BOOTSTRAP METHOD FOR ESTIMATION OF SPECTRAL BANDWIDTH WITH LIMITED OBSERVATIONS

R. Vandermeulen, Student M. ASCE, T. Kijewski, Student M. ASCE
and A. Kareem, M. ASCE
University of Notre Dame, Notre Dame, IN 46556
Rik.H.Vandermeulen.2@nd.edu, tkijewsk@nd.edu, kareem@nd.edu

Abstract

Characterizing the dynamic response to external forces is an important component of both the design and analysis of safe and serviceable structures, requiring accurate estimates of the natural frequency and critical damping ratio. The simplest methods for determining these parameters are based on the response power spectrum of the structure to an external loading, such as wind. The location and width of the power spectrum's peak are related to the natural frequency and critical damping ratio of the structure, respectively. Unfortunately, the amount of continuously stationary data required to create a response spectrum with reasonable statistical accuracy is nearly impossible to gather in the field. This paper explores using the bootstrap statistical processing technique to improve estimates of the damping ratio for structures with limited data records. The following will briefly describe the standard method for calculating the critical damping ratio of a structure, as well as listing the sources of error in those calculations. The basic principle of bootstrap resampling is explained, and various methods for applying it to this problem are suggested. Simulations highlight the improved bandwidth estimates via bootstrapped spectra for limited data.

Introduction

Spectral bandwidth serves as an important measure of signal content, indicative not only of the bandedness of the signal, but in the case of structures, bearing a unique relationship to viscous damping. As structural damping is a critical system parameter particularly affecting the performance of the structure from a serviceability perspective, its reliable estimation is paramount to the design of tall buildings. Though a host of techniques have evolved to address the accurate estimation of structural damping from systems for which exact inputs are not known (Desforges *et al.*, 1995), one of the most popular and straightforward approaches is based upon the structural response power spectrum. With the advent of the Fast Fourier Transform (FFT), the calculation of these spectra has been greatly expedited. In particular, as the excitations due to wind are assumed as white noise with constant power spectrum, this approach has become quite attractive for the analysis of structural systems under ambient loading. Unfortunately, to insure the accuracy of this approach, considerable amounts of data may be required. Further, as true of all spectral techniques, the data analyzed must be assumed stationary over the duration of the record, which is often not possible over such lengthy intervals. In light of these limitations, the following study will investigate the use of Bootstrapping Theory to improve the quality of spectral bandwidth measures and provide reliable estimates of variance. To examine the effectiveness of this approach, simulated response time histories will be analyzed via both standard and bootstrapped spectra.

Spectral Errors

Although there are a variety of measures of spectral bandwidth, the most common and simplified measure for structures is defined as the half-power bandwidth (HPBW) of the power spectrum (Bendat and Piersol, 1986), which is determined by identifying the two frequencies, f_1 and f_2, at which the power spectrum takes on half its peak value. The HPBW and its relationship to the system's vibration properties of natural frequency (f_N) and critical damping ratio (ξ) is given by

$$\beta = f_2 - f_1 = 2\xi f_N \tag{1}$$

As the bandwidth of lightly damped systems can be quite narrow, bias error becomes a governing concern, since a fine spectral resolution is required to adequately capture the rapidly varying spectral peak. Since large changes in the magnitude of the spectrum occur over a short frequency range, without sufficient frequency resolution, these changes can be missed or poorly estimated. As reflected by (1), systems with low natural frequency and damping have smaller HPBW and thus require finer frequency resolution.

The discrete frequencies at which the spectrum is calculated via the Fourier Transform is determined by $N_{FFT}=Tf_s$, where f_s is the sampling frequency and T is the length of the block of data. N_{FFT} is typically chosen as a power of two to expedite calculations by using the Fast Fourier Transform. These frequencies are equispaced, and the frequency resolution (Δf) will be equal to the smallest Fourier Frequency, inversely proportional to the length of the block of data. The bias error is calculated by (Bendat and Piersol, 1986)

$$\varepsilon_b = -\frac{1}{3}\left(\frac{\Delta f}{\beta}\right)^2 . \tag{2}$$

In common practice, to minimize bias to -2%, frequency resolution is chosen such that at least four spectral lines lie within the HPBW. In addition, the variance error must also be considered, which is the random sampling error involved with any statistical problem. Assuming a stationary process, the amount of variance will be reduced by averaging spectra estimated from n_D blocks of data. Thus, the variance error will approximately reduce by (Bendat and Piersol, 1986)

$$\varepsilon_r \approx 1/\sqrt{n_D} . \tag{3}$$

Bootstrapping Estimates of Variance

Bootstrapping will be exercised as an alternative means by which to estimate the true variance of the spectra (Efron and Tibshirani, 1993). Essentially, the bootstrap yields an estimate of a statistical parameter and its variance by generating new bootstrap samples of length n_D by randomly sampling, with replacement, from the measured data. These are averaged to form a bootstrap replicate. The process, shown in Figure 1, is repeated B

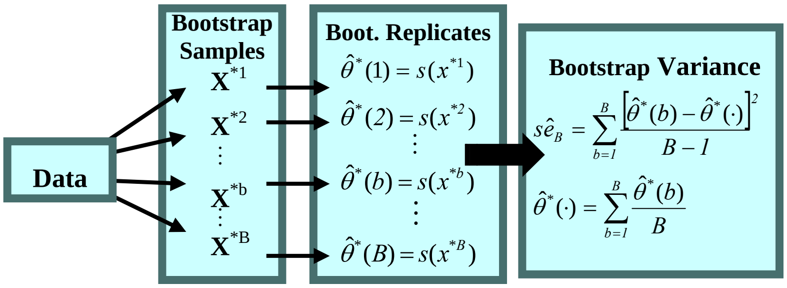

Figure 1. Schematic representation of bootstrapping notion.

times, and these replicates are used to estimate the variance of the data. In particular, the bootstrapping technique will be used in this study to provide an estimate of the spectral bandwidth and variance. It is hoped that the introduction of such a scheme will provide practitioners with a simple means by which to estimate the variance of their bandwidth estimates even when theoretical assumptions are not entirely met. In attempting to improve standard half-power bandwidth estimations, three different applications of the bootstrap were investigated.

Bootstrapping of Individual Bandwidths

In method 1, the half-power bandwidth of the spectra calculated from each block of data was determined. In this manner, the HPBW would be essentially resampled by the bootstrap strategy. Unfortunately, the mean bandwidth as predicted by resampling deviated significantly from theory, as the quality of a spectrum drawn from one block of data is quite poor. Preliminary simulations revealed a mean resampled HPBW of 0.005 Hz, well below the theoretical value of 0.008 Hz. This method was not used on subsequent simulations.

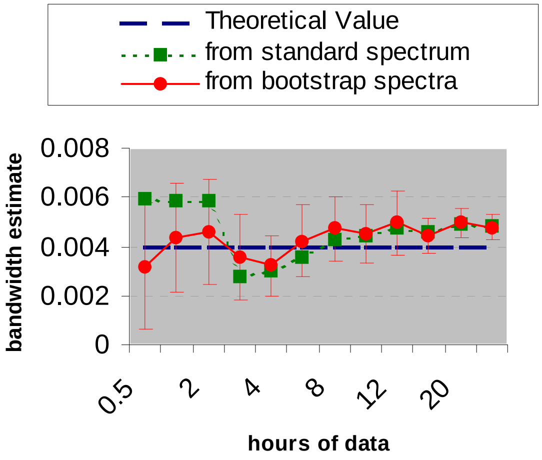

Figure 2. HPBW Estimates: 1% Damping, $N_{FFT}=8192$.

Bootstrapping of Spectra from Resampled Time History Blocks

In method 2, bootstrapping theory was applied to blocks of the time history. In this way, resampled time histories were generated, from which power spectra were calculated. Essentially, the variance estimate in this case gauges the implications of certain blocks of the time history be omitted, while others may be considered multiple times in the averaging for the final power spectrum. This approach, having greater physical meaning than the other methods, proved to be most successful. To expedite calculations, rather than resampling the blocks of the response history and repeating the Fourier transform process, the spectra from the individual blocks are simply resampled and averaged. Still, this method requires far greater computation than the first method. The number of bootstrap replicates, *B*, varied from 10 to 1000, but for most of the calculations 50 resamples were sufficient (Efron and Tibshirani, 1993).

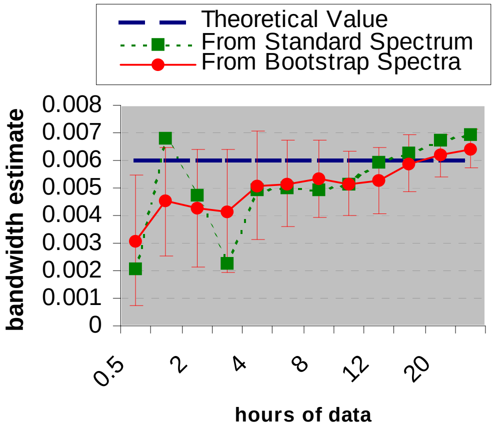

Figure 3. HPBW Estimates: 1.5% Damping, N_{FFT}=8192.

The third method considered was much more computationally demanding than the first two. In this method, new individual response spectra are created, then averaged and used to calculate the HPBW. At each frequency, one of the original individual spectra was randomly selected and its spectral density recorded. A new individual response spectrum is thus created by moving across the full frequency range, randomly selecting, at each frequency, the magnitude of the spectral density from one of the original spectra. This was repeated to create a full collection of n_D respampled spectra, which could then be averaged to produce a new bootstrap power spectrum and HPBW estimate. The process was then repeated to generate B bootstrap replicates of the HPBW. This process proved to be computationally intensive and when conducted successfully, yielded results similar to Method 2. As a result, Method 2 was deemed most feasible and used for the following simulations.

Simulation Results

The response of a SDOF oscillator with natural frequency of 0.2 Hz and variable levels of damping (1.0%, 1.5%, 2.0%) was simulated to yield 24 hours of data, sampled at 10 Hz. The data was divided into blocks in accordance with (2) to yield as many samples for averaging as possible. The number of samples averaged where then varied to illustrate the effects of limited observation data.

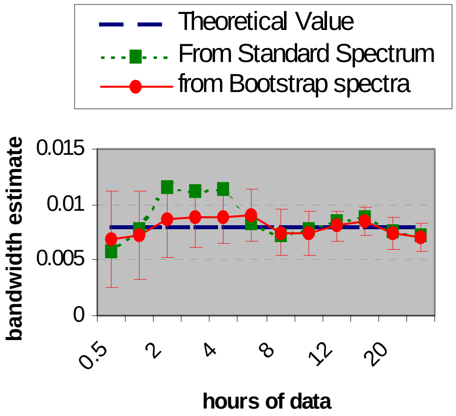

Figure 4. HPBW Estimates: 2% Damping, N_{FFT}=8192.

For the cases considered, 24 hours of data should be sufficient to insure approximately four spectral lines within the half-power bandwidth and at least 100 averages to bring the spectral variance to 10%. To meet the minimum bias error requirements, blocks of N_{FFT}=8192 data points were used.

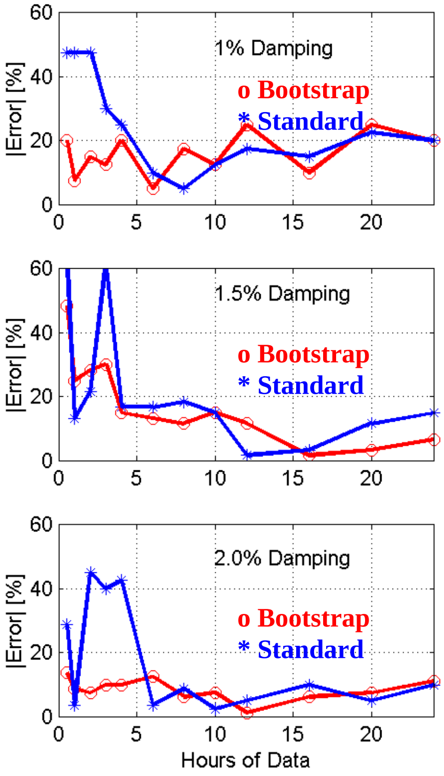

Figure 5. Error in Damping Estimates.

Figures 2-4 illustrate the bandwidth estimates using both the true spectrum and the bootstrapped spectra for three levels of damping. As expected, considering more hours of data increases the number of ensembles being averaged, thereby reducing the variance and improving the bandwidth estimates yielded by both approaches. As these figures further illustrate, the bootstrap standard deviation, denoted by the red bars, encompasses the theoretical bandwidth in nearly all cases, illustrating that the bootstrapped estimates are within one standard deviation of the true answer. One exception to this was using 0.5 hours of data with a damping ratio of 1.5%. However, it should be noted that 0.5 hours of data only allows two data blocks of 819.2 seconds each. With so few blocks, only a couple of unique bootstrap replicates can be created.

One interesting result of these simulations was the comparative response of the bandwidth estimate from the actual spectra and the bootstrap mean estimate as the number of hours of data decreases. The standard bandwidth estimate seemed to behave much more erratically to a change in the data record. In most of the simulations, the standard estimate fluctuated from well above to well below the theoretical value as the amount of data was reduced. By comparison, the bootstrapped estimates remained fairly constant, with gradually increasing standard deviations.

Figure 5 summarizes the implications of the bandwidth estimates on the actual estimate of damping for all three cases. Note that the bootstrapped estimate, denoted by red circles, typically yields superior results for limited amounts of data. As the amount of data being considered increases, the bootstrap and standard spectral estimates approach one another with some fluctuations. As expected, the error in the estimates is still significant as spectral variance is on the order of 10%. Additionally, the results are less accurate for smaller levels of damping due to the difficulty in resolving the narrow spectral peak. This is discussed below.

Limiting Factors

As mentioned earlier, with decreasing damping, the length of the data blocks must increase to control bias error. For the simulations with 1.5% or 2% damping, a block size of 8192 was sufficient to provide 2% bias error, yet this was not truly sufficient for the case with 1% damping. For a fair comparison, the N_{FFT} points or block size must be

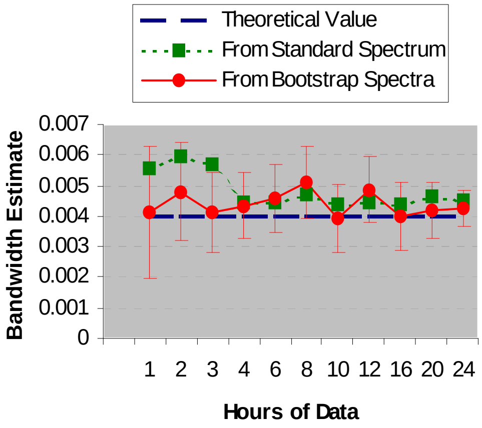

Figure 6. HPBW Estimates: 1% Damping, N_{FFT}=16,384.

increased for this latter case to 16,384 data points. For uniformity, this was initially ignored, and calculations on the 1% damping simulation were performed with blocks of 8192 points for comparison with the other cases, yet calculations using the full 24 hours of data showed significant error from the theoretical half-power bandwidth. The bandwidth failed even to lie within a standard deviation of the bootstrap mean.

To confirm that the bias error was responsible, new bandwidth estimates were calculated from the same simulation using longer blocks, with 16,384 data points. Using longer blocks decreases the number of blocks available, and correspondingly increases the variance error. Despite this fact, the new calculations showed improvement. The standard bandwidth estimate improved slightly, and the mean bootstrap estimate showed significant improvement. This suggests that bias error is of critical importance, even more so than variance, when estimating half-power bandwidths.

Conclusions

Based on computer simulations of structural response, applying bootstrap techniques to the estimation of damping ratios of structures with limited data records shows significant promise, providing better results than the traditional spectra in these situations. As the number of ensembles increases, the standard spectra and the bootstrapped spectra approach one another. With short data records, mean bootstrap estimates of the half-power bandwidth tend to be much more stable than standard methods, and offer a simple means for determining the true variance error. However, insuring adequate spectral resolution to minimize bias error became a governing consideration, as bootstrapped techniques cannot overcome poor spectra resolution.

Acknowledgements

The authors gratefully acknowledge the support provided by NSF Grant CMS-95-03779. The second author was supported by the National Defense Science and Engineering Graduate Fellowship and the National Science Foundation Graduate Research Trainee Fellowship.

References

Bendat, J.S. and A.G. Piersol (1986), *Random Data: Analysis and Measurement Procedures*, J. Wiley, New York, 2nd Ed.

Bradley, E. and R.J. Tibshirani (1993), *An Introduction to the Bootstrap*, Chapman & Hall, New York.

Desforges, M.J, Cooper, J.E., and J.R. Wright (1995), “Spectral and Modal Parameter Estimation from Output-Only Measurements,” *Mechanical Systems and Signal Processing*, **9**(2), 169-186.