TIME-FREQUENCY CHARACTERIZATION OF NONLINEAR DYNAMICAL SYSTEMS

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Abstract

Time-Frequency analysis techniques have received increasing attention for their ability to characterize nonlinear and nonstationary signal features. Among these techniques, the wavelet transform and a combination of Empirical Mode Decomposition and the Hilbert Transform have received considerable attention, though lacking an objective comparison of their suitability for analysis of many common nonlinear and nonstationary signals. The following study engages in such a comparison for a number of classical nonlinear systems, such as the Duffing oscillator and the Lorenz and Rössler systems, to shed new light on the manner in which the wavelet transform characterizes nonlinearities in frequency.

Introduction

While the Fourier transform remains a staple of signal analysis, its inability to handle nonlinear and nonstationary phenomenon has proven problematic, challenging analysts to develop alternative transform techniques, e.g., time-frequency analyses like the wavelet transform (WT) and Empirical Mode Decomposition with Hilbert Transform (EMD/HT). Both techniques provide a venue in which signal energy content can be characterized as a function of both frequency and time through localized bases.

While a more detailed discussion of EMD/HT-based analysis is presented in Huang *et al.* (1998), the approach can be conceptually envisioned as the decomposition of a signal, via sifting, into a series of temporally monocomponent intrinsic mode functions (IMFs). The resulting empirical bases can then be processed by the Hilbert transform to yield an estimate of the analytic signal (Gabor, 1946) allowing an instantaneous frequency estimate via the phase derivative.

The wavelet transform is defined as

$$W(a,t)=\frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} x(\tau)g^*\left(\frac{t-\tau}{a}\right)d\tau \qquad (1)$$

where a is the scale parameter, inversely proportional to frequency, and $g(t)$ is the parent wavelet, which can be selected from a wide body of known functions or engineered by the user. In the interest of space, readers are referred to any number of texts for more details on wavelet theory, e.g. Mallat (1998). For the purposes of this study, a Morlet wavelet is selected by virtue of its analogs to the Fourier basis and the direct relationship between its central frequency parameter f_o and frequency and scale. In the applications herein, the complex-valued wavelet coefficients along the ridge will be utilized directly to estimate the analytic signal (Staszewski, 1997) and specifically the instantaneous frequency. As such, wavelet coefficients will be presented not only in terms of the squared magnitude as the wavelet scalogram, but in a skeleton plot named the wavelet instantaneous frequency spectrum (WIFS), which can then be directly compared to the skeleton plot generated through EMD/HT. In addition, the instantaneous spectral bandwidth of the wavelet scalogram will be presented as a secondary measure of nonlinear characteristics, as discussed in Kijewski-Correa (2003). Finally, it should be noted that the selection of central frequency has great bearing on the resolution capabilities of the transform, as discussed in Kijewski and Kareem (2003).

Examples

The following examples demonstrate that the two approaches provide comparable evidence of nonlinear and nonstationary features and demonstrate that complete characterization of subtle nonlinearities arises through distinctly different measures, as the wavelet employs both instantaneous frequency and bandwidth. Examples of EMD/HT provided below are taken from Huang *et al*. (1998), while wavelet results are generated in accordance with the procedure in Kijewski-Correa (2003).

Duffing Oscillator

Huang *et al.* (1998) explored a variety of classical non-linear problems with distinctly different frequency nonlinearities using wavelets and EMD/HT, one being the Duffing oscillator under harmonic excitation, in accordance with the second-order differential equation

$$\frac{d^2x}{dt^2} + \left(1 + \varepsilon x^2\right) \cdot x = \gamma \cos \omega t \quad (2)$$

where ε, γ are constants and ω is the harmonic forcing frequency. Huang *et al.* (1998) investigated the solution to this problem using EMD/HT and identified 3 components characterized by marked intrawave oscillations of both long and short period and noted that the Morlet wavelet could not capture the intrawave frequency modulations.

In order to expand upon these findings, the response of the system in Equation 2 is generated using a 4th order Runga Kutta simulation to determine if the wavelet is sensitive enough to detect even subtler nonlinearities, embodied by a Duffing oscillator with ε = -0.22, γ = 0.1, forced at frequency of 1/50 Hz. As shown in Figure 1, the wavelet scalogram's warmest hues indicate the frequencies at which the signal concentrates, clearly at the dominant frequency of the oscillator. The higher frequencies of the scalogram manifest a rippling indicative of fluctuations in the frequency content with time. The instantaneous frequencies were identified from the ridges of the transform and reflect two dominant components with one near 0.12 Hz manifesting some oscillation and one corresponding to the forcing frequency near 0.02 Hz. The instantaneous frequency estimate from the wavelet phase, grounded in Fourier harmonics, manifests a smooth regular periodicity. As the wavelet analysis fits small waves to the data, the instantaneous frequency estimate is more representative of the best-fit frequency over some short time

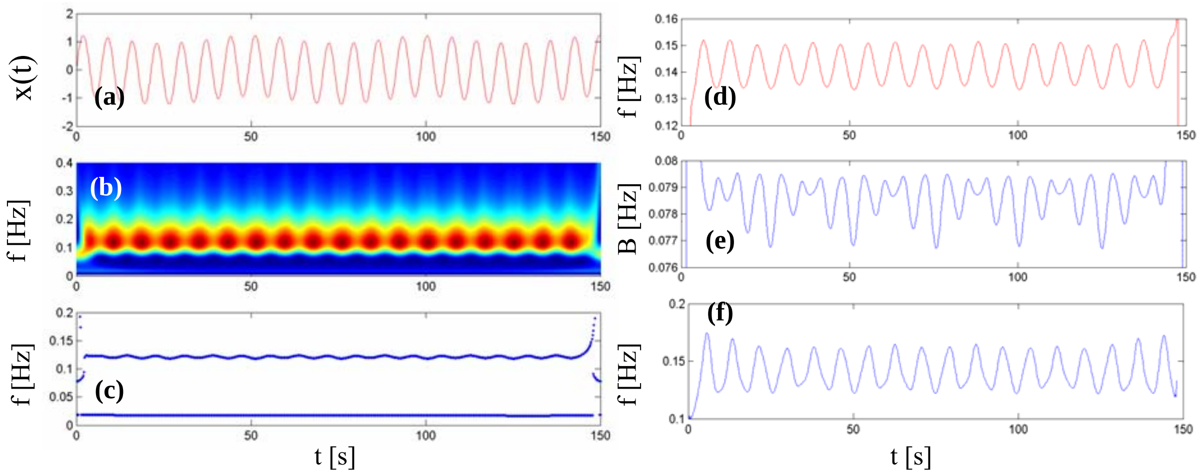

Figure 1. (a) Forced Duffing oscillator; (b) wavelet scalogram; (c) WIFS; (d) instantaneous frequency of high frequency component (WT phase); (e) instantaneous bandwidth of high frequency component; (g) estimate of instantaneous frequency by Hilbert Transform.

interval. Thus it is well suited for supercyclic oscillations but may not fully capture subcyclic characteristics. Turning to the wavelet instantaneous bandwidth, a different perspective with a far more oscillatory characteristic is evident, as would be expected for a measure that carries in it the deviation from this mean frequency at a specific time or the subcyclic oscillations. This result can be compared to the instantaneous frequency estimated via the Hilbert transform, also shown in Figure 1. This result shows more irregularity when compared to the wavelet instantaneous frequency, as expected. For the wavelet, any fluctuations or spread about this mean are carried in the bandwidth, while the Hilbert result does not separate these two components and provides a modulated instantaneous frequency as a result of intrawave fluctuations.

Lorenz System

The Lorenz system, initially proposed to study deterministic non-periodic flow, has become a fundamental system for the investigation of chaos and was considered by Huang *et al.* (1998) as another classic nonlinear system for investigation using EMD/HT. The system is described by

$$\dot{x} = -\sigma x + \sigma y \qquad \dot{y} = rx - y - xz \qquad \dot{z} = -bz + xy \tag{3}$$

where σ, r and b are positive constants, taken as 10, 20 and 3, respectively, for the purpose of this example. The system is released from its initial position of (10, 0, 0), resulting in the x-component response shown in Figure 2, illustrating the transient characteristics of the system. The EMD/HT result generated by Huang *et al.* (1998) is provided in Figure 2 and reveals the characteristic intrawave frequency modulation observed in other nonlinear systems as a result of subcyclic frequency variations. EMD/HT also manifests a low frequency component, which rapidly drops from 1 Hz and lingers around 0.1-0.2 Hz. This mode corresponds to the rapid transient drop occurring in the first few seconds of the Lorenz response also shown in Figure 2. After this transient

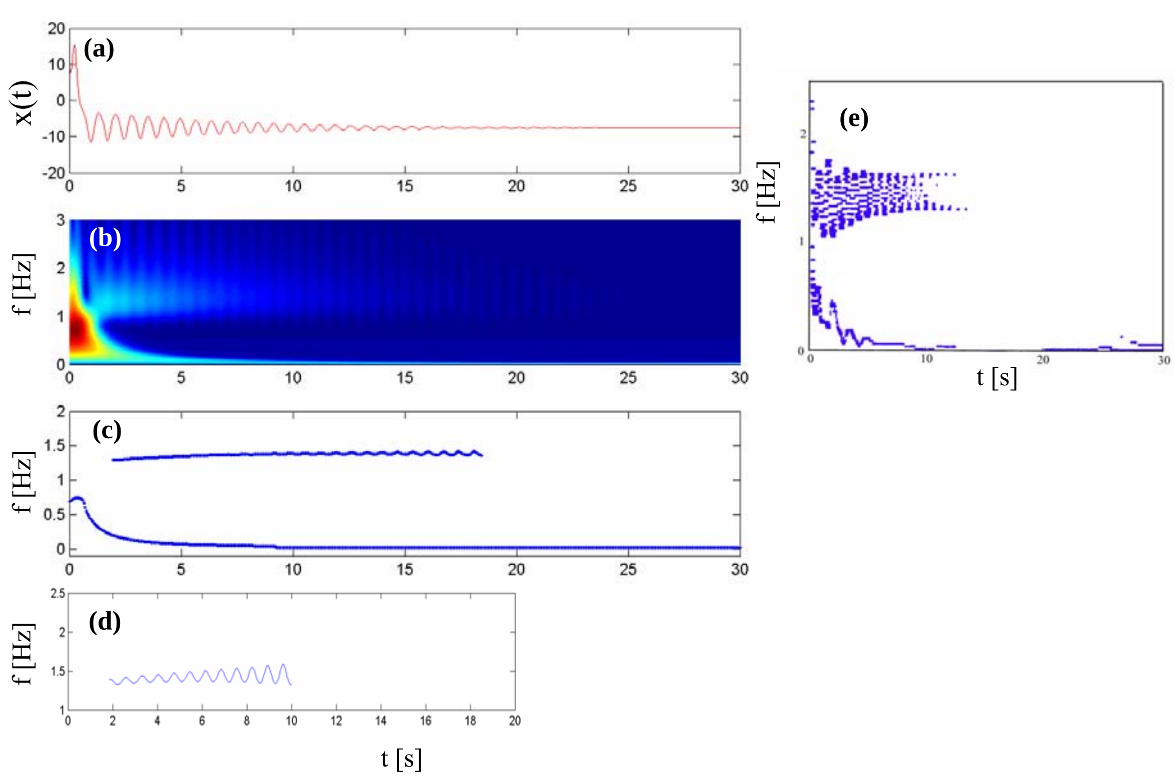

FIGURE 2. (a) Lorenz system x-component; (b) wavelet scalogram; (c) WIFS; (d) instantaneous frequency of high frequency component (via wavelet phase); (e) Huang *et al.*'s (1998) EMD/HT result.

period, the oscillator tends to behave as a damped nonlinear system. Note the decaying of the instantaneous frequency indicating supercyclic frequency modulation due to the damping characteristic as well as the oscillations about 1.4 Hz due to subcyclic nonlinear behavior. Linearization of the system by Huang *et al.* (1998) yielded a dominant frequency of about 1.46 Hz, consistent with the main frequency about which intrawave modulations occur. In terms of the wavelet result, also shown in Figure 2, while the scalogram indicates that the wavelet has captured the transient behavior, the WIFS provides a clearer representation of the behavior. Consistent with the EMD/HT, the Morlet wavelet with f_o=0.5 Hz, chosen for more precise temporal resolution, was capable of detecting the marked frequency shifts due to the transient behavior in the first few seconds of the signal. This is characterized by a shift in the low frequency component from about 0.5 Hz to 0.019 Hz. As these shifts result from significant amplitude changes, they can be detected by the wavelet instantaneous frequency. After this transition range, the wavelet detects a bi-modal response, with the low frequency response near 0.019 Hz and the oscillator's response about the linearized frequency of the system. The estimation of instantaneous frequency of this mode shows oscillation similar to the EMD/HT and begins to fade out and become less reliable as the signal's energy decays, consistent with the EMD/HT, which indicates that the instantaneous frequency estimate about 1.4 Hz is scarcely detected beyond 10 seconds, though the lower frequency response is still clearly present. This example demonstrates that the selection of wavelets with appropriate resolution can detect subcyclic characteristics.

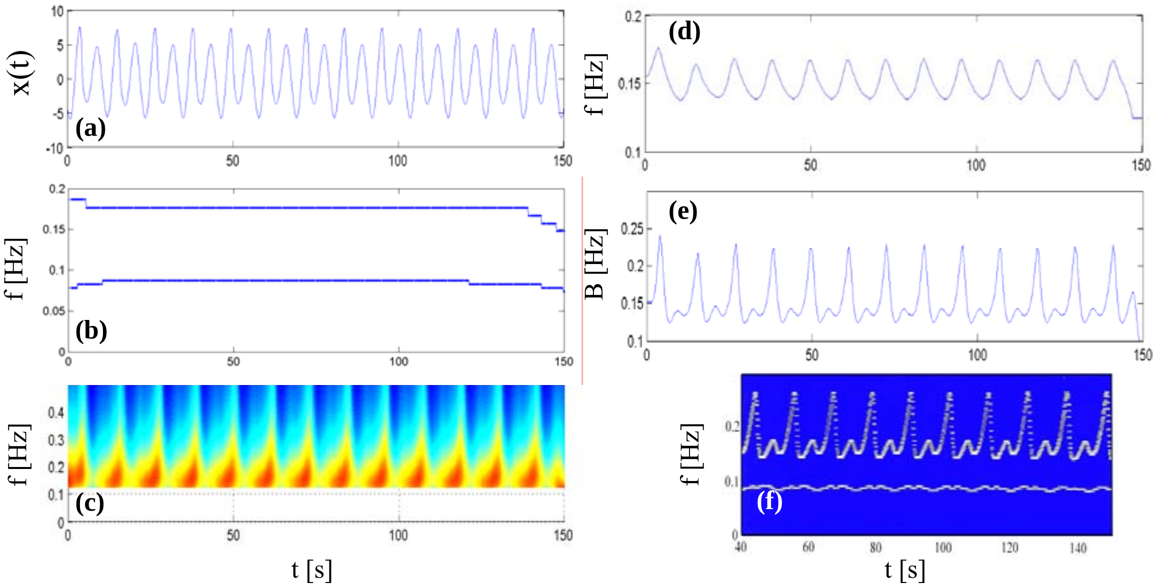

FIGURE 3. (a) Rössler system x-component; (b) WIFS - stage 1; (c) wavelet scalogram - stage 2; (d) WIFS - stage 2; (e) instantaneous bandwidth - stage 2; (f) Huang *et al.*'s (1998) EMD/HT result.

Rössler System

Next consider the Rössler system for the famous period doubling event:

$$\dot{x} = -(y + z) \qquad \dot{y} = x + \frac{1}{5}y \qquad \dot{z} = \frac{1}{5} + z(x + \mu) \tag{4}$$

where μ= 3.5. The system simulated by a first-order forward difference technique from the initial conditions {x(0) = -4, y(0) = 4, z(0) = 0} is shown in Figure 3. The EMD/HT of the x-component of the Rössler system, as performed by Huang *et al.* (1998), is provided in Figure 3f. Note that the EMD of the data yielded 2 meaningful IMF components, of interest is the mode oscillating between 0.17 and 0.25 Hz representing the dominant nonlinear characteristic of this system. As wavelet analyses can often be performed in stages, it is often desirable to first identify components within the signal by choosing a wavelet with sufficiently fine frequency resolution. This can be achieved using f_o = 5 Hz in stage 1 to identify two distinct components at around 0.09 and 0.17 Hz (Fig. 3b). This information then defines two frequency ranges over which to apply a more temporally refined analysis in stage 2 focusing on the higher frequency component. To track the time-evolving frequency content, a wavelet with fine time resolution must be chosen (f_o=0.25 Hz). Using this wavelet in a stage 2 analysis, the scalogram in Figure 3c is obtained. Note that the contours of the wavelet scalogram mimic the same oscillatory peaking of the frequency observed in the EMD/HT. This oscillatory pattern is represented in a mean sense by the WIFS in Figure 3d, which reveals that the energy is primarily concentrated in an oscillatory pattern between 0.1375 and 0.1665 Hz. Also shown is the instantaneous bandwidth analysis in Figure 3e that clearly demonstrates the behavior identified in the EMD/HT. Thus the dual identification of the instantaneous frequency and bandwidth from the wavelet transform illustrates its ability to characterize nonlinear behavior, albeit differently than the EMD/HT.

Conclusions

As the examples presented in this study demonstrate, there is a clear distinction between the wavelet transform and Empirical Mode Decomposition with Hilbert Transform: EMD/HT will in general detect subtle changes in frequency due to some nonlinearity, essentially identifying all nonlinearities (subcyclic and supercyclic) through a single instantaneous frequency measure, while the Morlet wavelet's instantaneous frequency detects changes in the instantaneous frequency in its truest mean sense, as it locally fits windowed sinusoids to the data. Dependent on the resolution chosen, the wavelet transform may only be capable of detecting nonlinearities evolving over entire cycles of oscillation or following significant changes in amplitude, termed supercyclic oscillations. In which case, the measure of subcyclic oscillations, rather being carried through the instantaneous frequency, is portrayed by the deviations from this time varying mean — the instantaneous bandwidth. However, adjustments to temporal resolutions of parent wavelets allow the wavelet to detect subcyclic nonlinearity even in its instantaneous frequency measure, whereas poor temporal resolution will reduce the Morlet wavelet analysis to a Fourier harmonic analysis. Thus, it becomes apparent that both EMD/HT and the wavelet transform can characterize nonlinear signal features, though they characterize them in fundamentally different ways, dependent greatly on the wavelet resolutions chosen.

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