to a well-known problem, attributed sometimes to Hopf and sometimes to Chern, to the effect that the Euler number $\chi(M^{2n})$ of a compact Riemannian manifold M^{2n} of negative sectional curvature must satisfy the inequality $(-1)^n\chi(M^{2n}) > 0$.

Dodziuk [Do] and Singer ([Yau], p. 672) have proposed to settle the Chern-Hopf conjectures using the Atiyah index theorem for coverings (see [At]). In this approach, one is required to prove a vanishing theorem for L^2 harmonic k-forms, $k \neq n$, on the universal covering of M^{2n}. The vanishing of these L^2 Betti numbers imply, by Atiyah's result, that $(-1)^n\chi(M^{2n}) \geq 0$. The strict inequality $(-1)^n\chi(M^{2n}) > 0$ follows provided one can establish the existence of non-trivial L^2 harmonic n-forms on the universal cover.

The program outlined above was carried out by Gromov [G] when the manifold in question is Kähler and is homotopy equivalent to a compact manifold with *strictly* negative sectional curvatures. The central idea in Gromov's approach is the notion of *d(bounded)* differential forms. As the terminology suggests, these are differential forms that are exterior derivatives of *bounded* forms, where boundedness is to be understood in the Riemannian sense. For instance, Gromov [G] points out that a bounded closed k-form, $k \geq 2$, on a complete simply-connected manifold whose sectional curvatures are bounded above by a negative constant is automatically d-bounded.

Following [G], a Kähler manifold is termed *Kähler-hyperbolic* if the Kähler form of its universal cover is d(bounded). Armed with the concept of Kähler-hyperbolicity, Gromov went on to prove the desired vanishing and existence theorems on the universal cover, thus establishing the inequality $(-1)^n\chi(M^{2n}) > 0$ when the Kähler manifold M^{2n} is homotopy equivalent to a compact manifold with negative curvature.

Although Kähler hyperbolicity is a powerful idea in the case of strictly negative curvature, it is clearly inadequate for the problem "$K \leq 0 \Longrightarrow (-1)^n\chi(M^{2n}) \geq 0$". This can be seen by taking a compact Kähler manifold M^{2n} satisfying $K \leq 0$ and $\chi(M^{2n}) = 0$ (e.g., a flat complex torus or, more generally, the product of such a torus and a Kähler manifold of negative curvature). The Kähler form on the universal cover of M^{2n} is not d(bounded). Otherwise, by Gromov's work one would have the correct vanishing and existence theorems of L^2 harmonic forms and, by Atiyah's theorem, the Euler number of M^{2n} would actually be non-zero.

Our contribution to this circle of ideas has been to single out a condition which is weaker than d-boundedness and can be applied to Kähler manifolds of *non-positive* curvature:

Definition 1. *A differential form α on a complete non-compact Riemannian manifold is called d(sublinear) if there exist a differential form β and $c > 0$ such that $d\beta = \alpha$ and $|\beta(x)| \leq c(1 + \rho(x, x_0))$, where $\rho(x, x_0)$ stands for the Riemannian distance between x and a base point x_0.*

The concept of d-sublinearity is both natural and flexible. This can be seen from the following results, both of which will be used to prove the Main Theorem in §3.

Theorem 1. *Let M^n be a complete simply-connected manifold of non-positive sectional curvature and α a bounded closed k-form on M^n, $k \geq 1$. Then α is d(sublinear).*

Theorem 2. *Let N^{2n} be a complete non-compact Kähler manifold of complex dimension n whose Kähler form is d(sublinear). If $k \neq n$ then any L^2 harmonic k-form on N^{2n} is identically zero.*

Other results on L^2 cohomology can be found in Anderson [An], Atiyah [At], Dodziuk [Do], Donnelly and Fefferman [DF], Donnelly and Xavier [DX], Elworthy and Rosenberg [ER] and Lott [L].

Extending Gromov's terminology, we propose the following:

Definition 2. *A Kähler manifold is Kähler-parabolic if the Kähler form on its universal cover is d(sublinear) but not d(bounded).*

Accordingly, our results imply that $(-1)^n \chi(M^{2n}) \geq 0$ if the compact manifold M^{2n} is Kähler-parabolic, with strict inequality holding if M^{2n} is Kähler-hyperbolic ([G]).

§1 Proof of Theorem 1: Solutions of $d\beta = \alpha$ with sublinear growth.

Throughout this section (M^m, g) will be a complete simply-connected manifold of non-positive curvature. Let also α be a bounded smooth closed k-form on M. Since M^m is diffeomorphic to $\mathbb{R}^m$ there exists a form β such that $d\beta = \alpha$. The purpose of this section is to show that β can be chosen to have sublinear growth, in the sense of the definition given in the introduction. The proof will follow from the Poincaré lemma by a comparison argument.

Fix $p \in M$ and denote by $exp_p : T_pM \to M$ the exponential map based at p.

Lemma 1. *Consider the maps $\tau_t : M \to M$, given by $x \to exp_p(t\, exp_p^{-1}(x))$, where $0 \leq t \leq 1$. Then*

$$|(\tau_t)_* \xi| \leq t|\xi| \tag{1}$$

for every tangent vector ξ.

Proof: Let $\sigma : [0, 1] \to M^n$ be the geodesic segment joining p to x, $\xi \in T_x M^n$ and $y = (exp_p)^{-1}(x) \in T_p M^n$. By a straightforward computation one has

$$\begin{aligned}(\tau_t)_* \xi &= (d\, exp_p)_{t(exp_p)^{-1}(x)}[td(exp_p^{-1})_{(x)}\xi] \\ &= (d\, exp_p)_{ty}\{t[d(exp_p)_y]^{-1}\xi\}.\end{aligned}$$

It is now manifest from the above formula that

$$J(t) := (\tau_t)_* \xi \tag{2}$$

is the Jacobi field along σ satisfying $J(0) = 0,\ J(1) = \xi$. On the other hand, since the sectional curvatures are non-positive, the function $f(s) := |J(s)|$ is convex ([BGS], p.5). In particular, for $s \geq t$, one has

$$f(s) \geq f(t) + (s-t)\frac{f(t) - f(0)}{t - 0} = \frac{s}{t} f(t).$$

Setting $s = 1$ one has $f(t) \leq t f(1)$ and the result follows from (2). □

Recall that if α is a k-form and Z is a vector field, then $(\alpha\lfloor_Z)$ is the $(k-1)$-form given by

$$(\alpha\lfloor_Z)(\xi_1, \cdots, \xi_{k-1}) = \alpha(Z, \xi_1, \cdots, \xi_{k-1}).$$

For the sake of completeness we give a proof of the following elementary result.

Lemma 2. *Let Ψ be a closed k-form in $\mathbb{R}^m$. Then the $(k-1)$-form Φ defined by*

$$\Phi(x) = r \int_0^1 [(\tau_t)^*(\Psi\lfloor_{\frac{\partial}{\partial r}})](x)dt \tag{3}$$

satisfies $d\Phi = \Psi$; here $\frac{\partial}{\partial r} = \sum_{i=1}^m \frac{x_i}{r}\frac{\partial}{\partial x_i}$, $r = (\sum_{i=1}^m x_i^2)^{1/2}$ and $\tau_t(x) = tx$.

Proof: By the standard proof of the Poincaré lemma ([ST], p.130), Φ can be taken to be

$$\Phi(x) = \sum_{i_1 < \cdots < i_k} \sum_{j=1}^k (-1)^{j-1} x_{i_j} \Big(\int_0^1 t^{k-1} \Psi_{i_1 \cdots i_k}(tx)dt\Big) dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_j}} \wedge \cdots \wedge dx_{i_k},$$

where $\Psi = \sum_{i_1 < \cdots < i_k} \Psi_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

In particular, one has

$$\begin{aligned}\Psi(x) &= \sum_{i_1 < \cdots < i_k} \sum_{j=1}^k x_{i_j} \Big(\int_0^1 t^{k-1} \Psi_{i_1 \cdots i_k}(tx)dt\Big)(dx_{i_1} \wedge \cdots \wedge dx_{i_k})\lfloor_{\frac{\partial}{\partial x_{i_j}}} \\ &= r \sum_{i_1 < \cdots < i_k} \Big(\int_0^1 t^{k-1} \Psi_{i_1 \cdots i_k}(tx)dt\Big)(dx_{i_1} \wedge \cdots \wedge dx_{i_k})\lfloor_{\frac{\partial}{\partial r}} \\ &= r \int_0^1 t^{k-1} (\Psi\lfloor_{\frac{\partial}{\partial r}})(tx)dt \\ &= r \int_0^1 [(\tau_t)^*(\Psi\lfloor_{\frac{\partial}{\partial r}})](x)dt,\end{aligned}$$

as desired. $\square$

Proof of Theorem 1: Let $(x_1, ..., x_n)$ be Euclidean coordinates in T_pM and consider the pull-back metric h of the metric g under $exp_p : T_pM \to M$. Observe that there are now two ways to interpret the map τ_t. The first interpretation comes from Lemma 1 with (M, g) being replaced by (T_pM, h); alternatively, one can think of τ_t as the self-map of T_pM, $(x_1, ..., x_n) \to t(x_1, ..., x_n)$, that appears in the Poincaré lemma (Lemma 2). It is an easy and yet basic observation that these two ways of thinking about τ_t give rise to the same map.

We may also replace the form α that appears in the statement of Theorem 1 by a closed form Ψ on T_pM which is bounded in the induced metric h. Let Φ be given by Lemma 2 and observe that, by Lemma 1,

$$|(\tau_t)^* \varphi(x)|_h \le C t^{k-1} |\varphi(\tau_t(x))|_h, \quad k \ge 1, \tag{4}$$

holds for any $(k-1)$-form φ on T_pM; here $|\cdot|_h$ is any one of the equivalent norms induced by h. Since $|\frac{\partial}{\partial r}| = 1$, it follows from (4) and Lemma 2 that

$$\begin{aligned} |\Phi(x)|_h &\le r \int_0^1 |[(\tau_t)^* (\Psi \lfloor_{\frac{\partial}{\partial r}})](x)|_h dt \\ &\le Cr \int_0^1 t^{k-1} |\Psi \lfloor_{\frac{\partial}{\partial r}} (tx)|_h dt \\ &\le C_1 r \int_0^1 t^{k-1} |\Psi(tx)|_h dt \\ &\le C_1 r \sup_{0 \le t \le 1} |\Psi(tx)|_h. \end{aligned}$$

In particular,

$$|\Phi(x)|_h \le C_1 \rho_h(0, x) \sup |\Psi|_h \le C_2 \rho_h(0, x).$$

Hence Φ is d(sublinear) and the proof of Theorem 1 is complete. $\square$

§2 Proof of Theorem 2: Vanishing of L^2 harmonic forms.

We begin the proof of Theorem 2 by recalling some basic facts in Hodge theory and Kähler geometry. If M^m is an oriented complete Riemannian manifold, let δ be the adjoint operator of d acting on the space of L^2 k-forms. Denote by $\Omega^k_{(2)}(M^m)$ and $\mathcal{H}^k_{(2)}(M^m)$ the spaces of L^2 k-forms and L^2 harmonic k-forms, respectively. By elliptic regularity and completeness of the manifold, a k-form in $\mathcal{H}^k_{(2)}(M^m)$ is smooth, closed and co-closed.

Suppose that M^{2n} is a complete Kähler manifold of complex dimension n with Kähler form w. Since w is bounded, the Lefschetz map $L : \Omega^k_{(2)}(M^{2n}) \to \Omega^{k+2}_{(2)}(M^{2n})$, $\alpha \to w \wedge \alpha$, is well-defined, it is injective if $k < n$ ([We], p.182) and satisfies $L\mathcal{H}^k_{(2)}(M^m) \subset \mathcal{H}^{k+2}_{(2)}(M^m)$ ([We], p.191). By duality, it is sufficient to consider the case $k < n$ in Theorem 2.

By hypothesis, there exists a 1-form β with $d\beta = w$ and

$$|\beta(x)| \leq c(1 + \rho(x, x_0)),$$

where c is an absolute constant.

In what follows we assume that the distance function $\rho(x, x_0)$ is smooth for $x \neq x_0$. The general case follows easily by an approximation argument.

Let $\eta : \mathbb{R} \to \mathbb{R}$ be smooth, $0 \leq \eta \leq 1$,

$$\eta(t) = \begin{cases} 1, & \text{if } t \leq 0 \\ 0, & \text{if } t \geq 1 \end{cases}$$

and consider the compactly supported function

$$f_j(x) = \eta(\rho(x, x_0) - j),$$

where j is a positive integer.

Let α be a harmonic k-form in L^2 , $k < n$, and consider the form $\Phi = \beta \wedge \alpha$. Observing that $\delta(w \wedge \alpha) = 0$ since $w \wedge \alpha \in \mathcal{H}^{k+2}_{(2)}(M^m)$ and noticing that $f_j\Phi$ has compact support, one has

$$0 = < \delta(w \wedge \alpha), f_j \Phi > = < w \wedge \alpha, d(f_j \Phi) >,$$

where $<,>$ stands for the global L^2 inner product.

We further note that, since $d\beta = w$ and $d\alpha = 0$,

$$\begin{aligned} 0 &= < w \wedge \alpha, d(f_j \Phi) > \\ &= < w \wedge \alpha, f_j d\Phi > + < w \wedge \alpha, df_j \wedge \Phi > \\ &= < w \wedge \alpha, f_j w \wedge \alpha > + < w \wedge \alpha, df_j \wedge \Phi > \\ &= < w \wedge \alpha, f_j w \wedge \alpha > + < w \wedge \alpha, df_j \wedge \beta \wedge \alpha > . \end{aligned} \tag{5}$$

Since $0 \leq f_j \leq 1$ and $\lim_{j \to \infty} f_j(x)(w \wedge \alpha)(x) = (w \wedge \alpha)(x)$, it follows from the dominated convergence theorem that

$$\lim_{j \to \infty} < w \wedge \alpha, f_j w \wedge \alpha > = \|w \wedge \alpha\|^2. \tag{6}$$

Since w is bounded, $\mathrm{supp}(df_j) \subset B_{j+1} \setminus B_j$ and $|\beta(x)| = O(\rho(x, x_0))$, one obtains

$$< w \wedge \alpha, df_j \wedge \beta \wedge \alpha > \leq (j+1)C \int_{B_{j+1} \setminus B_j} |\alpha(x)|^2 dx, \tag{7}$$

where C is a constant independent of j.

We claim that there exists a subsequence $\{j_i\}_{i \geq 1}$ such that

$$\lim_{i \to \infty} (j_i + 1) \int_{B_{j_i+1} \setminus B_{j_i}} |\alpha(x)|^2 dx = 0. \tag{8}$$

If not, there would exist a positive constant a such that

$$(j+1) \int_{B_{j+1} \setminus B_j} |\alpha(x)|^2 dx \geq a > 0, \ \ j \geq 1.$$

This inequality implies

$$\begin{aligned} &\int_{M^{2n}} |\alpha(x)|^2 dx = \sum_{j=0}^{\infty} \int_{B_{j+1} \setminus B_j} |\alpha(x)|^2 dx \\ &\geq a \sum_{j=0}^{+\infty} \frac{1}{j+1} = +\infty, \end{aligned}$$

a contradiction to the assumption $\int_{M^{2n}} |\alpha(x)|^2 dx < \infty$. Hence, there exists a subsequence $\{j_i\}_{i \geq 1}$ for which (8) holds. Using (7) and (8), one obtains

$$\lim_{i \to \infty} < w \wedge \alpha, df_{j_i} \wedge \beta \wedge \alpha >= 0. \tag{9}$$

It now follows from (5), (6) and (9) that $w \wedge \alpha = 0$. Since L is injective ($k < n$), $\alpha = 0$ as desired. □

§3 Proof of the Main Theorem

We begin with the following:

Lemma 3. *Let $F : M_1 \to M_2$ be a smooth homotopy equivalence between two compact Riemannian manifolds, $\pi : \widetilde{M_i} \to M_i$ the universal covering maps for $i = 1, 2$. Then, for any closed differential form α on M_2, $\pi^*(\alpha)$ is d(sublinear) on $\widetilde{M_2}$ if the form $(F \circ \pi)^*(\alpha)$ is d(sublinear) on $\widetilde{M_1}$.*

Proof: Assume that α is a closed form on M_2, and that $(F \circ \pi)^*(\alpha)$ is d(sublinear) on $\widetilde{M_1}$. Our goal is to show that $\pi^*(\alpha)$ is d(sublinear) on $\widetilde{M_2}$.

Since F is a homotopy equivalence, there exists a smooth map $G : M_2 \to M_1$ such that both $F \circ G$ and $G \circ F$ are homotopic to the identity maps.

Clearly, the maps F and G can be lifted to the universal covering spaces. Let then $\widetilde{F} : \widetilde{M_1} \to \widetilde{M_2}$ and $\widetilde{G} : \widetilde{M_2} \to \widetilde{M_1}$ be the lifted maps, so that the following diagram commutes:

$$\begin{tikzcd}
\widetilde{M_1} \arrow[r, "\widetilde{F}"] \arrow[d, "\pi"'] & \widetilde{M_2} \arrow[r, "\widetilde{G}"] \arrow[d, "\pi"'] & \widetilde{M_1} \arrow[d, "\pi"'] \\
M_1 \arrow[r, "F"] & M_2 \arrow[r, "G"] & M_1
\end{tikzcd}$$

Since M_1 is compact, α is bounded and π is a local isometry, $(F\circ\pi)^*(\alpha)$ is a bounded form on $\widetilde{M_1}$. By the assumption and the commutativity of the diagram, there exists β with sublinear growth such that

$$d\beta = (F \circ \pi)^*(\alpha) = (\pi \circ \widetilde{F})^*(\alpha) = \widetilde{F}^*(\pi^*(\alpha)).$$

In particular,

$$d(\widetilde{G}^*(\beta)) = \widetilde{G}^* \circ \widetilde{F}^*(\pi^*(\alpha)).$$

On the other hand, since $F \circ G$ is homotopic to the identity map, the closed forms α and $(F \circ G)^*(\alpha)$ are in the same de Rham cohomology class. Hence there exists a form Θ on M^{2n} such that $\alpha = (F \circ G)^*(\alpha) + d\Theta$. It follows that

$$\begin{aligned}
\pi^*(\alpha) &= d(\pi^*(\Theta)) + (\pi^* G^* F^*)(\alpha) \\
&= d(\pi^*(\Theta)) + (\widetilde{G}^* \pi^* F^*)(\alpha) \\
&= d(\pi^*(\Theta)) + (\widetilde{G}^* \widetilde{F}^* \pi^*)(\alpha) \\
&= d(\pi^*(\Theta)) + d(\widetilde{G}^*(\beta)) \\
&= d[\pi^*(\Theta) + \widetilde{G}^*(\beta)].
\end{aligned}$$

Since Θ is bounded, β is sublinear and $\widetilde{G}_*$ is bounded, it follows that $\pi^*(\alpha)$ is d(sublinear), as desired. ☐

Proof of Main Theorem: Suppose that $F : M_1^{2n} \to M_2^{2n}$ is a homotopy equivalence from a compact Riemannian manifold M^{2n} of non-positive sectional curvature to a compact Kähler manifold M_2^{2n}. By an approximation argument if necessary, one may assume that F is a smooth map.

Let $\pi : \widetilde{M_1^{2n}} \to M_1^{2n}$ be the universal covering map and w the Kähler form on M_2^{2n}. Since $(F \circ \pi)^*(w)$ is a bounded closed form on $\widetilde{M_1^{2n}}$, it follows from Theorem 1 that $(F \circ \pi)^*(w)$ is d-sublinear. By Lemma 3, the lifted Kähler form $\widetilde{w}$ on $\widetilde{M_2^{2n}}$ is d-sublinear as well. It follows from Theorem 2 that the L^2 cohomology of $\widetilde{M_2^{2n}}$ is concentrated in the middle dimension. The Atiyah index theorem for covers [At] then gives $(-1)^n \chi(M_2^{2n}) \geq 0$. Since $\chi(M_1^{2n}) = \chi(M_2^{2n})$, the conclusion follows. ☐

References

[An] Anderson, M., *L^2-harmonic forms on complete Riemannian manifolds*, Springer Lecture Notes in Math. vol. **1339** (1988), 1-19.

[At] Atiyah, M., *Elliptic operators, discrete groups and Von Neumann algebra*, Asterisque, vol. **32-33** (1976), 43-47.

[BGS] Ballmann, W, Gromov, M., and Schroeder, V., *Manifolds of non-positive curvature*, Birkhauser, Boston (1985).

[Ch] Chern, S. S., *On curvature and characteristic classes of a Riemannian manifold*, Abh. Math. Sem. Univ. Hamburg, vol. **20** (1955), 117-126.

[Do] Dodziuk, J., *L^2 harmonic forms on complete manifolds*, In Yau, S. T. (ed.) “Seminar on Differential Geometry”, Princeton University Press, Ann. of Math Studies, vol. **102** (1982), 291-302.

[DF] Donnelly, H. and Fefferman, C., *L^2-cohomology and the index theorem for the Bergman metric*, Annals of Math., vol. **118** (1983), 593-618.

[DX] Donnelly, H. and Xavier, F., *On the differential form spectrum of negatively curved Riemannian manifolds*, Amer J. Math., vol. **106** (1984), 169-185.

[ER] Elworthy, K.D., and Rosenberg, Steven, *The Witten Laplacian on negatively curved simply-connected manifolds*, Tokyo J. Math., vol. **16** (1995), 513-524.

[FJ] Farrell, F. T., Jones, L. E., *Topological rigidity for compact nonpositively curved manifolds*, Proceedings of Symposia in Pure Mathematics, American Math. Soc., vol. **54**, Part III (1993), 229-274.

[G] Gromov, M., *Kähler hyperbolicity and L^2-Hodge theory*, J. Diff. Geom., vol. **33** (1991), 263-292.

[GW] Greene, R. and Wu, H., *Function theory on manifolds which posses a pole*, Springer Lecture Notes in Math. , vol. **699** (1979).

[L] Lott, J., *The zero-in-the-spectrum question*, L'Enseign. Math., vol. **42** (1996), 341-376.

[ST] Singer, I and Thorpe, J., *Lecture Notes on Elementary Topology and Geometry*, Springer-Verlag, 1976.

[St] Stern, M., *L^2-cohomology of negatively curved Kähler manifolds*, Preprint, Institute for Advanced Study, Princeton (1989).

[We] Wells, R. L., Jr., *Differential analysis on complex manifolds*, Springer-Verlag, 1979.

[Yau] S. T. Yau, *Problem Section*, In Yau, S. T. (ed.) “Seminar on Differential Geometry”, Princeton University Press, Annals of Math Studies, vol. **102** (1982), 669-706.

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